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PART II.

LIST OF ILLUSTRATIONS.

XI. *Double Refraction and Dispersion in Iceland Spar: an Experimental Investigation, with a comparison with* HUYGHEN'S *Construction for the Extraordinary Wave.*

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Communicated by Professor J. CLERK MAXWELL, *M.A., F.R.S.*

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SECTION I.

Preliminary.

IN a paper read before the Royal Society, June 20, 1878, the results of an investigation into the truth of FRESNEL'S theory of double refraction in a biaxal crystal were stated. The comparison between theory and experiment was made by a method suggested by Professor STOKES (British Association Report, 1862), according to which the reciprocal of the velocity of wave propagation was determined by experiment and also on FRESNEL'S theory. The greatest difference between the two amounted to ·0009, and there appeared to be some connexion between the differences and the wave length of the light used. In the endeavour to follow up this connexion I undertook a series of similar experiments with light of different wave lengths, using three lines of the hydrogen spectrum and the sodium line. The extreme smallness of the arragonite prisms I had previously worked with led me to use, at first at least, Iceland spar, which could be obtained in large pieces with ease, and for which the theoretical calculations were greatly more simple. Professor STOKES had already made a series of experiments by the same method with this substance (Proceedings of the Royal Society, vol. 20, p. 443) and arrived at results confirming HUYGHEN'S construction. The details of his experiments are as yet unpublished, and I venture to think it might be useful to have arranged in tabular form a series of results, to serve in the future as a test of any theory of double refraction which might be proposed. The method of the experiments, as suggested by Professor STOKES (British Association Report, 1862), is as follows: A prism is cut from a piece of spar, and the position of its faces with reference to the cleavage faces carefully determined. The prism is mounted on a spectrometer, and the collimator adjusted so that the rays of a definite wave length falling on the prism are parallel, the edge of the prism being parallel to the axis of revolution of the reading telescope. The deviation of the light passing through the prism in any position is observed, also the position of the image of the

slit formed by reflexion at the face of incidence. From this and the known direction of the incident light we can calculate the angle of incidence.

Let this be ϕ. Let the deviation be D and the angle of the prism i. Let V be velocity of the light in air, v in the crystal. Let ψ be the angle of emergence, ϕ' ψ' the angles which the wave normal in the crystal makes with the faces of the prism.

Then we have

$$\left.\begin{aligned}\frac{\sin\phi}{V}&=\frac{\sin\phi'}{v}\\ \frac{\sin\psi}{V}&=\frac{\sin\psi'}{v}\end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$\left.\begin{aligned}\phi'+\psi'&=i\\ \phi+\psi&=D+i\end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (2)$$

$$\frac{\sin\phi}{\sin\psi}=\frac{\sin\phi'}{\sin\psi'}$$

$$\therefore \frac{\sin\phi+\sin\psi}{\sin\phi-\sin\psi}=\frac{\sin\phi'+\sin\psi'}{\sin\phi'-\sin\psi'}$$

$$\tan\frac{\phi'-\psi'}{2}=\tan\frac{\phi'+\psi'}{2}.\tan\frac{\phi-\psi}{2}.\cot\frac{\phi+\psi}{2} \quad \ldots \ldots \quad (3)$$

whence we can find $\phi'-\psi'$, and since $\phi'+\psi'$ is known, we can get at once ϕ' and ψ', and then v is given by either of the formulæ

$$\frac{V}{v}=\frac{\sin\phi}{\sin\phi'}=\frac{\sin\psi}{\sin\psi'} \quad \ldots \ldots \ldots \ldots \quad (4)$$

But since we know the position of the faces of the prism with reference to the optic axis, we can find the angle between the wave normal and the optic axis, and if μ_1, μ_2 be the reciprocals of the principal velocities, μ that of a velocity in a direction making an angle θ with the optic axis, we have by HUYGHEN'S construction,

$$\frac{1}{\mu^2}=\frac{\cos^2\theta}{\mu_1^2}+\frac{\sin^2\theta}{\mu_2^2} \quad \ldots \ldots \ldots \ldots \quad (5)$$

and from this μ_1, μ_2, θ being known, we can find μ.

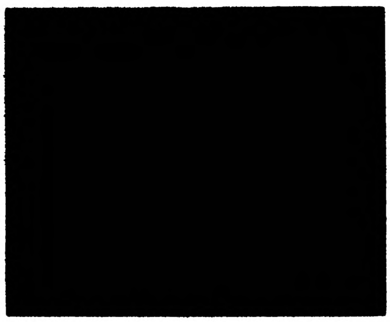
Section II.

I. *Description of crystal.*
II. *Account of experiments with the results.*

It was my object in carrying out the work to secure a series of observations for values of θ from 0° to 90°, differing by about 1° 30′ or rather less. This I found could be obtained by the use of four prisms of 44° or thereabouts, each having its edge perpendicular to the optic axis, which would therefore lie in the principal plane of each prism, the prisms being so cut that the optic axis made angles of −32°, 14°, 38°, and 64°, with the outward drawn normal to one of the faces; the angles are considered positive when the optic axis falls on the same side of the normal as the edge of the prism. Prisms cut in this manner would, I found, enable me to work over a range extending from about 5° on one side of the optic axis to about 100° on the other.

Iceland spar, as is well known, cleaves readily so as to form an oblique rhombohedron.

Fig. 1.



Let A B C D E F G, fig. 1, represent a rhomb of spar, and let A be a solid angle, such that each of the three plane angles B A D, D A F, F A B is obtuse. The optic axis is equally inclined to each of the faces B A D, D A F, F A B, the angle of inclination being 26° 15′ 30″ about. It is, therefore, perpendicular to the interior bisectors of the acute angles G F A, G B A. I procured a large rhomb of spar, which was cut by A. Hilger, 196, Tottenham Court Road, into four prisms, the edge of each being nearly parallel to the interior bisectors of the acute angle of the same rhombic face. The angle of each prism was about 44°, and the faces were cut so as to be inclined to the optic axis as stated above.

We proceed now to describe the experiments and give the results for each of these four prisms numbered I., II., III., and IV. In each case let P, Q denote the faces of the prism, i the angle between them, ϕ' ψ' the angles which the wave normal in the prism makes with the normals to P, Q respectively, ϕ ψ the corresponding angles in air; ϕ is the angle of incidence or emergence according as the light is incident on P or Q, and *vice versâ* for ψ.

The values of the angle of incidence on one face extend from nearly grazing incidence

to the position of minimum deviation, forming an arithmetic progression of which the common difference is 4°. The prism was then reversed so that the face of incidence became that of emergence, and another set of results obtained, extending from minimum deviation to nearly grazing incidence on that face.

Each set of experiments was taken twice, and only in two or three cases were the differences between the results of the two measurements, usually made on different days, greater than 20″. In about 18 per cent. of the measurements the differences amounted to 20″, in the rest it was less, so that in comparatively few cases is the difference between the mean and an extreme observation as great as 10″.

The spectrometer was the same as that used in the experiments with arragonite, and was kindly lent me by Professor STOKES. The method of taking the measurements and the means adopted to secure the parallelism of the edge of the prism and the axis of rotation of the telescope are described at length (Phil. Trans., 1879). The collimator and telescope were focused for parallel rays by means of a method suggested by Dr. SCHUSTER (Phil. Mag., February, 1879).

The focusing was done once for each prism, and remained untouched during the experiments with that prism. All the adjustments were made for the red hydrogen line C. When the rays from this line were parallel no appreciable alteration was required to render the sodium rays parallel.

The other hydrogen rays F and g were very nearly parallel, but probably not quite so.

The experiments were performed in the spectroscope room at the Cavendish Laboratory, which was kindly placed at my disposal by Professor MAXWELL during February, March, and April of the present year.

The value given for the angle of the prism is in each case the mean of 10 measures, no two of which differed by more than 20″.

In the course of the preliminary work I found that variations in temperature of 5° or 6° C., to which the room was subject during the months of February and March, produced a very appreciable effect in the value of the angles between some of the faces. In making the final measurements, therefore, I was careful to keep the room at a nearly constant temperature of about 13° C. by means of a gas stove.

For each position of the prism an observation of the deviation of each of the four rays C, D, F, g was taken so that there are four values of deviation, corresponding respectively to these four rays, to each angle of incidence.

Tables I., II., III., and IV. give the results of experiment for the red line C of the hydrogen spectrum in the four prisms.

The error in the result, due to an error in one of the observed quantities, is greatest near the position of minimum deviation. If we assume an error of 10″ in the values of the angle of incidence and the deviation taken so as to produce the maximum error in the result, that error amounts to about ·00005 when a maximum. The probable error of the experiments is considerably less than this.

Table I.—Prism I., Ray C.

$i = 43° 56' 20''$.

D+*i*.	ϕ.	ϕ'.	μ.
89° 25′ 5″	76° 8′	35° 57′ 5″	1·65367
86 48 5	72 8	35 7 56	1·65393
84 30 20	68 8	34 7 39	1·65416
82 31 0	64 8	32 56 58	1·65438
80 48 40	60 8	31 36 52	1·65431
79 22 35	56 8	30 8 6	1·65395
78 12 30	52 8	28 31 16	1·65335
77 17 15	48 8	26 47 25	1·65223
76 37 25	44 8	24 56 58	1·65078
76 12 35	40 8	23 0 48	1·64873
76 4 0	36 8	20 59 21	1·64623
D+*i*.	ψ.	ψ'.	
76° 4′ 1″	39° 51′ 48″	22° 54′ 48″	1·64627
76 12 26	43 51 48	24 56 23	1·64335
76 37 36	47 51 48	26 52 44	1·64021
77 18 41	51 51 48	28 43 11	1·63684
78 15 56	55 51 48	30 26 47	1·63341
79 29 26	59 51 48	32 2 45	1·62991
81 0 36	63 51 48	33 29 49	1·62669
82 49 46	67 51 48	34 47 10	1·62356
84 58 36	71 51 48	35 53 35	1·62095
87 27 31	75 51 48	36 48 20	1·61862
90 17 56	79 51 48	37 30 24	1·61680

Table II.—Prism II., Ray C.

$i = 43° 36' 19''$.

D+*i*.	ϕ.	ϕ'.	
84° 39′ 59″	71° 51′ 30″	35° 46′ 5″	1·62580
82 16 14	67 51 30	34 47 11	1·62352
80 9 4	63 51 30	33 38 11	1·62064
78 18 19	59 51 30	32 19 22	1·61734
76 42 24	55 51 30	30 51 43	1·61345
75 21 29	51 51 30	29 15 29	1·60919
74 14 19	47 51 30	27 31 37	1·60438
73 20 59	43 51 30	25 40 30	1·59912
72 41 14	39 51 30	23 42 54	1·59350
72 15 24	35 51 30	21 39 16	1·58745
72 4 29	31 51 30	19 30 7	1·58107
D+*i*.	ψ.	ψ'.	μ.
72° 8′ 56″	44° 8′ 16″	26° 15′ 2″	1·57449
72 30 6	48 8 16	28 21 12	1·56820
73 7 36	52 8 16	30 21 6	1·56239
74 1 1	56 8 16	32 13 56	1·55692
75 11 1	60 8 16	33 58 32	1·55187
76 38 41	64 8 16	35 33 35	1·54733
78 25 21	68 8 16	36 57 46	1·54350
80 31 41	72 8 16	38 10 8	1·54019

Table III.—Prism III., Ray C.

$i=43°\ 53'\ 57''$.

D+i.	ψ.	ψ'.	μ.
93° 2′ 0″	86° 6′ 24″	39° 29′ 24″	1·56883
89 31 5	82 6 24	39 10 47	1·56789
86 19 45	78 6 24	38 39 37	1·56640
83 27 15	74 6 24	37 56 21	1·56431
80 52 50	70 6 24	37 1 26	1·56163
78 35 20	66 6 24	35 55 34	1·55827
76 35 5	62 6 24	34 38 42	1·55468
74 50 15	58 6 24	33 11 58	1·55058
73 20 10	54 6 24	31 35 58	1·54608
72 3 55	50 6 24	29 51 24	1·54116
71 1 50	46 6 24	27 58 36	1·53619
70 13 0	42 6 24	25 58 29	1·53093
69 37 35	38 6 24	23 51 34	1·52567
69 16 5	34 6 24	21 38 30	1·52042
69 9 10	30 6 24	19 19 56	1·51523
D+i.	ϕ.	ϕ'.	μ.
69° 11′ 4″	36° 53′ 26″	23° 17′ 35″	1·51805
69 11 24	40 53 26	25 38 16	1·51293
69 27 29	44 53 26	27 53 54	1·50833
69 59 9	48 53 26	30 4 0	1·50435
70 45 59	52 53 26	32 5 50	1·50084
71 48 34	56 53 26	34 0 6	1·49786
73 7 39	60 53 26	35 45 1	1·49537
74 43 49	64 53 26	37 19 39	1·49331
76 38 39	68 53 26	38 42 35	1·49172
78 52 44	72 53 26	39 53 6	1·49044
81 27 44	76 53 26	40 49 54	1·48962
84 23 49	80 53 26	41 32 40	1·48882
87 42 39	84 53 26	42 0 16	1·48841

Table IV.—Prism IV., Ray C.

$i=43°\ 51'$.

D+i.	ϕ.	ϕ'.	μ.
69° 2′ 48″	47° 53′ 9″	29° 50′ 17″	1·49092
68 16 23	43 53 9	27 44 57	1·48887
67 44 18	39 53 9	25 32 30	1·48726
67 25 43	35 53 9	23 14 7	1·48583
67 22 28	31 53 9	20 50 13	1·48499
D+i.	ψ.	ψ'.	μ.
67° 22′ 16″	33° 6′ 43″	21° 34′ 43″	1·48534
67 25 16	37 6 43	23 58 48	1·48470
67 43 16	41 6 43	26 17 24	1·48457
68 15 41	45 6 43	28 29 52	1·48491
69 1 41	45 6 43	30 35 34	1·48554

Column 1 gives the value of $D+i$, D being the observed deviation, and i the angle of the prism. (In the calculations D occurs only in the form $D+i$, therefore $D+i$ is given in the tables instead of D.) Column 2 the observed angle of incidence. Column 3 the angle which the wave normal in the crystal makes with the normal to the faces of incidence calculated from the formulæ

$$\psi'+\phi'=i$$

$$\tan\frac{\phi'\sim\psi'}{2}=\tan\frac{\phi\sim\psi}{2}\cot\frac{\phi+\psi}{2}\tan\frac{i}{2}$$

already proved, and column 4 the values of μ or $\frac{V}{v}$ calculated from

$$\frac{V}{v}=\frac{\sin\phi}{\sin\phi'}=\frac{\sin\psi}{\sin\psi'}$$

On comparing the results for the ray C with theory I found so close an agreement that I thought it hardly requisite to work out all the calculations for the rays F and g. I therefore completed the calculations for only about a third of the observations, giving a series of values of μ in directions inclined at angles of about 4° to each other, extending in an almost continuous arc from the optic axis to directions perpendicular to it.

These are contained in Tables V. and VI.

The middle column in each case gives the angle of incidence. The columns on the right refer to the ray g, those on the left to the ray F.

For Table V., Prism II., the results for the angle of incidence ϕ have been calculated for the value 46° 36′ 53″ of the angle of the prism instead of 46° 36′ 19″ the value used for the results in which the angle of incidence is denoted by ψ. The reasons for this will be discussed in connexion with the theory.

This closes the experimental part of the work.

TABLE V.—Results of the Experiments.

Ray F. Prism I.			Angle of incidence.	Ray g. $i=43° 56' 20''$.		
μ.	ϕ'.	D + i.	ϕ.	D + i.	ϕ'.	μ.
1·66780	32° 39′ 3″	83° 11′ 15″	64° 8′	83° 34′ 30″	32° 28′ 52″	1·67557
1·66663	28 16 25	78 52 40	52 8	. .	. .	. .
1·66385	24 44 25	77 18 15	44 8	77 42 5	24 37 14	1·67143
μ.	ψ'.	D + i.	ψ.	D + i.	ψ'.	μ.
1·65978	22° 42′ 59″	76° 47′ 31″	39° 51′ 48″	77° 12′ 6″	22° 36′ 27″	1·66736
1·64996	28 28 13	77 58 16	51 51 48	78 20 6	28 20 6	1·65720
1·64287	31 45 48	80 8 11	59 51 48	80 30 6	31 36 22	1·65022
1·63360	35 34 20	85 36 51	71 51 48	85 58 11	35 23 46	1·64067
1·62930	37 10 12	90 56 6	79 51 48	91 17 56	36 58 48	1·63643
Prism II.			$i=43° 36' 19''$.	Prism II.		
μ.	ϕ'.	D + i.	ϕ.	D + i.	ϕ'.	μ.
1·63455	34° 30′ 58″	82° 50′ 18″	67° 51′ 30″	83° 8′ 48″	34° 22′ 2″	1·64087
1·61974	29 2 58	75 53 43	51 51 30	76 11 28	28 55 58	1·62570
1·60336	23 33 38	73 12 43	39 51 30	73 30 18	23 28 21	1·60903
1·59058	19 22 50	72 36 58	31 51 30	72 54 43	19 18 48	1·59590
μ.	ψ'.	D + i.	ψ.	D + i.	ψ'.	μ.
1·58487	26° 3′ 57″	72° 40′ 1″	44° 8′ 16″	72° 57′ 41″	25° 57′ 44″	1·59072
1·56653	32 0 38	74 29 21	56 8 16	74 45 36	31 53 6	1·57205
1·54924	37 54 21	80 59 16	72 8 16	81 15 1	37 45 26	1·55442

TABLE VI.—Results of the Experiments.

Ray F.			Angle of incidence.	Ray g.		
Prism III.				$i=43°\ 53'\ 5$		
μ.	ψ'.	D+i.	ψ.	D+i.	ψ'.	
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1·57014	36 47 24	81 18 40	70 6 24	81 33 30	36 39 26	1·57502
1·55876	33 0 11	75 14 35	58 6 24	75 28 30	32 53 30	1·56345
1·54914	29 41 15	72 27 35	50 6 24	72 41 10	29 35 29	1·55371
1·53312	23 44 11	70 0 25	38 6 24	70 13 40	23 39 56	1·53745
1·52241	19 14 15	69 32 35	30 6 24	69 45 50	19 11 5	1·52644
μ.	ψ'.	D+i.	ϕ.	D+i.	ϕ'.	
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1·52573	23 10 8	69 34 39	36 53 26	69 48 14	23 5 54	1·53014
1·51571	27 45 2	69 49 24	44 53 26	70 1 34	27 40 10	1·51981
1·50475	33 49 30	72 9 4	56 53 26	72 20 49	33 43 28	1·50870
1·50005	37 7 53	75 4 19	64 53 26	75 15 59	37 1 15	1·50389
1·49610	40 37 1	81 48 14	76 53 26	82 0 4	40 29 39	1·49982
	Prism IV.		$i=43°\ 51'$.		Prism IV.	
	ϕ'.	D+i.	ϕ.	D+i.		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1·49507	27 37 28	59 7 35	43 53 9	58 57 15	27 33 16	1·49856
1·49114	20 44 52	60 0 35	31 53 9	59 49 40	20 41 49	1·49460
μ.	ψ'.	D+i.	ψ.	D+i.	ψ'	
	° ′ ″	° ′ ″	° ′ ″			
1·49074	26 10 22	68 1 36	41 6 43	68 12 11		1·49430

SECTION III.

I. *Determination of the position of the principal plane of the prism.*

II. *Proposition proved.—The principal plane of prisms I., III., and IV. may be treated as if it passed through the optic axis.*

III. *Theoretical calculations for the reciprocal of the wave velocity.*

Our next step will be the determination of the position of the faces of the prisms with reference to the optic axis.

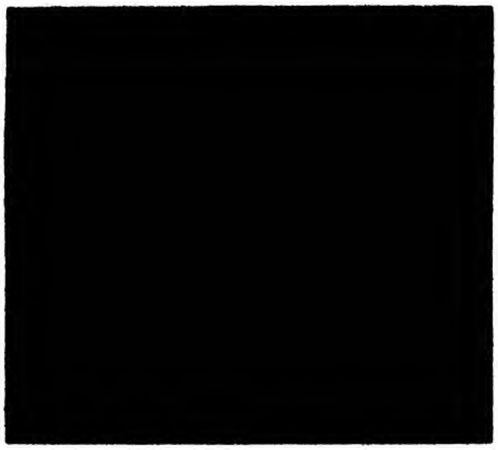
This was accurately determined for each prism by measuring the angles between them and two of the rhombic faces of the crystal.

The angle between these faces and also the angle between the cut faces of each of the prisms were accurately observed.

Let us take point O within the crystal as origin, and from it draw normals to the

faces of the rhomb. Let the normals, drawn in directions making acute angles with each other, meet in R_1 R_2 R_3 a sphere centre O. Then $R_1 R_2 = R_2 R_3 = R_3 R_1$, and if the optic axis meet the sphere in S, $SR_1 = SR_2 = SR_3$.

Fig. 2.



Let P Q be the points in which the normals to the two faces, P Q, of one of the prisms meet the sphere. Let us take the plane R_1 R_2 as plane of x y, the internal and external bisectors of the angle R_1 OR_2 as axes of x and y respectively, the axis of z being perpendicular to the plane, x y. Then R_3 and S lie in the plane, z x.

Let

$$PR_1 = \theta_1$$
$$PR_2 = \theta_2$$
$$R_1R_2 = 2\mu$$

θ_1 θ_2 μ are known from experiment. Let α β γ be the direction angles of O P. Then from triangles P x R_1 P x R_2

$$\cos\theta_1 = \cos\alpha\cos\mu + \sin\alpha\sin\mu\cos PxR_1$$
$$\cos\theta_2 = \cos\alpha\cos\mu - \sin\alpha\sin\mu\cos PxR_1$$

$$\cos\alpha = \frac{\cos\theta_1 + \cos\theta_2}{2\cos\mu}$$

$$= \frac{\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}}{\cos\mu} \quad \ldots\ldots\ldots\ldots \quad (1)$$

From triangles P y R_1 P y R_2

$$\cos\theta_1 = \cos\beta\cos\left(\frac{\pi}{2}+\mu\right) + \sin\beta\sin\left(\frac{\pi}{2}+\mu\right)\cos PyR_2$$

$$\cos\theta_2 = \cos\beta\cos\left(\frac{\pi}{2}-\mu\right) + \sin\beta\sin\left(\frac{\pi}{2}-\mu\right)\cos PyR_2$$

$$\cos\theta_2 - \cos\theta_1 = 2\cos\beta\sin\mu$$

$$\cos\beta=\frac{\cos\theta_2-\cos\theta_1}{2\sin\mu}$$

$$=\frac{\sin\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}}{\sin\mu} \quad (2)$$

These formulæ give us the values of α and β.

2μ or the angle between the normals to the rhombic faces was observed in three pieces of the crystal used. The values were

74° 55′ 37″ Mean of four measures. Maximum difference, 10″.
74° 55′ 34″ Mean of five measures.
74° 55′ 35″ Mean of ten measures. Maximum difference, 25″.

We may therefore put with great accuracy

$$2\mu=74°\ 55'\ 35''$$

The temperature indicated by a thermometer placed almost in contact with the crystal, and shaded from the direct radiation of the light used to read the vernier, was from 14° C. to 13° C. Each of the angles θ_1 θ_2 was observed ten times for each face and the mean taken, the temperature being kept as nearly as possible at 13° C. The greatest variation between any two observations never exceeded 40″.

TABLE VII.—The position of the normals to the faces of the prisms.

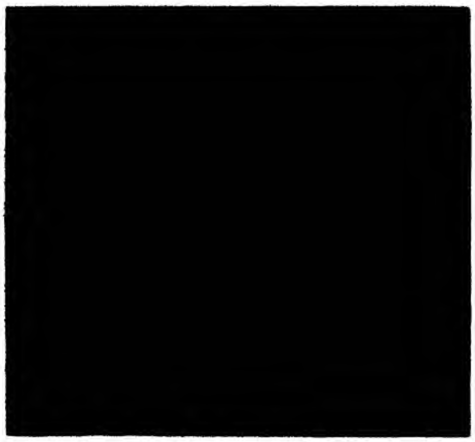
Face and direction of normal.	θ_1	θ_2	α	β
		Prism I.		
P outwards. . .	65° 48′ 35″	65° 41′ 32″	58° 50′ 25″	89° 54′ 43″
Q inwards . . .	39 50 28	39 59 43	14 55 34	90 4 53
		Prism II.		
P inwards . . .	85° 48′ 20″	85° 39′ 20″	84° 37′ 5″·5	89° 52′ 37″·7
Q outwards . .	53 31 35	52 53 20	41 0 55	89 34 49
		Prism III.		
P inwards . . .	70° 14′ 35″	70° 10′ 25″	64° 44′ 56″	89° 56′ 47″
Q outwards . .	104 40 25	104 43 25	108 38 34	90 2 23
		Prism IV.		
P inwards . . .	96° 40′ 55″	96° 38′ 5″	98° 23′ 59″	89° 57′ 41″
Q outwards . .	128 58 25	128 45 20	142 14 8	89 51 37·5

Table VII. gives the results of these calculations. The first column gives the face to which the normal considered is drawn and its direction with reference to the crystal prism. The next two columns give the values of θ_1 θ_2, the last two those of α β.

The values of β show that the principal plane of the prism which contains the normals to the faces P and Q is nearly coincident with the plane z O x.

We proceed to find the position of the line of junction of these planes and the angle between them.

Fig. 3.



Let P Q (fig. 3) meet z x in M.

Draw Q K, P L arcs perpendicular to z x.

Then from triangle P M L

$$\sin \mathrm{PL} = \sin \mathrm{PM} \sin \mathrm{PML}$$

From triangle Q K M

$$\sin \mathrm{QK} = \sin (\mathrm{PM} + \mathrm{PQ}) \sin \mathrm{PML}$$

Whence

$$\frac{\sin (\mathrm{PM} + \mathrm{PQ})}{\sin \mathrm{PM}} = \frac{\sin \mathrm{QK}}{\sin \mathrm{PL}}$$

and we have

$$\cot \mathrm{PM} \sin \mathrm{PQ} = \frac{\sin \mathrm{KQ}}{\sin \mathrm{LP}} - \cos \mathrm{PQ} \quad \ldots \ldots \ldots \quad (3)$$

If P and Q are on opposite sides of z x, we get

$$\cot \mathrm{PM} \sin \mathrm{PQ} = \frac{\sin \mathrm{KQ}}{\sin \mathrm{LP}} + \cos \mathrm{PQ} \quad \ldots \ldots \ldots \quad (4)$$

In these formulæ, P Q, P L, Q K being known from Table VII. and the angle of the prism, we can find P M.

Then by substitution in the formula

$$\sin \text{PML} = \frac{\sin \text{PL}}{\sin \text{PM}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we obtain P M L.

If we call P M δ, and the angle P M L χ, we have for the four prisms respectively

TABLE VIII.

	Prism I.	II.	III.	IV.
δ	22° 53′ 23″	14° 23′ 11″	25° 23′ 36″	13° 26′ 29″
χ	0° 13′ 35″	0° 29′ 41·7″	0° 7′ 31″	0° 9′ 57″

We shall now prove that in the case of prisms I., III., and IV. we may neglect the inclination of the plane of the prism to the plane $z\,x$. For S being the optic axis, N the point in which any wave normal meets the sphere, M the intersection of P Q and z O x.

Let

$$\text{NS}=\theta \quad \text{NM}=\psi$$
$$\text{SM}=\lambda \quad \text{SMN}=\chi$$

we have

$$\cos\theta = \cos\lambda\cos\psi + \sin\lambda\sin\psi\cos\chi$$
$$= \cos(\lambda-\psi) - 2\sin\lambda\sin\psi\sin^2\frac{\chi}{2}$$
$$= \cos(\lambda-\psi) - x \text{ say}$$
$$\cos^2\theta = \cos^2(\lambda-\psi) - 2x\cos(\lambda-\psi)$$

neglecting x^2.

This we may do, for x^2 is $<(\cdot 004)^4$

$$\frac{1}{\mu^2} = \frac{\cos^2\theta}{\mu_1^2} + \frac{\sin^2\theta}{\mu_2^2}$$
$$= \frac{1}{\mu_2^2} - \left(\frac{1}{\mu_2^2} - \frac{1}{\mu_1^2}\right)\cos^2\theta$$
$$= \frac{1}{\mu_2^2} - \left(\frac{1}{\mu_2^2} - \frac{1}{\mu_1^2}\right)\cos^2(\lambda-\psi) + 2x\left(\frac{1}{\mu_2^2} - \frac{1}{\mu_1^2}\right)\cos(\lambda-\psi)$$

In neglecting the mutual inclination of these planes, *i.e.*, in putting $\chi=0$, we omit a term in $\frac{1}{\mu^2}$ of the value

$$2x\cos(\lambda-\psi)\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)$$

and in μ of the value

$$\frac{x\cos(\lambda-\psi)\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)}{\left\{\frac{1}{\mu_2^2}-\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)\cos^2(\lambda-\psi)\right\}^{\frac{3}{2}}}$$

This is not greater than

$$\mu_1^3\, x\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)$$

x is not greater than

$$\frac{\chi^2}{2}$$

Term neglected is not greater than

$$\mu_1^3\,\frac{\chi^2}{2}\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)$$

In the three cases considered χ is less than 14′.
The circular measure of 14′ is ·004.

$$\therefore \frac{\chi^2}{2}=\cdot000008$$

Using RUDBERG's values of μ_1, μ_2 we have

μ_1 is less than 1·7

$\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}$ is less than ·1

μ_1^3 is less than 5

Therefore, greatest difference is less than

$$\cdot5\times\cdot000008$$
or ·000004

Hence neglecting χ or supposing the plane of each of the prisms I., III., and IV. to coincide with the plane $z\,x$ will never produce any change in the fifth decimal figure in the value of μ.

In the case of prism II. the value of χ is nearly 30′, and we may have to take account of the obliquity.

For prisms I., III., and IV. the value of θ is given by formulæ of the form

$$\theta=\lambda\pm\phi'$$

ϕ' having the meaning attached to it in the results of experiment.

To determine λ we require to know the position of the optic axis with reference to x.

The optic axis is equally inclined to R_1 R_2 R_3 (fig. 2). Hence each of the angles subtended at S by the arcs R_1 R_2, R_2 R_3, R_3 R_1 is 120°.

Therefore R_2 S x is 60°

$$\sin R_2x = \sin SR_2 \sin R_2Sx$$

$$\sin SR_2 = \frac{2}{\sqrt{3}} \sin R_2x$$

$$2R_2x = 74° \; 55' \; 35''$$

whence

$$SR_2 = 44° \; 36' \; 57'' \quad \ldots \ldots \ldots \ldots \quad (6)$$

$$= SR_3 = SR_1$$

Again, from the right-angled triangle R_3 x R_2

$$\cos R_3R_2 = \cos R_3x \cos xR_2$$

and

$$R_3R_2 = 2\mu = 2R_2x$$

$$\therefore \cos R_3x = \cos 2\mu \sec \mu$$

Substituting the value

$$2\mu = 74° \; 55' \; 35''$$

we have

$$R_3x = 70° \; 52' \; 28''$$

$$\therefore Sx = R_3x - R_3S$$

$$= 26° \; 15' \; 31'' \quad \ldots \ldots \ldots \ldots \quad (7)$$

and

$$\lambda = Sx \backsim \alpha$$

where α refers to the face P.

The position of the wave normal is also given by

$$\lambda' \pm \psi'$$

where

$$\lambda' = Sx \backsim \alpha'$$

α' being the x direction angle of the face Q.

From these equations we get the following table of values to determine θ the angle between the optic axis and any wave normal.

TABLE IX.

		λ or λ'.	θ.
Prism I.	Face P.	32° 36′ 23″	32° 36′ 23″$-\phi'$
	„ Q.	11° 19′ 57″	$\psi'-$11° 19′ 57″
Prism III.	„ P.	38° 29′ 25″	38° 29′ 25″$+\phi'$
„	„ Q.	82° 23′ 22″	82° 23′ 22″$-\psi'$
Prism IV.	„ P.	72° 8′ 28″	72° 8′ 28″$+\phi'$
	„ Q.	115° 59′ 28″	115° 59′ 28″$-\psi'$

Tables X., XI., XII., and XIII. give the values of θ calculated from these formulæ from the values of ϕ' ψ' given in Tables I., II., III., IV., V., and VI.

For prism II. we must calculate the value of θ from the formula

$$\cos\theta = \cos \mathrm{SM} \cos \mathrm{NM} + \sin \mathrm{SM} \sin \mathrm{NM} - \cos\chi$$

$$= \cos \mathrm{SM}\{\cos \mathrm{NM} + \tan \mathrm{SM} \sin \mathrm{NM} \cos\chi\}$$

$$= \frac{\cos \mathrm{SM}}{\cos\nu} \cos(\mathrm{NM}-\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

if

$$\tan\nu = \tan \mathrm{SM} \cos\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now we have from the triangle M Q K (fig. 3)

$$\sin \mathrm{MK} = \tan \mathrm{QK} \cot \mathrm{PML}$$

We know that

$$\mathrm{QK} = 0^\circ\ 25'\ 11''$$

$$\mathrm{PML} = 0^\circ\ 29'\ 41''\cdot 7 \quad \text{[Tables VII. and VIII.]}$$

$$\therefore \mathrm{MK} = 57^\circ\ 59'\ 23'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\cos \mathrm{AQ} = \cos \mathrm{AK} \cos \mathrm{KQ}$$

$$\mathrm{AQ} = 41^\circ\ 0'\ 55''$$

$$\therefore \mathrm{AK} = 41^\circ\ 0'\ 49'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\therefore \mathrm{AM} = \mathrm{AK} + \mathrm{MK} = 99^\circ\ 0'\ 12''$$

But

$$\mathrm{AS} = 26^\circ\ 15'\ 31'' \quad \text{[From Section III. (7).]}$$

$$\therefore \mathrm{SM} = 72^\circ\ 44'\ 41'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Also from Table VIII.

$$NM=14^\circ\ 23'\ 11''+\phi'$$

Hence

$$\nu=72^\circ\ 44'\ 39''\quad \text{[From Section III. (9).]}$$

$$\nu-MN=58^\circ\ 21'\ 28''-\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

From these values we can obtain the values of θ corresponding to the angles of incidence in Tables I., II., III., and IV.

TABLE X.—Theoretical results for the line C.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
	° ′ ″			
I.	3 20 42	1·65368	1·65367	− 1
I.	2 31 33	1·65393	1·65393	0
I.	1 31 16	1·65422	1·65416	− 6
I.	0 20 35	1·65435	1·65438	+ 3
I.	0 59 31	1·65430	1·65431	+ 1
I.	2 28 17	1·65399	1·65395	− 4
I.	4 5 7	1·65335	1·65335	0
I.	5 48 58	1·65231	1·65223	− 8
I.	7 39 25	1·65082	1·65078	− 4
I	9 35 35	1·64883	1·64873	−10
I.	11 34 51	1·64635	1·64627	− 8
I.	11 37 2	1·64631	1·64623	− 8
I.	13 36 26	1·64340	1·64335	− 5
I.	15 32 41	1·64018	1·64021	+ 3
I.	17 23 14	1·63678	1·63684	+ 6
I.	19 6 50	1·63332	1·63341	+ 9
I.	20 42 48	1·62989	1·62991	+ 2
I.	22 9 42	1·62660	1·62669	+ 9
II.	22 35 37	1·62560	1·62580	+20
I.	23 27 13	1·62355	1·62356	+ 1
II.	23 34 31	1·62326	1·62352	+26
I.	24 33 38	1·62085	1·62095	+10
II.	24 43 31	1·62044	1·62064	+20
I.	25 28 23	1·61855	1·61862	+ 7
II.	26 2 20	1·61711	1·61734	+23
I.	26 10 27	1·61676	1·61680	+ 4
II.	27 29 59	1·61329	1·61345	+16
II.	29 6 13	1·60897	1·60919	+22
II.	30 50 5	1·60418	1·60438	+20
II.	32 41 11	1·59893	1·59912	+19
II.	34 38 48	1·59326	1·59350	+24
II.	36 42 26	1·58721	1·58745	+24
II.	38 51 35	1·58078	1·58107	+29
II.	41 0 26	1·57443	1·57449	+ 6
III.	42 53 57	1·56879	1·56883	+ 4
II.	43 6 36	1·56817	1·56820	+ 3
III.	43 12 34	1·56787	1·56789	+ 2
III.	43 43 44	1·56634	1·56640	+ 6
III.	44 27 0	1·56420	1·56431	+11

TABLE X.—continued.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
II.	45° 6′ 30″	1·56226	1·56239	+13
III.	45 21 55	1·56151	1·56163	+12
III.	46 27 47	1·55830	1·55827	− 3
II.	46 59 20	1·55677	1·55692	+15
III.	47 44 39	1·55459	1·55468	+ 9
II.	48 43 56	1·55175	1·55187	+12
III.	49 11 23	1·55044	1·55058	+14
II.	50 18 59	1·54726	1·54733	+ 7
III.	50 47 23	1·54594	1·54608	+14
II.	51 43 10	1·54334	1·54350	+16
III.	52 31 57	1·54113	1·54116	+ 3
II.	52 55 32	1·54006	1·54019	+13
III.	54 24 45	1·53608	1·53619	+11
III.	56 24 52	1·53087	1·53093	+ 6
III.	58 31 47	1·52560	1·52567	+ 7
III.	60 44 51	1·52033	1·52042	+ 9
III.	61 47 0	1·51797	1·51805	+ 8
III.	63 3 25	1·51516	1·51523	+ 7
III.	64 7 41	1·51288	1·51293	+ 5
III.	66 23 19	1·50830	1·50833	+ 3
III.	68 32 48	1·50428	1·50435	+ 7
III.	70 35 15	1·50079	1·50084	+ 5
III.	72 29 31	1·49782	1·49786	+ 4
III.	74 14 26	1·49534	1·49537	+ 3
III.	75 49 4	1·49331	1·49331	0
III.	77 12 0	1·49170	1·49172	+ 2
IV.	78 1 15	1·49083	1·49092	+ 9
III.	78 22 31	1·49046	1·49044	− 2
III.	79 19 19	1·48955	1·48962	+ 7
IV.	80 2 5	1·48890	1·48882	− 8
IV.	80 6 35	1·48885	1·48887	+ 2
III.	80 29 41	1·48852	1·48841	−11
IV.	82 10 2	1·48715	1·48726	+11
IV.	84 33 25	1·48586	1·48583	− 3
IV.	85 35 15	1·48542	1·48534	− 8
IV.	87 1 19	1·48495	1·48499	+ 4
IV.	87 59 15	1·48474	1·48470	+ 4
IV.	90 17 56	1·48457	1·48457	0
IV.	92 30 24	1·48484	1·48491	+ 7
IV.	94 36 6	1·48549	1·48554	+ 5

θ having been determined μ is given by

$$\frac{1}{\mu^2}=\frac{\cos^2\theta}{\mu_1^2}+\frac{\sin^2\theta}{\mu_2^2}$$

$$=\frac{1}{\mu_2^2}\left\{1-\mu_2^2\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)\cos^2\theta\right\}$$

$$=\frac{1}{\mu_2^2}\left\{1-\frac{\mu_1^2-\mu_2^2}{\mu_1^2}\cos^2\theta\right\}$$

$$=\frac{1}{\mu_2^2}\cos^2\delta \qquad (14)$$

If

$$\sin^2 \delta = \frac{\mu_1^2 - \mu_2^2}{\mu_1^2} \cos^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\therefore \mu = \mu_2 \sec \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and we require to find μ_1, μ_2.

μ_1 is the maximum radius vector of the spheroidal sheet of the surface of wave slowness.

This is given by $\theta = 0$.

From Table X. we have, considering at present the line C, when

$$\theta = 0^\circ\ 20'\ 35'' \qquad \mu = 1{\cdot}65438$$

μ_1 is also the refractive index of the ordinary wave. Its value was determined by observations on the angle of incidence and deviation of the ordinary ray in prisms I., III., and IV.

The values were

$$\begin{array}{ll} 1{\cdot}65438 & \text{Prism I.} \\ \left.\begin{array}{l} 1{\cdot}65438 \\ 1{\cdot}65433 \end{array}\right\} & \text{Prism III.} \\ 1{\cdot}65433 & \text{Prism IV.} \end{array}$$

We take then as the value of μ_1,

$$1{\cdot}65436$$

Observations of the minimum deviation were made to determine μ_1 from the usual formula

$$\mu = \frac{\sin \frac{D+i}{2}}{\sin \frac{i}{2}}$$

D being the minimum deviation.

The mean of these was

$$1{\cdot}65441$$

but the error of this last method is much greater than that in the former, and as any error in the observed value of D would probably increase D, through the prism not being exactly in the position of minimum deviation, we should expect to get a value for μ_1 rather in excess of the true.

The values given by MASCART and RUDBERG are respectively

$$1{\cdot}65452$$

and

$$1{\cdot}65446$$

To determine μ_2 we must consider the minimum radius vector of the spheroidal sheet; this is given by

$$\theta = 90^\circ$$

Now when

$$\theta = 90^\circ\ 17'\ 56''$$

we see from Table X.

$$\mu = 1{\cdot}48457$$

But

$$\frac{1}{\mu^2} = \frac{\cos^2\theta}{\mu_1^2} + \frac{\sin^2\theta}{\mu_2^2}$$

$$\therefore \frac{1}{\mu_2^2} = \left\{\frac{1}{\mu^2} - \frac{\cos^2\theta}{\mu_1^2}\right\} \operatorname{cosec}^2\theta$$

Substituting the values of μ, μ_1, and θ we get

$$\mu_2 = 1{\cdot}48456$$

The values given by MASCART and RUDBERG are

1·48455
1·48474

The middle column of Table X. gives the values of μ in the directions given by the first column for the values

$$\mu_1 = 1{\cdot}65436$$
$$\mu_2 = 1{\cdot}48456$$

The Roman numerals I., II., &c., in the first column refer to the tables of experimental results from which the values of μ in the fourth column are taken. The fifth column gives the excess of experiment over theory.

These differences it will be seen are much greater in the case of prism II. than for any of the others.

They are also greater for the first part of the results in Table II., in which the face of incidence was P, than for the latter, when the light was incident on the face Q.

Postponing, then, for the present the consideration of this point, let us compare the differences between theory and experiment for prisms I., III., and IV. We notice at once their extreme smallness—the greatest is only ·00014, and only in eight out of the sixty measurements taken do they amount to as much as ·0001. The mean irrespective of sign is ·000055. The differences are, on the whole, negative near the major axis. They tend to become least at about 15° away from either axis. From that point they are positive and reach a maximum value at from 45° to 50° away from the major axis. So that the curve given by experiment would, though very nearly coincident with an ellipse, lie inside the ellipse near the major axis, cut it at about 15° from that axis, and lie outside for the rest of its course.

The difference, however, between the radii vectors to the two curves drawn in the same direction would never be greater than $\frac{1}{10000}$th part of either.

My first inference from these results was that HUYGHEN'S construction represented nature to a degree of exactness comparable with the probable error of experiment,

Before considering the results for the rays F and g we must return to the experiments with prism II.

The large differences it gives, overlapping as they do values given by experiments on prism I., in which the differences are small, pointed clearly to errors of experiment. On referring to my note-book containing the direct results of experiment, I found that the observation of deviation and incidence for the face P had been made on March 29, while the observations for the face Q and the angle between the faces were made on April 1.

It seemed possible that the temperature of the prism had been different on the two occasions, and that this was the cause of the error. I therefore proceeded to observe afresh the angle between the faces P Q of the prism. The result differed by 34″ from that found on April 1. I therefore recalculated the experimental results for the prism II. so far as the face P was concerned.

Table X. (A) gives the results of the calculations. In Table X. (B) these are compared with the theory.

TABLE X. (A).—Prism II. Ray C.

$i = 43°\ 36'\ 53''$

D + i.		ϕ'.	
84 39 59	71 51 30	35 46 16	1·62569
82 16 14	67 51 30	34 47 22	1·62341
80 9 4	63 51 30	33 38 22	1·62053
78 18 19	59 51 30	32 19 32	1·61724
76 42 24	55 51 30	30 51 53	1·61332
75 21 29	51 51 30	29 15 39	1·60906
74 14 19	47 51 30	27 31 46	1·60425
73 20 59	43 51 30	25 40 41	1·59897
72 41 14	39 51 30	23 43 3	1·59336
72 15 24	35 51 30	21 39 24	1·58730
72 4 29	31 51 30	19 30 14	1·58091

TABLE X. (B).—Theory for same.

	μ From Theory.	μ From Experiment.	Excess of Experiment.
22 35 27	1·62560	1·62569	+ 9
23 34 21	1·62326	1·62341	+15
24 43 20	1·62043	1·62053	+10
26 2 10	1·61711	1·61724	+13
27 29 48	1·61329	1·61332	+ 3
29 6 1	1·60897	1·60906	+ 9
30 49 54	1·60418	1·60425	+ 7
32 40 59	1·59894	1·59897	+ 3
34 38 36	1·59326	1·59336	+10
36 42 14	1·58721	1·58730	+ 9
38 51 23	1·58083	1·58091	+ 8

Thus this variation has tended to decrease the differences between observation and theory, and has reduced them to almost the same magnitude as those given by the face Q of the prism. They now agree more nearly with the results of prisms I., III., and IV., though even yet the differences observed are greater than in any of the other prisms.

Prism II., however, was at first cut wrongly from the crystal, and when recut it was so small that I formed the intention of not using it at all, and leaving a gap in my series of observations between the values $\theta=27°$ and $\theta=41°$. I found, however, on a second and more careful trial, that the images formed by it were clearer and brighter than I had thought, and so determined to take a series of observations with it. When I observed a second time the angle of prism II., I took a series of measurements of deviation, &c., which lead to results in agreement with Tables X. (A), X. (B), so that on the whole the results given by this prism are in accordance with those already arrived at in prisms I., III., and IV.

Our next step is to consider the theory for the rays F and g.

The position of the plane containing the two normals to the faces of the prism is of course the same, and therefore so also are the formulæ which give the relations between θ and ϕ', θ and ψ'.

The values of the axes of spheroid on HUYGHEN'S theory are, however, different.

Let us take the green line, F, first.

μ_1 is, as before, the value of the ordinary refractive index.

We have as for the line C the four values

$\mu_1=1{\cdot}66780$	Prism I.
$1{\cdot}66776$ / $1{\cdot}66778$	Prism III.
$1{\cdot}66783$	Prism IV.

We take $\mu_1=1{\cdot}66779$. μ_2 is the value of μ when $\theta=90°$ in Table XI.

Now for $\theta=89°\ 49'\ 6''$ experiment gives

$$\mu=1{\cdot}49074$$

we take this as the value of μ_2.

Hence for F we have

$$\mu_1=1{\cdot}66779$$

$$\mu_2=1{\cdot}49074$$

MASCART and RUDBERG give respectively

$$1{\cdot}66802,\ 1{\cdot}49075$$

and

$$1{\cdot}66793,\ 1{\cdot}49084$$

For g we have, as before,

$$\mu_1 = 1{\cdot}67557 \quad \text{Prism I.}$$
$$1{\cdot}67545 \quad \text{Prism III.}$$
$$1{\cdot}67556 \quad \text{Prism IV.}$$

Whence

$$\mu_1 = 1{\cdot}67553$$

and for μ_2 when

$$\theta = 89^\circ\ 53'\ 4'', \text{ Table XII.}$$

we have

$$\mu = 1{\cdot}49430.$$

Whence

$$\mu_2 = 1{\cdot}49430.$$

Thus for g

$$\mu_1 = 1{\cdot}67553$$
$$\mu_2 = 1{\cdot}49430$$

Tables XI. and XII. give the results of the calculations.

TABLE XI.—Results of Theory for F.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
	° ′ ″			
I.	0 2 40	1·66779	1·66780	+ 1
II.	4 19 58	1·66660	1·66663	+ 3
I.	7 51 58	1·66387	1·66385	− 2
I.	11 23 12	1·65967	1·65978	+11
I.	17 8 26	1·64987	1·64996	+ 9
I.	20 26 1	1·64279	1·64287	+ 8
II.	23 50 45	1·63451	1·63455	+ 4
I.	24 14 23	1·63351	1·63360	+ 9
I.	25 49 35	1·62934	1·62930	− 4
II.	29 18 42	1·61965	1·61974	+ 9
II.	34 48 0	1·60336	1·60336	0
II.	38 58 47	1·59048	1·59058	+10
II.	40 49 21	1·58478	1·58487	+ 9
III.	45 45 57	1·57000	1·57014	+14
II.	46 46 2	1·56645	1·56653	+ 8
III.	49 23 10	1·55861	1·55876	+15
III.	52 42 6	1·54902	1·54914	+12
III.	58 39 10	1·53303	1·53312	+ 9
III.	61 39 33	1·52570	1·52573	+ 3
III.	63 9 6	1·52228	1·52241	+13
III.	66 14 27	1·51579	1·51571	− 8
III.	72 18 55	1·50476	1·50475	− 1
III.	75 36 18	1·50009	1·50005	− 4
III.	79 6 26	1·49612	1·49610	− 2
IV.	80 14 4	1·49507	1·49507	0
IV.	87 6 40	1·49112	1·49114	+ 2
IV.	89 49 6	1·49074	1·49074	0

TABLE XII.—Results of Theory for g.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
I.	0° 7′ 31″	1·67553	1·67557	+ 4
		No experiment.		
I.	7 59 9	1·67138	1·67143	+ 5
I.	11 16 40	1·66735	1·66736	+ 1
I.	17 0 9	1·65740	1·65720	−20
I.	20 16 35	1·65023	1·65022	− 1
II.	23 59 41	1·64098	1·64087	−11
I.	24 3 49	1·64080	1·64067	−13
I.	25 39 1	1·63655	1·63643	−12
II.	29 25 42	1·62579	1·62570	− 9
II.	34 53 17	1·60917	1·60903	−14
II.	39 2 49	1·59606	1·59590	+16
II.	40 43 8	1·59071	1·59072	+ 1
III.	45 43 55	1·57486	1·57502	+16
II.	46 38 30	1·57203	1·57205	+ 2
III.	49 29 51	1·56329	1·56345	+16
III.	52 47 52	1·55354	1·55371	+17
III.	58 43 25	1·53729	1·53745	+16
III.	61 35 19	1·53016	1·53014	− 2
III.	63 12 16	1·52637	1·52644	+ 7
III.	66 9 35	1·51992	1·51981	− 9
III.	72 12 53	1·50877	1·50870	− 7
III.	75 30 40	1·50396	1·50387	− 9
III.	78 59 4	1·49991	1·49982	− 9
IV.	80 18 16	1·49865	1·49856	− 9
IV.	87 9 43	1·49467	1·49460	− 7
IV.	89 53 4	1·49430	1·49430	0

TABLE XIII.—Results of Theory for C.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
I.	0° 20′ 35″	1·65435	1·65438	+ 3
II.	4 5 7	1·65335	1·65335	0
I.	7 39 25	1·65082	1·65078	− 4
I.	11 34 51	1·64635	1·64627	− 8
I.	17 23 14	1·63678	1·63684	+ 6
I.	20 42 48	1·62989	1·62991	+ 2
II.	23 34 21	1·62326	1·62341	+15
I.	24 33 38	1·62085	1·62095	+10
I.	25 28 23	1·61855	1·61862	+ 7
II.	29 6 1	1·60897	1·60906	+11
II.	34 38 36	1·59326	1·59336	+10
II.	38 51 23	1·58083	1·58091	+12
II.	41 0 26	1·57443	1·57449	+ 6
III.	45 51 25	1·56151	1·56163	+12
II.	46 59 20	1·55677	1·55692	+15
III.	49 11 23	1·55044	1·55058	+14
III.	52 31 57	1·54113	1·54116	+ 3
III.	58 31 47	1·52560	1·52567	+ 7
III.	61 47 0	1·51797	1·51805	+ 8
III.	63 3 25	1·51516	1·51523	+ 7
III.	66 23 19	1·50830	1·50833	+ 3
III.	72 29 31	1·49782	1·49786	+ 4
III.	75 49 4	1·49331	1·49331	0
III.	79 19 19	1·48955	1·48962	+ 7
IV.	80 6 35	1·48885	1·48887	+ 2
IV.	87 1 19	1·48495	1·48499	+ 4
IV.	89 42 4	1·48457	1·48457	0

Table XIII. gives the results for the ray C for the same values of the angle of incidence as those given in Tables XI. and XII. for F and *g*. This enables a comparison of the results to be more easily made for the three rays than if it were requisite to refer to X. In each case the results are similar.

The differences are least near the axes, being negative for F near the minor axis, and for *g* near both major and minor.

For C the errors are positive throughout, so that a small increase of the axes of the curve given by theory would, on the whole, bring theory and experiment into closer agreement.

For F the differences near the minor axis being negative, we should require to decrease the minor axis of the ellipse. This would increase slightly the positive errors, and render, on the whole, the variation from FRESNEL'S spheroid more marked, and greater than the variation of the red ray.

While for the violet ray, g, the differences near both axes are negative. To bring the two curves into agreement then we should require to decrease both the axes μ_1, μ_2. This would produce a corresponding increase in all the positive errors and render the variation from FRESNEL'S theory near the middle of the arc more marked than in the case of the red or green rays.

In fact, while for the red, supposing the variations in μ_1, μ_2 contemplated above to have been adopted, the greatest difference between theory and experiment would be about

$$\cdot 0001$$

for the green ray F it would rise to

$$\cdot 00015$$

and for the violet, g, to

$$\cdot 0002.$$

SECTION IV.

I. *Comparison with previous experiments.*
II. *Effect of variation of constants.*

As an additional proof of the accuracy of the experiments it may be worth while giving the results of a series of measurements covering the same ground as the second part, Table I., made some months previously. Since the prism did not occupy exactly the same position relative to the instrument as it did during the experiments in Table I., the values of the angle of incidence, and therefore of ψ', were slightly different to those in Table I.

In making the comparison, therefore, the results of calculation had to be altered by interpolation to give the values of μ corresponding to the values of ψ' in Table I.

The result is contained in Table XIV.

TABLE XIV.

μ from Table I.	μ from Experiments in December, 1878.	Difference.
1·64627	1·64627	0
1·64335	1·64332	3
1·64021	1·64010	11
1·63684	1·63681	3
1·63341	1·63339	3
1·62991	1·62994	3
1·62669	1·62663	6
1·62356	1·62365	9
1·62095	1·62097	2
1·61862	1·61859	3
1·61680	1·61680	0

The agreement between the two results is striking, and seems to show that we may assume the experimental results to be true with an error which is not greater than ·00005.

In my paper on a biaxal crystal I was able to show that the assumption of certain errors in the determination of the position of the plane of the prism with reference to the crystallographic axes led to results rather more in agreement with experiment than those obtained at first.

In the present instance this is impossible, for any change in the position of the plane of the prism would produce effects of almost exactly the same amount in the values of μ for the lines C and g; but the error we wish to correct in C is only half as great as that in g, and hence no change in the position of the plane can produce the required effect.

But, again, the telescopes used were not perfectly achromatic for the red and violet rays. I found usually little or no difference in the position of the focus for the lines C and F, but there was an appreciable difference between C and g.

If the collimator be focused so that the rays from the line C emerge parallel, those from g will be divergent. This may produce a variation in the angle of incidence between the waves C and g.

For the prisms were so placed that by turning the table on which they rested without altering the position of the collimator, either face of the prism could be made a face of incidence. To secure this the edge of the prism passed nearly through the axis of the collimator, and in most positions of the prism the light from only about half the collimator lens reached it.

A figure will make this clearer. S C (fig. 5) is the axis of the collimator, A P B the prism, A P being the face of incidence.

Fig. 5.

Almost all the light incident on A P passes through the upper part, C E, of the lens of the collimator. If the prism be turned round K, a point in the axis S C produced, so that B Q becomes the face of incidence, then only the lower portion, C F, of the collimator lens will be used.

Again, since the collimator is focused for red rays, they will be incident on the face A P in direction C P, and if P N be the normal to A P the angle of incidence will be C P N.

The violet rays, however, diverging as they do from a point on C S, will be incident on A P at various angles, most of which, however, will be less than C P N. By assuming then the violet rays to issue parallel from the object glass, we have made the angle of incidence for the violet too great.

Again, if Q R be any emergent ray, we have assumed the deviation to be measured by D Q R.

In the case of the violet rays this again will be too great, and too great by the same amount as the angle of incidence.

We must therefore consider the effect of decreasing the angle of incidence and the deviation by the same amount.

If ϕ be the angle of incidence, ψ the angle of emergence,

$$\psi = D + i - \phi$$

$$\delta\psi = \delta D - \delta\phi = 0$$

$$\delta\psi' + \delta\phi' = 0$$

$$\delta\phi' \text{ is negative since } \delta\phi \text{ is so}$$

$$\therefore \delta\psi' \text{ is positive.}$$

Hence ψ is unchanged, ψ' is increased.

The value of μ will therefore in all cases be decreased.

Now the experimental values of μ are already too great. Hence this alteration will tend to bring them more nearly into agreement with theory. The amount of error introduced depends on the angle of incidence. To find a general expression for it would be a work of difficulty owing to the complicated nature of the formulæ involved.

Let us therefore consider the effect of decreasing the angle of incidence and the deviation by 1′, (a) near minimum deviation, (b) near grazing incidence for prism I. The effects will be much the same for all the prisms.

We have from Table I.

ϕ	76° 8′	36° 8′
$D+i$	89° 25′ 5″	76° 4′
ϕ'	35° 57′ 5″	20° 59′ 21″
μ	1·65367	1·64623

By decreasing ϕ and $D+i$ each by 1′ we have the following values:-

ϕ'	35° 57′ 3″	20° 59′ 5″
μ	1·65355	1·64591

Thus near minimum deviation the change produced in μ amounts to about ·0003 while at grazing incidence it is only about ·0001.

Of course an error of the same kind occurs in the values of μ_1, μ_2. They, however, were determined from observations at nearly grazing incidence.

They may then be slightly too great.

To correct them completely for this error we should have to reduce the theoretical values by a small quantity nearly the same for all; while the experimental values require reducing by quantities which are greatest near minimum deviation, and decrease as we approach grazing incidence until they reach about the values of the corrections applied to μ_1 and μ_2.

The greatest error for the ray g does not exceed ·0002, so that the results of theory and experiment for g would be brought into very close agreement by supposing the violet rays of the light emerging from the collimator to be inclined to the red at angles not greater than 45″.

Thus, allowing for this probable divergency of the green and violet rays, it appears that HUYGHEN's construction represents the result of experiment for the three rays of the hydrogen spectrum to a degree of approximation comparable with the probable error of the experiments.*

* In the abstract printed in the 'Proceedings of the Royal Society,' I had assumed that the violet rays issuing from an achromatic lens, focused so as to make the orange rays parallel, were convergent. From this it followed that the correction for want of parallelism tended to increase the difference between theory and experiment, and led me to the inference that HUYGHEN's construction might be true for the red rays and yet differ appreciably from the truth for light of shorter wave length. Professor STOKES has since pointed out to me that the violet rays are in reality divergent, and that the error introduced by assuming them to be parallel really tends to correct the differences observed between theory and experiment, and so leads to the inference in the text that HUYGHEN's construction is true for the three hydrogen rays within the limits of experimental error.

XII. *On the Normal Paraffins.*—Part III.

By C. SCHORLEMMER, *F.R.S., Professor of Organic Chemistry in Owens College, Manchester.*

Received August 2,—Read November 20, 1879.

THE isomeric monochlorides obtained from the normal paraffins existing in petroleum yield by the abstraction of hydrochloric acid a mixture of olefines, one portion of which readily combines with hydrochloric acid in the cold, whilst the other is not changed, even if it be left in contact with the acid for weeks, and only unites with it on heating.*

The chlorides which are formed in the cold boil with partial decomposition and at a lower temperature than the others, which distil without undergoing any change, and, as MORGAN has shown, have the general formula, $CH_3-CHCl-C_nH_{2n+1}$.† They are, therefore, derivatives of the olefines having the constitution $CH_2{=}CH-C_nH_{2n+1}$ or $CH_2{=}CHR$, and which, as LE BEL‡ has also shown, combine with hydrochloric acid only on heating.

The constitution of the olefines from petroleum which unite with the acid in the cold is not yet known. I have formerly pointed out that possibly they are not derived from normal paraffins, and this view seems to find confirmation in the observation of LE BEL that the property of combining with cold hydrochloric acid belongs to the hydrocarbons which have the general formulæ $CH_2{=}CRR'$ and $CHR{=}CRR'$.

On the other hand, I showed that the formation of isomeric olefines might be explained without making the assumption that the normal paraffins from petroleum contained an admixture of isomerides, but that in order to decide this question an absolutely pure paraffin must be used.§

For reasons which are not far to seek, I selected for this investigation the normal hexane from mannite, which possibly might, by the action of chlorine, yield the following monochlorides:—

$$(1)\quad CH_3-CH_2-CH_2-CH_2-CH_2-CH_2Cl.$$
$$(2)\quad CH_3-CH_2-CH_2-CH_2-CHCl-CH_3.$$
$$(3)\quad CH_3-CH_2-CH_2-CHCl-CH_2-CH_3.$$

* Journ. Chem. Soc., 1873, p. 323.

† *Ibid.*, 1875, p. 301.

‡ Bull. Soc. Chim. (2), vol. xxviii., p. 460.

§ Journ. Chem. Soc., 1875, p. 306.

I have already shown that the first two of these compounds are thus formed. The following method seemed to me capable of determining whether the third is also produced.

On heating the chlorides with an alcoholic solution of potash they are converted into olefines, and thus the three following hexylenes might be formed :—

Butylethylene.	$CH_3-CH_2-CH_2-CH_2-CH=CH_2$
Methylpropylethylene	$CH_3-CH_2-CH_2-CH=CH-CH_3$
Diethylethylene	$CH_3-CH_2-CH=CH-CH_2-CH_3$

The first of these does not combine with hydrochloric acid in the cold, as MORGAN has found, nor does the hexylene which is obtained by decomposing the secondary hexyl iodide from mannite, as LE BEL and WASSERMANN have shown. This hydrocarbon, however, is not butylethylene, as they assumed, but consists of methylpropylethylene.*

If, therefore, normal hexane from mannite yields a hexylene combining with hydrochloric acid in the cold, it could be only diethylethylene, which might further be identified by conversion into the corresponding secondary alcohol. This ought to consist of ethylpropyl carbinol, and should yield on oxidation only propionic acid.

This is the programme which I intended to follow in my research; the results obtained were, however, quite unexpected.

In order to prepare pure hexane, I distilled mannite with an excess of fuming hydriodic acid and the addition of amorphous phosphorus. A good yield of secondary hexyl iodide is thus easily obtained. The iodide was reduced by the method already described, and the hexane separated from some hexylene and dodecane, which are formed at the same time.†

By the action of chlorine on the pure hydrocarbon a mixture of monochlorides, boiling between 121° and 134°, was obtained, which was heated with an alcoholic solution of potash to 100°. The reaction went on rather quickly, and was soon completed. The product consisted principally of hexylene, but contained also a mixture of ethylhexyl ethers.

The hexylene was left in contact with an excess of fuming hydrochloric acid for several weeks, the mixture being contained in a well-closed bottle and kept in the dark. The excess of acid being removed, the product was, after washing and drying, distilled. Of course I expected to find a considerable quantity of uncombined hexylene to be present; but, to my great surprise, *the whole product boiled constantly and without the least decomposition* at 124—125°, while the chloride which MORGAN obtained by the action of cold hydrochloric acid on the hexylenes from petroleum distilled at 116—118° with the evolution of hydrochloric acid.

* O. HECHT, Deut. Chem. Ges. Ber., Bd. xi., p. 1152.

† Phil. Trans., Vol. 162, p. 111.

In order to elucidate the constitution of the chloride, it was heated with anhydrous lead acetate and glacial acetic acid to 125°. A complete decomposition was thus easily effected, and, besides hexyl acetate, a crystalline compound was formed, consisting of lead acetochloride Pb $(C_2H_3O_2)$ Cl, a salt which has been discovered by CARIUS.

The hexyl acetate was thrown out of solution by the addition of water, and without further purification converted into the alcohol by heating it with a concentrated solution of potash. The product was found to contain a little hexylene, which was easily got rid of; the alcohol, after being dried over ignited potassium carbonate, distilled between 130° and 140°. By repeated fractional distillation and drying over potassium carbonate, it was separated into a portion boiling at 130—135°, which was a little larger than the other, boiling between 135° and 140°; but no body having a constant boiling point could be isolated, although the chloride boiled within one degree. Perhaps it was after all a mixture, and, if so, the alcoholic liquid would be a mixture of methylbutyl carbinol and ethylpropyl carbinol.

In order to decide this question each portion was subjected to fractional oxidation, as I expected that, in case two ketones were formed, one might be more readily oxidised than the other. I employed, therefore, in each operation only so much of the chromic acid solution that but a small quantity of fatty acids could be formed, and separated these in the same way as described in my former papers.

The results were, however, the same as those formerly obtained: the formation of acetic acid and butyric acid could be easily proved both qualitatively and quantitatively, but no propionic acid could be found; and if any was present, its quantity must have been exceedingly small.

I have already stated that in the preparation of hexylene a small quantity of ethylhexyl ethers is formed. As they could not be separated by fractional distillation, I heated them with hydriodic acid, and thus obtained, besides ethyl iodide, a liquid boiling between 168° and 178°—*i.e.*, the boiling points of the common secondary and the normal primary hexyl iodide.

The results of this research confirm thus far those which I had already obtained seven years ago—viz.: that by the action of chlorine on the normal hexane from mannite, normal hexyl chloride is formed, together with a larger quantity of the secondary chloride $C_4H_9.CHCl.CH_3$. But it is certainly very remarkable that the hexylene obtained from them, or perhaps only from the secondary compound, combines so readily with hydrochloric acid in the cold, while the hexylene obtained by perfectly analogous reactions from normal petroleum hexane unites with the acid only by heating under pressure; and yet the two chlorides thus obtained have apparently the same constitution.

I have already, in my first paper, pointed out some other differences existing between the two hexanes, and said:—

"The hexane from mannite and some of the derivatives boil a few degrees higher

than the corresponding compounds from petroleum, and also the barium salts of the two caproic acids exhibit a decided difference. The hexane from petroleum is certainly not a pure compound; but whether this is the cause of the difference between the two hydrocarbons, or whether we have here a case of fine isomerism for which an explanation has to be found, it is at present impossible to decide."*

I believe the results of my present research speak strongly in favour of the latter view. Further experiments are, however, required to elucidate these points, as well as the constitution of the olefines from petroleum which combine readily with hydrochloric acid in the cold.

Another point which wants clearing up is the fact that the normal paraffins from petroleum have a higher specific gravity than those obtained from any other source. This is probably due to some admixture, for I found that on treating hexane with hot nitric acid† the specific gravity of the non-oxidised portion decreased and became at last constant, being 0·663 at 17°, or the same as that of hexane from mannite and of normal dipropyl.‡ The normal paraffins from petroleum appear, therefore, not to be pure compounds; and for several reasons I am inclined to belive that petroleum, after being freed from olefines, aromatic hydrocarbons, and those of other series, consists of an inextricable mixture of isomeric and homologous paraffins, in which, however, the normal hydrocarbons preponderate. This would certainly explain why it is so extremely difficult to isolate from it bodies having a constant boiling point,§ but not why the normal hexylene obtained from petroleum will not unite with hydrochloric acid under the same conditions as that prepared from mannite.

A continuation of these researches has been already commenced. My friend THORPE, who made the most interesting discovery that the terebinthinate exudation of *Pinus Sabiniana* contains a large quantity of normal heptane,‖ has kindly invited me to join him in the chemical investigation of this hydrocarbon. At the same time we shall compare it with other "normal" heptanes from different sources.

* Phil. Trans., Vol. 162, p. 119.

† Proc. Roy. Soc., vol. xvi., p. 372.

‡ Phil. Trans., Vol. 162, p. 120.

§ Journ. Chem. Soc., 1875, p. 306.

‖ Journ. Chem. Soc., 1879, p. 296.

XIII. *On the Motion of Two Spheres in a Fluid.*

By W. M. Hicks, M.A., *Fellow of St. John's College, Cambridge.*
Communicated by Professor J. Clerk Maxwell, *F.R.S.*

Received May 16—Read June 19, 1879.

The general theory of the motion of a single rigid body through an infinite incompressible fluid is well known, chiefly through the work of Thomson and Tait* and Kirchhoff,† and we are able to calculate numerically the results in the case of the sphere, the ellipsoid, and a large number of cylindrical surfaces. The theory of the motion of two or more bodies in a fluid has naturally not made the same progress, and we are unable to determine the form of the expressions involved for the general motion of any particular solids. So far as I am aware, the first attempt was made by Stokes, in a paper read before the Cambridge Philosophical Society in 1843, entitled "On some cases of Fluid Motion."‡ In this paper, amongst other problems, he considers the case of two spheres. He determines the instantaneous velocity potential for two concentric spheres and for two concentric cylinders with fluid between them, and finds that the effect of the fluid is to increase the inertia of the inner sphere by a mass $=\frac{1}{2}.\frac{a^3+2b^3}{b^3-a^3}$ of the mass of the fluid displaced, and that of the inner cylinder by a mass $\frac{b^2+a^2}{b^2-a^2}$ of the mass displaced, a, b, being the radii of the spheres or cylinders. He also approximates to the cases where one sphere is moving in the presence of another in an infinite fluid; and also in the presence of a plane, the method used being first to calculate the velocity potential for any motion of the points of the plane, and then suppose them actually animated with velocities equal and opposite to the normal velocities of the fluid motion at those points if the plane had been removed. He applies the same method also to the consideration of the motion of two spheres. In a note in the Report of the British Association at Oxford, 1847, he states the theorem given by me in § 4 relating to the image of a doublet whose axis passes through the centre, and mentions that this will easily serve to determine the motion. In 1863 Herr Bjerknes communicated a paper to the Society of Sciences at Christiana, on the motion of a sphere which changes its volume, and in

* Nat. Phil., p. 264, new edition, p. 330.

† Borchardt, Bd. 71.

‡ Camb. Phil. Trans., vol. viii.

which he approximates for the motion of two spheres. I have not been able to see this paper, nor some others which he presented to the same Society at some later periods; but he has given an account of his researches in the 'Comptes Rendus,'* together with some historical notices on the development of the theory. He does not seem, however, to have been acquainted with the important paper of STOKES above referred to.† In 1867 THOMSON and TAIT'S 'Natural Philosophy' appeared, containing general theorems on the motion of a sphere in a fluid bounded by an infinite plane, viz.: that a sphere moving perpendicularly to the plane moves as if repelled by it, whilst if it moves parallel to it it is attracted. In a paper on vortex motion in the same year (Edin. Trans., vol. xxv.), THOMSON proved that a body or system of bodies passing on one side of a fixed obstacle move as if attracted or repelled by it, according as the translation is in the direction of the resultant impulse or opposite to it. In the 'Philosophical Magazine' for June, 1871, Professor GUTHRIE publishes some letters from Sir W. THOMSON on the apparent attraction or repulsion between two spheres, one of which is vibrating in the line of centres. Results only are given, and he states that if the density of the free globe is less than that of the fluid, there is a "critical" distance beyond which it is attracted, and within which it is repelled. The problem of two small spheres is also considered by KIRCHHOFF in his 'Vorles. ü. Math. Phys.,' pp. 229, 248. In his later papers BJERKNES takes up the question of "pulsations" as well as vibrations. Of solutions for other cases than spheres, KIRCHHOFF has considered‡ the case of two thin rigid rings, the axes of the rings being any closed

* 'Comptes Rendus,' tom. lxxxiv., p. 1222, &c.

† Not only Herr BJERKNES, but several writers on the Continent seem to be unacquainted with this paper of STOKES, and also with GREEN'S papers. KIRCHHOFF, in his 'Vorlesungen über Mathematische Physik' (second edition, p. 227), says that DIRICHLET first treated the motion of a sphere in a fluid in the Monatsberichte der Berl. Akad.' in 1852, and CLEBSCH that of the ellipsoid in 1856, in 'Crelle,' Bd. 52. BJERKNES also repeats the same statement, and CLEBSCH in his paper regards DIRICHLET as the first to solve for the sphere. In his paper DIRICHLET says: "Wie es scheint, ist bis jetzt für keinen noch so einfachen Fall der Widerstand, den ein in einer ruhenden Flüssigkeit fortbewegter fester Körper von dieser erleidet, aus den seit Euler bekannten allgemeinen gleichungen der Hydrodynamik abgeleitet worden." The fact is that GREEN in a paper read before the Royal Society of Edinburgh in 1833, entitled "Researches on the Vibrations of Pendulums in Fluid Media" (Trans. Roy. Soc. Edin.; also published in the Reprint of his papers, p. 313), and written without the knowledge of POISSON'S paper of 1831, "Sur les mouvements simultanés d'un pendule et de l'air environnant," treated of the motions of an ellipsoid moving parallel to one of its axes. He obtains the velocity potential as an elliptic integral for a motion parallel to an axis, which also of course contains implicitly that for the sphere. He shows that it is necessary to suppose the density of the body augmented by a quantity proportional to the density of the fluid. For the case of the spheroids moving in their equatorial planes or parallel to their axes he completely determines this quantity, whilst for the sphere he finds that it is one-half the mass of the fluid displaced. The first place in which I have been able to find the well known form of the velocity potential for a sphere is in STOKES' paper of 1843 before mentioned. He obtains it as a particular case of a more general problem, and refers to it as the "known" value for the sphere. The equations of the lines of flow were, I believe, first given by DIRICHLET.

‡ BORCHARDT, Bd. 71.

curves and the sections by planes perpendicular to the axis being small circles of constant radii, and he arrives at the result that their action on one another may be represented by supposing electric currents to flow round them; and I have recently solved the problem of the motion of two cylinders in any manner with their axes always parallel. The velocity potentials for the motion of the two cylinders are found in general as definite integrals, which, when the cylinders move as a rigid body, are expressed in a simple finite form as elliptic functions of bipolar coordinates. The functions involved in the coefficients of the velocities in the expression for the energy have a close analogy with those for spheres arrived at in the following investigation.

1. Our first aim will be to find the velocity potential for the motion of the fluid in which a sphere is fixed and in which a source of fluid exists. By the image of the source in general is meant that collocation of sources or sinks within the sphere which produces outside of it a fluid motion which in conjunction with the original source has no normal motion across the sphere: in other words, that "mass" of positive or negative sources which produces across the surface of the sphere a normal flow equal and opposite to that of the outside source. When this "image" is found, the way is *theoretically* clear to finding the velocity potential when two spheres are fixed in the fluid, and thence, by distributing over the surface of the spheres sources proportional to the normal motion of the surface at that point, to determine the velocity potential when the two spheres are moving in any manner. In the case of an electrical point the image is, as is well known, a negative point at the inverse point of the other. In the case of fluid motion the image is, as will be shown, a positive source at the inverse point, together with a negative line sink stretching from this point to the centre of the sphere.

Fig. 1.

2. Take O the centre of the sphere for origin and let the axis of z pass through the source S. Let the radius of the sphere be a, and the distance of S from the centre be b. Then the velocity potential will clearly be symmetrical about O S. The velocity potential for the unit source at S can be expanded in the series

$$-\frac{1}{R}=-\frac{1}{\sqrt{r^2-2br\cos\theta+b^2}}=-\frac{1}{b}-\Sigma_1^\infty\frac{r^n}{b^{n+1}}P_n$$

which holds good for points where $r<b$, whence when $r=a(<b)$ the flow *into* the sphere at any point (θ) is

$$\Sigma_1^\infty \frac{na}{b^{n+1}} P_n$$

Expand the potential due to the sources, &c., inside the spheres in a series of spherical harmonics

$$V=\Sigma_0^\infty \frac{a^n}{r^{n+1}} Y_n (r>a)$$

Hence the flow *out* of the sphere, for points just outside, is

$$-\Sigma_0^\infty (n+1)\frac{1}{a^2} Y_n$$

and this must be equal to the other, whence

$$Y_n=-\frac{n}{n+1}\left(\frac{a}{b}\right)^{n+1} P_n \text{ and } Y_0=0$$

and

$$V=-\Sigma_1^\infty \frac{n}{n+1} \frac{a^{2n+1}}{(br)^{n+1}} P_n$$

$$=-\Sigma_1^\infty \frac{a^{2n+1}}{(br)^{n+1}} P_n+\Sigma_1^\infty \frac{1}{n+1} \frac{a^{2n+1}}{(br)^{n+1}} P_n$$

Consider

$$\chi=\frac{\mu'}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}=\mu'\Sigma_0^\infty \frac{\lambda^n}{r^{n+1}} P_n \begin{pmatrix}\lambda<a\\ r>a\end{pmatrix}$$

the potential for a source μ' at a point on O S inside the sphere at a distance λ from the centre. Then

$$\int_0^\lambda \left(\chi-\frac{\mu'}{r}\right) d\lambda=\mu'\Sigma_1^\infty \frac{1}{n+1} \frac{\lambda^{n+1}}{r^{n+1}} P_n$$

Comparing this with the expression for V, we see that if we make $\lambda=\frac{a^2}{b}$ and $\mu'=\frac{a}{b}\times$ source

$$V=-\frac{a}{b}\Sigma_1^\infty \frac{\lambda^n}{r^{n+1}} P_n+\frac{1}{\lambda}\int_0^\lambda \chi d\lambda-\frac{\mu'}{r}$$

$$=-\frac{a}{b}\frac{1}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}+\frac{1}{a}\int_0^\lambda \frac{d\lambda}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}$$

i.e., V is the potential of a source at the distance $\frac{a^2}{b}$ from O whose magnitude is equal

to $\frac{a}{b}$ of the source at S, together with a line sink extending from O to the distance $\frac{a^2}{b}$, the line density of the sink being $\frac{1}{a}\times$ source at S.

Performing the integration for V, we find finally that the whole velocity potential for a unit source at S is

$$\phi=-\frac{1}{\mathrm{SP}}+\mathrm{V}=-\frac{1}{\sqrt{r^2-2br\cos\theta+b^2}}-\frac{a}{b}\frac{1}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}$$
$$+\frac{1}{a}\log\frac{\lambda-r\cos\theta+\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}{r(1-\cos\theta)}$$

where $\lambda=\frac{a^2}{b}$

It is easy to verify this value for ϕ by direct differentiation.

If we apply the same method to find the velocity potential for the motion of fluid inside a sphere under the influence of a source inside, the integral becomes infinite unless the source is zero. The case is of course physically impossible since if fluid is generated within the sphere it must pass through the boundary. But if we also place an equal sink at any point within, the motion is then possible, and the expression becomes finite. S being the source let S′ be its inverse point with reference to the sphere, and S″ any point on the line S S′ produced to infinity. Then the "image" of S is a source $\frac{\mu a}{b}$ at S′, and a line distribution of sinks of line density $\frac{\mu}{a}$ from S′ to infinity. Let S_1 be an equal sink, then its image and that of S will produce potentials with finite derivatives. In fact, the potential at P will be

$$\phi=\mu\left\{\frac{1}{\mathrm{SP}}-\frac{1}{\mathrm{S_1P}}+\frac{a}{b}\cdot\frac{1}{\mathrm{S'P}}-\frac{a}{b_1}\cdot\frac{1}{\mathrm{S'_1P}}-\frac{1}{a}\log\frac{\mathrm{OS'}-r\cos\theta+\mathrm{S'P}}{\mathrm{OS'_1}-r\cos\theta_1+\mathrm{S'_1P}}\cdot\frac{1-\cos\theta_1}{1-\cos\theta}\right\}$$

where θ, θ_1 are the angles O P makes respectively with O S, O S_1.

3. The expression found for the motion when there is a single source outside the sphere enables us to deduce the velocity potential for a single sphere moving through an infinite fluid. Taking the direction of motion as the axis of x, from which we will suppose θ measured, we may arrange a surface distribution of sources proportional to $\cos\theta d\mathrm{S}$ and integrate over the surface of the sphere, or we may employ the simpler method used in a paper in the 'Quarterly Journal of Mathematics' for March, p. 128. The first gives us the velocity potential when the sphere moves by an integration which would be laborious. The other gives directly the potential, when the sphere is fixed and the fluid moves past it, by means of an easy differentiation. Putting a source at $x=b$ and an equal sink at $x=-b$, let these move off to infinity, increasing indefinitely as they do so, yet so that the motion at a finite distance from the origin is finite. In the limit we clearly get the case of fluid flowing past the sphere.

We have to find the limit when $b=\infty$ $\frac{\mu}{b^2}=k$ of

$$\phi=-\mu\left\{\frac{1}{\sqrt{r^2-2br\cos\theta+b^2}}-\frac{1}{\sqrt{r^2+2br\cos\theta+b^2}}\right.$$
$$+\frac{a}{b}\left(\frac{1}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}-\frac{1}{\sqrt{r^2+2\lambda r\cos\theta+\lambda^2}}\right)$$
$$\left.-\frac{1}{a}\log\frac{\lambda-r\cos\theta+\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}{\lambda+r\cos\theta+\sqrt{r^2+2\lambda r\cos\theta+\lambda^2}}\cdot\frac{1+\cos\theta}{1-\cos\theta}\right\}$$

When b is large and λ small this is easily shown to be

$$\phi=-\frac{\mu}{b^2}\left\{2r\cos\theta+\frac{a^3\cos\theta}{r^2}+\frac{A}{b}+\ldots\right.$$

Hence the limit is

$$\phi=-k\left(2x+\frac{a^3x}{r^3}\right.$$

If the velocity of the fluid at an infinite distance parallel to x is u towards the origin, then

$$2k=u$$

Also impressing on the whole system a velocity u, the sphere moves with velocity u in an infinite fluid, and the potential function is

$$\phi=-\frac{a^3ux}{2r^3}=-\frac{a^3u\cos\theta}{2r^2}$$

The well-known form of ϕ in this case.

4. If now two spheres A, B are present in the fluid, and we consider the series of images resulting from the first image in A, we see that they very rapidly become extremely complicated, *e.g.*, the first image is a source and line sink; the image of this in B consists of (1) a source and line sink, (2) the image of the first line sink or a line sink (segment of a circle), and an area source bounded by this last line sink and two straight lines from the centre. It is, therefore, hopeless in this way to find first the velocity potential for a source in the presence of the two spheres, and thence the potential for any motion of the spheres. But now suppose A fixed and B moving in any direction. If A were not present the velocity potential of B would be that due to a doublet at its centre, whose axis lies in the direction of the motion of B. The effect of the introduction of A will be to produce a series of images of this doublet, lying inside A and B. This method dispenses with the necessity of integrating over the spheres when we have found the velocity potential for the doublet. In the special

case where B is moving in the line joining the centres, the image becomes simplified and reduces to a single doublet. For let us find the image of a doublet whose axis passes through the centre of a sphere.

The doublet is formed by allowing an equal source and sink P, P′ to indefinitely approach one another, their magnitudes increasing indefinitely, yet so that $\mu.\overline{PP'}$ is finite. Now let P, P′ lie on the line through the centre of the sphere, and let Q, Q′ be their inverse points; moreover, let the limit of $\mu.\overline{PP'}=k$. Then the image of P, P′ consists of a source $\frac{\mu a}{OP}$ at Q, a sink $\frac{\mu a}{OP'}$ at Q′, and a line source (supposing P outside P′, and therefore Q′ outside Q) along Q Q′ with line density $\frac{\mu}{a}$, also the quantity $\frac{\mu}{a}.QQ'$, together with the sink at Q′, is equal and opposite to the source at Q, and we may suppose it added to the sink at Q′, when they become equal. Now as P, P′ approach to coincidence so do Q, Q′, and the image of the doublet k at P becomes the doublet at Q, whose magnitude is the limit of

$$\frac{\mu a}{OP}.QQ'=k.\frac{a}{OP}.\frac{QQ'}{PP'}=-k.\frac{a^3}{OP^3},$$

i.e., one of opposite sign and magnitude $\left(\frac{a}{OP}\right)^3\times$ that at P. The same result can easily be shown to follow from the analytical formula in § 2.

The case where the doublet has its axis perpendicular to the line joining the centres has more analogy with the case of a source. The image here consists of a doublet of the *same* sign at the inverse point, with a trail of doublets of opposite sign extending to the centre.

Fig. 2.

Let, as before, P, P′ be equal source and sink, Q, Q′ their inverse points with respect to the circle.

Then at Q, Q′ we have a source and sink of magnitude $\frac{\mu a}{OP}$, and in the limit we have a doublet

$$\frac{\mu a\overline{QQ'}}{OP}=k\left(\frac{a}{OP}\right)^3$$

Also, if R, R′ be corresponding points on O Q, O Q′, we have a line density $-\frac{\mu}{a}$ at

R and $+\frac{\mu}{a}$ at R′. Consequently when P, P′ approach indefinitely so do R, R′, and we get a line doublet along O Q, whose line magnitude at any point R is the limit of

$$-\frac{\mu}{a}.\mathrm{RR}'=-\frac{\mu}{a}.\frac{\mathrm{OR}}{\mathrm{OP}}.\mathrm{PP}'=-\frac{k}{a}.\frac{\mathrm{OR}}{\mathrm{OP}}$$

i.e., proportional to the distance from the centre.

Fig. 3.

5. Supposing that the positions of all the images of the doublets and their magnitudes are known when the sphere A is moving along the line B A, we proceed to find an expression for the kinetic energy. Let ρ_n be the distance of the n^{th} image in A from A, and σ_n the distance of the n^{th} image in B. Also let the magnitudes of the doublets there be μ_n, ν_n respectively. Let ϕ be the velocity potential of the motion, and ϕ_n, ϕ'_n the parts of ϕ due to μ_n and ν_n. Then denoting the kinetic energy by T

$$2\mathrm{T}=-\int[\phi]\frac{\delta\phi}{\delta n}d\mathrm{S}=-2\pi a^2 u\int_0^\pi[\phi]\sin\theta\cos\theta d\theta$$

where $[\phi]$ is the value of ϕ at any point $(a.\theta)$ on the sphere. Now $\phi=\Sigma\phi_n+\Sigma\phi'_n$ and the part of T due to ϕ_n will be

$$2\mathrm{T}=-2\pi a^2 u\int_0^\pi\frac{\mu_n(a\cos\theta+\rho_n)\sin\theta\cos\theta}{\{a^2+2\rho_n a\cos\theta+\rho_n{}^2\}^{\frac{3}{2}}}d\theta$$

$$=-2\pi a^2\mu_n u\int_{-1}^{+1}\frac{(\rho_n+a\mu)\mu d\mu}{(a^2+\rho^2_n+2\rho_n a\mu)^{\frac{3}{2}}}$$

Now

$$\int_{-1}^{+1}\frac{(\rho+a\mu)\mu d\mu}{\{a^2+\rho^2+2\rho a\mu\}^{\frac{3}{2}}}=-\frac{d}{d\rho}\int_{-1}^{+1}\frac{\mu d\mu}{\sqrt{a^2+\rho^2+2\rho a\mu}}$$

$$\frac{d}{d\rho}\cdot\frac{1}{3\rho^2a^3}\{(\rho+a)(\rho^2+a^2-\rho a)-(\rho-a)(\rho^2+a^2+\rho a)\}$$

When $\rho=\rho_n$ $\rho_n<a$ and the above becomes

$$\frac{d}{d\rho}\cdot\frac{2\rho}{3a^2}=\frac{2}{3a^2}$$

Similarly when $\rho \equiv \sigma_n$ $\sigma_n > a$ and it becomes

$$\frac{d}{d\rho} \cdot \frac{2a}{3\rho^2} = -\frac{4a}{3\rho^3}$$

and

$$2\mathrm{T} = -\tfrac{4}{3}\pi . u \Sigma_0^\infty \mu_n + \tfrac{8}{3}\pi a^3 u \Sigma_1^\infty \left(\frac{\nu_n}{\sigma_n^3}\right)$$

Also μ the original doublet $= -\frac{a^3 u}{2}$, and if M_1 be the mass of fluid displaced by the sphere A

$$2\mathrm{T} = \tfrac{1}{2}\mathrm{M}_1 u^2 \left\{1 + \Sigma_1^\infty \frac{\mu_n}{\mu}\right\} + \mathrm{M}_1 u^2 \Sigma_1^\infty \left(-\frac{\nu_n a^3}{\mu \sigma_n^3}\right)$$

But from what has been shown before

$$\mu_n = -\frac{\ }{\sigma_n^3}\nu_n$$

Hence

$$2\mathrm{T} = \tfrac{1}{2}\mathrm{M}_1 u^2 \left\{1 + 3\Sigma_1^\infty \left(\frac{\mu_n}{\mu}\right)\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By § 4 we have, if c is the distance between the centres,

$$\mu_n = -\frac{\ }{\sigma_n^3}\nu_n$$

$$\nu_n = -\left(\frac{b}{c - \rho_{n-1}}\right)^3 \mu_{n-1}$$

Also

$$\rho_n = \frac{a^2}{\sigma_n} \qquad c - \sigma_n = \frac{b^2}{c - \rho_{n-1}}$$

Hence

$$\mu_n = \left(\frac{ab}{\sigma_n(c - \rho_{n-1})}\right)^3 \mu_{n-1} = \left(\frac{b\rho_n}{a(c - \rho_{n-1})}\right)^3 \mu_{n-1}$$

$$\left(\frac{b}{a}\right)^{3n} \left\{\frac{\rho_n \rho_{n-1} \ldots \ldots \rho_1}{(c - \rho_{n-1}) \ldots \ldots (c - \rho_1)c}\right\}^3 \mu$$

Again

$$\rho_n = \frac{a^2}{\sigma_n} = \frac{a^2}{c - \dfrac{b^2}{c - \rho_{n-1}}}$$

whence

$$\rho_n \rho_{n-1} - \frac{c^2 - b^2}{c}\rho_n - \frac{a^2}{c}\rho_{n-1} + a^2 = 0$$

Put $\rho_n = u_n + x$, and choose x so as to make the constant term vanish. To find x we have

$$x^2 - \frac{a^2+c^2-b^2}{c}x + a^2 = 0$$

Fig. 4.

Now let C_1 C_2 be the inverse points of the spheres, and O the middle point of C_1 C_2. Put $C_1 C_2 = 2\lambda$. Then

$$OA = \sqrt{\lambda^2 + a^2} = \frac{a^2+c^2-b^2}{2c} = r_1 \text{ say} \quad \ldots \ldots \ldots \quad (2)$$

$$\lambda = \sqrt{r_1^2 - a^2} \quad \ldots \ldots \ldots \ldots \ldots \quad (3)$$

$$c = r_1 + r_2$$

Further, P being any point on the sphere A, denote the constant ratio $\frac{C_2P}{C_1P}$ by q_1 and let q_2 be the similar constant for the sphere B. Then

$$\left.\begin{aligned} q_1 &= \frac{\lambda+r_1-a}{\lambda-r_1+a} = \frac{\lambda+r_1}{a} = \frac{a}{r_1-\lambda} \\ q_2 &= \frac{\lambda-r_2+b}{\lambda+r_2-b} = \frac{b}{\lambda+r_2} = \frac{r_2-\lambda}{b} \end{aligned}\right\} \quad (4)$$

The equation to determine x now becomes

$$x^2 - 2r_1x + a^2 = 0$$

The roots of which are $r_1 \pm \lambda$.

Choosing the positive sign, the equation of differences becomes

$$u_n u_{n-1} - \left(x_2 - \frac{a^2}{c}\right)u_n + \left(x_1 - \frac{a^2}{c}\right)u_{n-1} = 0$$

Now $a^2 = x_1 x_2$ whence writing $\frac{1}{v_n}$ for u_n we get

$$v_n - \frac{x_2(c-x_1)}{x_1(c-x_2)}\, v_{n-1} = -\frac{c}{x_1(c-x_2)}$$

Here

$$\frac{c-x_2}{c-x_1}=\frac{r_2+\lambda}{r_2-\lambda}=q_2^{-2}$$

and

$$\frac{x_1}{x_2}=\frac{r_1+\lambda}{r_1-\lambda}=q_1^{2}$$

$$\frac{\quad}{x_1(c-x_2)}=\frac{\quad}{(r_1+\lambda)(r_2+\lambda)}$$

Whence writing $\frac{q_2}{q_1}\equiv q$

$$(\mathrm{E}-q^2)v_n=-\frac{\quad}{(r_1+\lambda)(r_2+\lambda)}$$

and

$$v_n=\mathrm{A}q^{2n}-\frac{1}{2\lambda}$$

and

$$\rho_n=aq_1+\frac{1}{-\frac{1}{2\lambda}+\mathrm{A}q^{2n}}$$

But

$$n=0\ \rho=0$$

$$\therefore \mathrm{A}=\frac{1}{2\lambda}-\frac{1}{\lambda+r_1}=\frac{r_1-\lambda}{2\lambda(r_1+\lambda)}=\frac{1}{2\lambda q_1^{2}}$$

and

$$\rho_n=aq_1-\frac{2\lambda}{1-q_1^{-2}q^{2n}}$$

$$=(r_1-\lambda)\frac{1-q^{2n}}{1-q_1^{-2}q^{2n}}$$

Also

$$c-\rho_n=r_1+r_2-(r_1+\lambda)+\frac{2\lambda}{1-q_1^{-2}q^{2n}}$$

$$=(r_2+\lambda)\frac{1-q^{2n+2}}{1-q_1^{-2}q^{2n}}$$

$$\therefore \frac{b}{a}\cdot\frac{\rho_n}{c-\rho_{n-1}}=q\cdot\frac{1-q_1^{-2}q^{2n-2}}{1-q_1^{-2}q^{2n}}=q\frac{p_{n-1}}{p_n}\ \text{say}$$

and

$$\mu_n=\left\{q^n\cdot\frac{p_{n-1}}{p_n}\,\frac{p_{n-2}}{p_{n-1}}\,\cdots\,\frac{p_0}{p_1}\right\}^3\mu$$

$$=\left\{\frac{(1-q_1^{-2})q^n}{1-q_1^{-2}q^{2n}}\right\}^3\mu$$

Whence

$$2T=\tfrac{1}{2}M_1u^2\left\{1+3(1-q_1^{-2})^3\Sigma_1^\infty\left(\frac{q^n}{1-q_1^{-2}q^{2n}}\right)^3\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We shall denote, in what follows,

$$(1-q_1^{-2})^3\Sigma_1^\infty\left(\frac{q^n}{1-q_1^{-2}q^{2n}}\right)^3$$

by the functional symbol $Q\left(\frac{1}{q_1}, q\right)$.

6. If the sphere B is also moving along the line AB, the kinetic energy of the fluid will be of the form

$$2T=\tfrac{1}{2}M_1u_1^2\left\{1+3Q\left(\frac{1}{q_1},q\right)\right\}+\tfrac{1}{2}M_2u_2^2\{1+3Q(q_2,q)\}+Lu_1u_2$$

It remains, then, to find the value of L.

It is easily seen that L depends on the part of ϕ belonging to the images of B's motion taken over the sphere A, together with that belonging to the images of A's motion taken over the sphere B. Let now dashed letters apply to the images, &c., of the B system, then using the results in § 5, the part of L due to the integration over A is

$$=-\tfrac{4}{3}\pi u_1\Sigma_1^\infty\mu'_n+\tfrac{8}{3}\pi a^3u_1\Sigma_0^\infty\left(\frac{\nu'_n}{\sigma'^3_n}\right)$$

But as before, remembering that now the original doublet is in B,

$$\mu'_n=-\left(\frac{a}{\sigma'_{n-1}}\right)^3\nu'_{n-1}$$

$$\mu'_1=-\left(\frac{a}{c}\right)^3\nu'$$

and

$$L_1=-\tfrac{4}{3}\pi u_1\Sigma_1^\infty(3\mu'_n)$$

$$=-2\pi u_1u_2\frac{a^3b^3}{c^3}\Sigma_1^\infty\left(\frac{\mu'_n}{\mu'_1}\right)$$

and

$$-\left\{\frac{b\rho'_n}{a(c-\rho'_{n-1})}\right\}^3$$

$$=\left(\frac{b}{a}\right)^{3n-3}\left\{\frac{\rho'_n\rho'_{n-1}\ldots\ldots\rho'_2}{(c-\rho'_{n-1})\ldots\ldots(c-\rho'_1)}\right\}^3\mu'_1$$

Now as before

$$\rho'_n=aq_1+\frac{1}{\frac{1}{2\lambda}+Aq^{2n}}$$

and determining A by the condition that $\rho_1'=\frac{a^2}{c}$ we shall find

$$\rho'_n=aq_1-\frac{2\lambda}{1-q^{2n}}=(r_1-\lambda)\frac{1-q_1^2q^{2n}}{1-q^{2n}}$$

and

$$c-\rho'_{n-1}=(r_2+\lambda)\frac{1-q_1^2q^{2n}}{1-q^{2n-2}}$$

$$\frac{b}{a}\frac{\rho'_n}{c-\rho'_{n-1}}=q\cdot\frac{1-q^{2n-2}}{1-q^{2n}}$$

$$\mu'_n=\left\{\frac{(1-q^2)q^{n-1}}{1-q^{2n}}\right\}^3\mu'_1$$

and

$$L_1=-2\pi u_1u_2\Sigma\left\{\frac{ab(1-q^2)}{c}\cdot\frac{q^{n-1}}{1-q^{2n}}\right\}^3$$

$$=-16\pi u_1u_2\lambda^3\Sigma_1\left(\frac{q^n}{1-q^{2n}}\right)^3$$

Similarly $L_2=$ same quantity.

Therefore, denoting by M′ the mass of fluid contained in a sphere of radius unity

$$L=-4\pi u_1u_2Q'(q)=-3M'u_1u_2Q'(q)$$

where

$$Q'(q)=\Sigma_1^\infty\left(\frac{2\lambda q^n}{1-q^{2n}}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Tables for Q and Q′ are given at the end of the paper for equal spheres, and for the case of $a=2b$.

7. When the sphere A is moving perpendicularly to B A, the original doublet is one perpendicular to the line B A, as also its images. Suppose A is moving along the axis of x, A B being the axis of z. Then the normal velocity at a point P on the sphere A is $v\sin\theta\cos\chi$, $(a.\theta.\chi)$ being the polar coordinates of P; and the kinetic energy is given by

$$2T=-a^2v\int_0^\pi\int_0^{2\pi}[\phi]\sin^2\theta\cos\chi\, d\theta d\chi$$

Let μ be the magnitude of a doublet at a point distant ρ from the centre of A; the part of ϕ depending on this is

$$\frac{\mu r\sin\theta\cos\chi}{\{r^2+\rho^2+2\rho r\cos\theta\}^{\frac{3}{2}}}$$

and the part of 2T depending on this is

$$-\mu a^3 v\int_0^\pi\int_0^{2\pi}\frac{\sin^3\theta\cos^2\chi d\theta d\chi}{\{a^2+\rho^2+2\rho a\cos\theta\}^{\frac{3}{2}}}$$

$$=-\mu\pi a^3 v\int_0^\pi\frac{\sin^3\theta d\theta}{\{a^2+\rho^2+2\rho a\cos\theta\}^{\frac{3}{2}}}$$

The integral of which is

$$-\frac{2\mu\pi v}{3\rho^3}[(\rho^2+a^2)\{(\rho+a)-(\rho\sim a)\}-\rho a\{\rho+a+(\rho\sim a)\}]$$

Writing ν and σ for μ, ρ for doublets outside the sphere A, we obtain

$$-\tfrac{4}{3}\pi\mu v \text{ and } -\frac{4\pi a^3\nu v}{3\sigma^3}$$

whence

$$2T=-M_1 v\Sigma\left\{\frac{\mu}{a^3}+\frac{\nu}{\sigma^3}\right\}$$

Now any ν at the distance σ produces an image in A consisting of a doublet $\nu\left(\frac{a}{\sigma}\right)^3$ at a distance $\frac{a^2}{\sigma}$, together with a line sink stretching from this to the centre, whose line magnitude is $-\frac{\nu}{a\sigma}\times$ distance from the centre. Hence the whole amount of the image is

$$\nu\left(\frac{a}{\sigma}\right)^3-\frac{1}{2}\frac{\nu}{a\sigma}\left(\frac{a^2}{\sigma}\right)^2=\frac{1}{2}\nu\left(\frac{a}{\sigma}\right)^2$$

Now every μ except μ_0 forms part of an image of some ν, and of that ν only. Hence

$$\Sigma\frac{\nu}{\sigma_3}=2\Sigma\frac{\mu}{a^3}-2\frac{\mu_0}{a^3}$$

and

$$2T=-\frac{M_1 v}{a^3}\{\mu_0+3\Sigma\mu\}$$

$$=\tfrac{1}{2}M_1 v^2\left\{1+3\Sigma\left(\frac{\mu}{\mu_0}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The Σ extending to the whole mass of *images* inside A.

8. If A has also a motion along B A, together with one perpendicular to it, T has no term depending on u, v; for it is clear that if the sign of v is changed, then the kinetic energy must be the same as before.

If B moves also perpendicularly to B A, T will have additional terms in v_2^2, and v_1, v_2. The coefficient of v_2^2 will be analogous to that for v_1^2, whilst that for v_1, v_2, as

in the case of u_1, u_2, consists of two parts, depending on the integration over the two spheres. As in the case of L this coefficient L′ is

$$L'v_1v_2 = -\tfrac{4}{3}\pi v_1\Sigma\left\{\mu' + \nu'\left(\frac{a}{\sigma'}\right)^3\right\} - \tfrac{4}{3}\pi v_2\Sigma\left\{\nu + \mu\left(\frac{b}{\sigma}\right)^3\right\}$$

dashed letters referring to the motion of B, μ, ν referring to images within A and B respectively, and σ' σ denoting distances from the centres of A and B. This may be reduced as in the former case to

$$L'v_1v_2 = -4\pi v_1\Sigma_1(\mu') - 4\pi v_2\Sigma_1(\nu)$$

the μ' being the images in A of B's motion perpendicular to A B, and ν the images in B of A's motion perpendicular to A B.

$$L'v_1v_2 = -4\pi v_1\nu_0\Sigma_1\left(\frac{\mu'}{\nu_0}\right) - 4\pi v_2\mu_0\Sigma\left(\frac{\nu}{\mu_0}\right)$$

where ν_0, μ_0 are the original doublets at the centres of B, A, *i.e.*

$$\nu_0 = -\frac{b^3v_2}{2} \quad \mu_0 = -\frac{a^3v_1}{2}$$

whence

$$L' = \tfrac{8}{3}M_2\Sigma_1\left(\frac{\mu'}{\nu_0}\right) + \tfrac{8}{3}M_1\Sigma\left(\frac{\nu}{\mu_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which last the ratios $\frac{\mu'}{\nu_0}$, $\frac{\nu}{\mu_0}$ dq not contain v_1, v_2.

In the case of general motion of two spheres, each will have three components of velocity, u_1, v_1, w_1; u_2, v_2, w_2; and, in general, the expression for the kinetic energy will contain 21 terms. In the case in question we can easily see that the coefficients of 12 of these vanish. For consider the term in u_1, w_2—suppose v_1, w_1, u_2, v_2 all zero; the energy, from the symmetry of the motion, must clearly be unaltered if we reverse the direction of w_2. And this can only happen if the coefficient of $u_1w_2=0$. In this way we find the terms all vanish except those in u_1^2, v_1^2, w_1^2, u_2^2, v_2^2, w_2^2, u_1 u_2, v_1 v_2, w_1 w_2. Also from symmetry the coefficients of v_1^2, v_2^2, v_1 v_2, are equal respectively to the coefficients of w_1^2, w_2^2, w_1 w_2.

In what has gone before we have expressed the coefficient of u_1^2, u_2^2, u_1 u_2 in terms of quantities determined by the radii and distance of the spheres, and have shown how the coefficients of the other terms depend on the images of the motion, whereby we can without much difficulty approximate to their values when the distance of the spheres is large compared with their radii—or the distance between their surfaces is large compared with the radius of one of the spheres. We pass on to

consider the case where one sphere is inside the other. An approximation to the value of the coefficients of $v_1{}^2$ and $v_1\,v_2$ is given in § 15. It is remarkable that in the case of two cylinders the coefficients of the terms in u^2, v^2 are equal, while those of $u_1\,u_2$ and $v_1\,v_2$ are equal and opposite. But this is due to the fact that in a cylinder the image of a doublet (or a source) is a single doublet, whatever be the direction of the axis of the original doublet.

9. If S S_1 be in a line through the centre the infinite trail of images of § 2 cuts out, and we are left with an image source and sink, and a line sink between them, supposing S to be outside S_1. Let now S and S_1 approach together and become a doublet whose strength is μ. Then we shall get a single doublet as its image whose strength $=\frac{\mu a}{b}\mathrm{L}.\frac{\mathrm{S'S'_1}}{\mathrm{SS_1}}=-\frac{\mu a^3}{b^3}$ as in the former case. This we might have deduced at once from the case of the external doublet in § 4, considered as the image of its image.

Fig. 5.

If we proceed to find the kinetic energy, as in the previous case, we must clearly be led to the same *form* for the result, viz.: when A is moving with a velocity u from B

$$2\mathrm{T}=\tfrac{1}{2}\mathrm{M}_1u^2\left\{1+3\Sigma_1^\infty\left(\frac{\mu_n}{\mu}\right)\right\}$$

where μ_n . . . . are the strengths of the doublets inside A alone. But in this case the relations between the μ, ρ, σ are given by the equations (a, b being the radii of sphere)

$$\mu_n=-\frac{}{\sigma_n{}^3}\nu_n,$$

$$\nu_n=-\left(\frac{b}{c+\rho_{n-1}}\right)^3\mu_{n-1}$$

$$\rho_n=\frac{}{\sigma_n}\qquad c+\sigma_n=\frac{b^2}{c+\rho_{n-1}}$$

whence

$$\rho_n=\frac{a^2}{\dfrac{b^2}{c+\rho_{n-1}}-c}$$

$$\rho_n\rho_{n-1}+\frac{c^2-b^2}{c}\rho_n+\frac{a^2}{c}\rho_{n-1}+a^2=0$$

which differs from the equation for external spheres in having $-\rho$ for ρ for all values of n. We may therefore use the same solution and writing here

$$\mathrm{OA}=\sqrt{\lambda^2+a^2}$$
$$\sqrt{\lambda^2+b^2}-\sqrt{\lambda^2+a^2}=c$$

$$\mathrm{OA}=\frac{b^2-a^2-c^2}{2c}=r_1$$

$$\mathrm{OB}=\frac{b^2+c^2-a^2}{2c}=r_2$$

$$c=r_2-r_1$$

$$q_1=\frac{r_1-\lambda}{a}=\frac{a}{r_1+\lambda}$$

$$q_2=\frac{r_2-\lambda}{\lambda}=\frac{b}{r_2+\lambda}$$

$$q=\frac{q_1}{q_2}$$

$$\rho_n=(r_1-\lambda)\frac{1-q^{2n}}{1-q_1^2q^{2n}}$$

which is the same *form* as before, only q is the inverse of its former value.

And, as before,

$$2\mathrm{T}=\tfrac{1}{2}\mathrm{M}_1u^2\left\{1+3(1-q_1^2)^3\Sigma_1^\infty\left(\frac{q^n}{1-q_1^2q^{2n}}\right)^3\right\}$$
$$=\tfrac{1}{2}\mathrm{M}_1u^2\{1+3\mathrm{Q}(q.q_1)\}$$

A table for Q when $b=2a$ is given at the end of the paper.

10. It will be well here, before passing on to the consideration of the motion, to make a short digression on the properties of the functions Q and Q′. In the first place it is easily seen that the series for the Q and Q′ functions are both convergent, even up to the case when the spheres touch, or $q=1$; for the ratio of the n^{th} term to the $n-1^{\text{th}}$ is

$$\left\{q\frac{1-q_1^{-2}q^{2n-2}}{1-q_1^{-2}q^{2n}}\right\}^3$$

and this is always less than q^3, which is less than unity, except in the case when the spheres touch. In this particular case the n^{th} term tends to the limit

$$\left(1-n\frac{dq}{dq_1}\right)$$

and the series is still convergent. The value of $\frac{dq}{dq_1}$ is the limit when $\lambda=0$ of

$$\frac{q_1-q_2}{1-q_1}=-\frac{a+b}{b}.$$

Hence when the spheres are in contact

$$Q=\Sigma_1^\infty\left(\frac{b}{n(a+b)+b}\right)^3=x^3\Sigma_1^\infty\left(\frac{1}{n+x}\right)^3=-\tfrac{1}{2}x^3\frac{d^3}{dx^3}\log\Gamma(1+x)\quad . \quad . \quad . \quad . \quad (9)$$

if $x=\frac{b}{a+b}$. The values of this may be found from LEGENDRE'S table of the log Γ-functions.

If the spheres be equal $x=\frac{1}{2}$ and

$$Q=\Sigma_1\frac{1}{(2n+1)^3}=S'_3-1\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now

$$S_3=1+\frac{1}{2^3}+\frac{1}{3^3}+\ldots\ldots=1\cdot202056903159\ldots\ldots$$

$$=S'_3+\tfrac{1}{8}S_3$$

whence

$$S'_3=\tfrac{7}{8}S_3=1\cdot051799790264$$

When the spheres are equal $q_2=\frac{1}{q_1}$. If in this case q denote either q_2 or $\frac{1}{q_1}$

$$Q=(1-q^2)^3\Sigma_1^\infty\left(\frac{q^{2n}}{1-q^{4n+2}}\right)^3$$

11. We may easily express the general term in terms of r, a, b. For writing it in the form

$$Q=\left\{\frac{q_1-\frac{1}{q_1}}{\frac{q_1}{q^n}-\frac{q^n}{q_1}}\right\}^3=u_n^3\text{ suppose,}$$

we get at once from the relations (2), (3), (4)

$$u_n=\frac{2\lambda a^nb^n}{(r_1+\lambda)^{n+1}(r_2+\lambda)^n-(r_1-\lambda)^{n+1}(r_2-\lambda)^n}$$

which, since $r_2=r-r_1$ and $r_1^2-\lambda^2=a^2$

$$\frac{2\lambda a^nb^n}{(r_1+\lambda)\{r(r_1+\lambda)-a^2\}^n-(r_1-\lambda)\{r(r_1-\lambda)-a^2\}^n}$$

Now

$$rr_1-a^2=\frac{r^2+a^2-b^2}{2}-a^2=\frac{r^2-a^2-b^2}{2}=\frac{x^2}{2}\text{ suppose,}$$

also

$$\lambda^2r^2=(r_1{}^2-a^2)r^2=\frac{(r^2+a^2-b^2)^2-4a^2r^2}{4}=\frac{x^4-4a^2b^2}{4}$$

We shall further write $4a^2b^2=\alpha^4$

Then

$$u_n=\frac{2\lambda a^nb^n}{r_1\left\{\left(\frac{x^2}{2}+\lambda r\right)^n-\left(\frac{x^2}{2}-\lambda r\right)^n\right\}+\lambda\left\{\left(\frac{x^2}{2}+\lambda r\right)^n+\left(\frac{x^2}{2}-\lambda r\right)^n\right\}}$$

$$=\frac{2^na^nb^n}{2rr_1\Sigma\left\{\frac{\underline{|n}}{\underline{|2p+1}\,\underline{|n-2p-1}}\cdot x^{2n-4p-2}(2\lambda r)^{2p}\right\}+\Sigma\frac{\underline{|n}}{\underline{|2p}\,\underline{|n-2p}}x^{2n-4p}(2\lambda r)^{2p}}$$

$$=\frac{2^na^nb^n}{\Sigma\frac{\underline{|n}}{\underline{|2p+1}\,\underline{|n-2p}}\{(n+1)x^2+2(n-2p)a^2\}x^{2n-4p-2}(x^4-\alpha^4)^p}=\frac{2^na^nb^n}{v_n}\quad . \quad . \quad (11)$$

and

$$v_n=\Sigma\Sigma\frac{\underline{|n}\,\underline{|p}\,(-1)^q}{\underline{|2p+1}\,\underline{|n-2p}\,\underline{|q}\,\underline{|p-q}}\{\overline{n+1}x^2+2(n-2p)a^2\}x^{2n-4q-2}\alpha^{4q}$$

Denote

$$\Sigma_{p=q}\frac{\underline{|n}}{\underline{|2p+1}\,\underline{|n-2p-1}}\cdot\frac{\underline{|p}}{\underline{|q}\,\underline{|p-q}}$$

by $S_{n.q}$

Then

$$v_n=\Sigma\overline{-1}|^q\{S_{n+1,q}.x^{2n-4q}+2S_{n.q}a^2x^{2n-4q-2}\}\alpha^{4q}$$

Let

$$y=\frac{(1+\sqrt{x})^n-(1-\sqrt{x})^n}{2\sqrt{x}}=n+\frac{n(n-1)(n-2)}{\underline{|3}}x+\ldots.$$

$$+\frac{\underline{|n}}{\underline{|2p+1}\,\underline{|n-2p-1}}x^p+\ldots.$$

Then

$$S_{n.0}=\text{value of } y \text{ when } x \text{ is } 1=2^{n-1}$$

$$S_{n.1}=\text{value of } \frac{dy}{dx} \text{ when } x \text{ is } 1=(n-1)2^{n-2}$$

and in general

$$S_{n.q}=\text{value of } \frac{1}{\underline{|q}}\frac{d^qy}{dx^q} \text{ when } x=1$$

Now $q<n$. Hence

$$\frac{d^q}{dx^q}\frac{(1-\sqrt{x})^n}{\sqrt{x}}=0 \text{ when } x=1$$

Therefore

$$S_{n.q}=\frac{1}{2\lfloor q}\left[\frac{d^q}{dx^q}\cdot\frac{(1+\sqrt{x})^n}{\sqrt{x}}\right]_{x=1}$$

Also we see that v_n is a rational integral function of $(r^2-a^2-b^2)$

When one sphere is inside the other the series for Q is still convergent.

12. When the spheres are concentric

$$r_1=r_2=\infty \quad \lambda=\infty$$

$$q_1=0 \quad q_2=0$$

$$q=\frac{q_1}{q_2}=\frac{a}{b}$$

and

$$Q=\left(\frac{a}{b}\right)^3\frac{1}{1-\left(\frac{a}{b}\right)^3}=\frac{a^3}{b^3-a^3}$$

whence

$$2T=\tfrac{1}{2}\cdot\frac{b^3+2a^3}{b^3-a^3}M_1u^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which agrees with the result found by STOKES in his paper of 1843 before referred to.

When the inner touches the outer $\lambda=0$ and

$$Q=x^3\Sigma\frac{1}{(n+x)^3}=-\tfrac{1}{2}x^3\frac{d^3}{dx^3}\log_e \Gamma(1+x)$$

$$=-1\cdot15129x^3\frac{d^3}{dx^3}\log_{10}\Gamma(1+x) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$x=\frac{b}{b-a}$$

If x is an integer $= m$ say

$$Q=m^3\left\{S_3-\Sigma_1^m\frac{1}{n^3}\right\}=m^3\left\{\cdot2020569-\Sigma_2^m\frac{1}{n^3}\right\}$$

a finite expression, and in this case

$$a=\frac{m-1}{m}b$$

In the particular cases

$$a=\tfrac{1}{2}b \qquad Q=\ \cdot61645$$

$$a=\tfrac{2}{3}b \qquad Q=1\cdot08054$$

If x is of the form $\frac{2m+1}{2}$,

$$Q=(2m+1)^3\left\{S'_3-\Sigma_0^m\left(\frac{1}{2n+1}\right)^3\right\}$$

$$=(2m+1)^3\left\{\cdot 0517998-\Sigma_1^m\left(\frac{1}{2n+1}\right)^3\right\}$$

Also a finite expression, and in this case

$$a=\frac{2m-1}{2m+1}b$$

In the particular cases

$$a=\tfrac{1}{3}b \quad Q=\cdot 39859$$

$$a=\tfrac{3}{5}b \quad Q=\cdot 84535$$

The expressions for Q directly in terms of r, a, b are the same functions of $a^2+b^2-r^2$ as the corresponding expressions for external spheres are of $r^2-a^2-b^2$.

13. The series for Q′ may, as in the case of Q, be shown to be convergent.

When the spheres are in contact

$$Q'=\frac{a^3b^3}{r^3}S_3=\left(\frac{ab}{a+b}\right)^3S_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Also the general term in r, a, b is given by

$$u_n=\frac{(2ab)^n}{2r\Sigma\frac{\lfloor n}{\lfloor 2p+1\lfloor n-2p-1}x^{2n-4p-2}(x^4-a^4)^p}=\frac{2^{n-1}a^nb^n}{r\Sigma(-1)^qS_{n\cdot q}a^{4q}x^{2n-4q-2}} \quad . \quad . \quad (15)$$

It is easily seen that both Q, Q′ for external spheres diminish as x—*i.e.*, r—increases.

Hence for external spheres $\frac{dQ}{dr}$, $\frac{dQ'}{dr'}$ are both negative.

When one sphere is inside the other, Q decreases as x increases—*i.e.*, as r diminishes.

Hence in this case $\frac{dQ}{dr}$ is positive.

The values of the first three terms of Q, Q′ are

$$\begin{array}{ll}\text{for Q} & \left\{\frac{ab}{r^2-b^2}\right\}^3,\ \left\{\frac{a^2b^2}{(r^2-b^2)^2-a^2r^2}\right\}^3,\ \left\{\frac{a^3b^3}{x^6+a^2x^4-2a^2b^2x^2-a^4b^2}\right\}^3 \\ \text{and for Q}' & \left(\frac{ab}{r}\right)^3,\ \left(\frac{a^2b^2}{rx^2}\right)^3,\ \left\{\frac{a^3b^3}{r(x^4-a^2b^2)}\right\}^3\end{array} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

14. We may easily find $\frac{dQ_1}{dr}, \frac{dQ'}{dr}$ at contact of the spheres. If Q_n denote the n^{th} term of Q_1, then it may be shown that, x denoting $\frac{b}{a+b}$

$$\frac{dQ_n}{dr}=-\frac{n(n+1)(n-1+3x)}{a(n+x)^4}x^2$$

$$\frac{dQ'_n}{a^3dr}=-\frac{n^2-1+3x(1-x)}{an^3}x^2$$

both of which are of the order $\frac{1}{n}$. Hence the values of $\frac{dQ_1}{dr}, \frac{dQ_2}{dr}, \frac{dQ'}{dr}$ at contact are $=-\infty$. But though this is the case, the value of $\frac{d}{dr}(Q_1-\frac{1}{a^3}Q')$ at contact is finite. The n^{th} term is

$$\frac{n^3(n+1)(n-4)+3(1-x)(n+x)^4+(n^2-1)(6n^2+4nx+x^3)x}{an^3(n+x)^4}x^3$$

which is of the order $\frac{1}{n^2}$, and therefore the whole sum is finite. Also when $n \gtreqless 2$ the n^{th} term is positive, when $n=1$ the sign depends on the value of x. But by considering the values of Q, &c., in terms of r, expanding them in ascending inverse powers of r, it can be shown that $\frac{d}{dr}\left(Q-\frac{1}{a^3}Q'\right)$ is positive always. Further, at contact $Q-\frac{Q'}{a^3}$ is a negative quantity, whilst at an infinite distance it is zero. Hence, *on the whole*, it must increase with r, and if this takes place continuously, $\frac{d}{dr}\left(Q-\frac{1}{a^3}Q'\right)$ would always be positive. Though I have convinced myself that such is the case, I have not been able to prove it in general. When the spheres are at a great distance the values of Q and Q′ depend only on their first terms, and $Q-\frac{1}{a^3}Q'$ only on the term of Q′, which is of the order $\frac{b^3}{r^3}$. Hence here also the differential coefficient is positive. I have calculated and laid down curves representing the magnitudes of the Q and Q′-functions in the case of equal spheres, and when the radius of one sphere is twice that of the other, and in both cases the value for $\frac{d}{dr}\left(Q-\frac{1}{a^3}Q'\right)$ comes out positive for all distances. In what follows we shall suppose that this quantity is always positive, but it must be understood throughout as only *proved* for the case of equal spheres and the case in which the radius of one sphere is double that of the other.

15. Although the rapidly increasing complexity of the successive images when the spheres move perpendicularly to their line of centres would lead us to regard the

problem of finding the energy in this case as almost hopeless, yet we can carry the approximation to any number of images with less labour than might at first sight appear. For suppose we wish to take into account $2n$ images in A, due to A's motion, that is on the whole $4n$ reflections. We need only first calculate the distribution of doublets for a *general* position of the original one, in the n^{th} image in A, and find the amount of the first n images. We can then treat the second portion of the $2n$ images as the images resulting from the different parts of the n^{th} image, and employ our first result to find the amount of the second portion by a single integration. Suppose we proceed as if we were going on indefinitely: we suppose an original doublet in A at a distance ρ and calculate the density of the parts of the first image in A, say $f(r)$ at a distance r, and thence its amount. We employ this result to find the density at any point of the second image, regarding it as made up of images of the different parts of the first, and this we do by using the expression found before, substituting for the original doublet at ρ, an amount $f(r)dr$ at a distance r, and integrating with respect to r over the first image. Thus we find the distribution for the second image and its amount, and therefore the amount for the first two images together. Starting now from this, and proceeding in the same way, we find the distribution and amount of the first four images, then of the first eight, and so on. Thus to find the distribution of the 2^pth image we only require $\overline{p+1}$ operations, and to find its amount only p operations. Even with this method of proceeding the work would be exceedingly laborious. But for all practical purposes the first two images in A, *i.e.*, the motion due to *four reflections*, will be sufficient—except when the spheres are in contact. We proceed then to find the values of the coefficient of v^2 and of $v_1 v_2$ to this degree of approximation.

Suppose we have at P inside A a doublet k at a distance ρ_1 from A, whose axis is perpendicular to A B.

i. *First image in* B.—Then we have at Q_1, its inverse point in B, a doublet $\left(\frac{b}{BP_1}\right)^3 k$ and a line doublet thence to B, whose line density $=-\frac{k}{b}\cdot\frac{r}{BP_1}$.

ii. *First image in* A.—The image of this in A consists of two parts, that depending on the single doublet in B, and that depending on the line doublet.

(α) *Image of* Q_1.—A doublet at P_2 $\left(AP_2=\frac{a^2}{AQ_1}\right)$ whose magnitude is $\left(\frac{ab}{AQ_1.BP_1}\right)^3 k$, and a negative line doublet from P_2 to A whose line density $=-\frac{k}{a}\cdot\left(\frac{b}{BP_1}\right)^3\frac{R}{AQ_1}$

(β) *Image of negative line doublet.*—At a distance r from B we have a negative doublet $=-\frac{k}{b}\cdot\frac{r}{BP_1}dr$. This has (1) a negative doublet at a distance from $A=\frac{a^2}{c-r}=R$ equal to $-\left(\frac{a}{c-r}\right)^3\frac{k}{b}\cdot\frac{r}{BP_1}dr$. That is from P_2 to A we have a negative line doublet whose density at a distance R is

$$-\left(\frac{a}{c-r}\right)^3 \frac{k}{b} \cdot \frac{r}{BP_1} \frac{dr}{dR}$$

and $c-r=\frac{a^2}{R}$

$$\therefore \frac{dr}{dR}=\frac{a^2}{R^2}$$

$$\therefore \text{density} = -\frac{k}{ab} \cdot \frac{cR-a^2}{BP_1}$$

(2) a line doublet image of each portion. The doublet $-\frac{k}{b} \cdot \frac{r}{BP_1} dr$ produces a positive line doublet from a distance $R'=\frac{a^2}{c-r}$ to A, whose line density $=\frac{k}{ab} \cdot \frac{rdr}{BP_1} \cdot \frac{R}{c-r}$.

Hence the density at a distance R, due to this part, from the whole line doublet in B

$$=\int_{c-\frac{a^2}{R}}^{BQ_1} \frac{k}{ab} \cdot \frac{R}{BP_1} \cdot \frac{rdr}{c-r}$$

$$=\frac{k}{ab} \cdot \frac{R}{BP_1}\left\{c \log \frac{\rho_2}{R}-a^2\left(\frac{1}{R}-\frac{1}{\rho_2}\right)\right\}$$

Hence finally the density at a distance R from A of the resultant line doublet

$$=-\frac{k}{a} \cdot \left(\frac{b}{BP_1}\right)^3 \frac{R}{AQ_1}-\frac{k}{ab} \cdot \frac{cR-a^2}{BP_1}+\frac{k}{ab} \cdot \frac{R}{BP_1}\left\{c \log \frac{\rho_2}{R}-a^2\left(\frac{1}{R}-\frac{1}{\rho_2}\right)\right\}$$

$$=-\frac{k}{a} \cdot \frac{R}{AQ_1} \cdot \left(\frac{b}{BP_1}\right)^3+\frac{k}{ab} \cdot \frac{R}{BP_1}\left\{c \log \frac{\rho_2}{R}-BQ_1\right\}$$

and the whole amount

$$=-\frac{k}{a}\int_0^{\rho_2}\left\{\frac{1}{AQ_1} \cdot \left(\frac{b}{BP_1}\right)^3+\frac{BQ_1}{b.BP_1}-\frac{c}{b.BP_1} \log \frac{\rho_2}{R}\right\} RdR$$

$$=-k\left\{\frac{1}{2}\left(\frac{ab}{AQ_1.BP_1}\right)^3+\frac{1}{2} \frac{BQ_1.a^3}{b.BP_1.AQ_1^2}-\frac{1}{4} \frac{a^3c}{b.BP_1.AQ_1^2}\right.$$

So that the whole amount of the image is

$$\frac{1}{2} \cdot \left(\frac{ab}{AQ_1.BP_1}\right)^3 k+\frac{1}{4} \frac{a^3}{b.BP_1.AQ_1^2}(c-2BQ_1)k$$

or substituting for AQ_1, &c., in terms of ρ_1

$$\frac{1}{2}\left(\frac{ab}{c^2-b^2-c\rho_1}\right)^3 k+\frac{a^3(c^2-c\rho_1-2b^2)}{4b(c^2-b^2-c\rho_1)^2} k$$

For the first image in A, $\rho_1=0$

Hence

$$\mu_1=\tfrac{1}{2}\left\{\left(\frac{ab}{c^2-b^2}\right)^3+\frac{a^3(c^2-2b^2)}{2b(c^2-b^2)^2}\right\}k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The density at any point of the first image is

$$-\frac{kb}{a}\left\{\frac{1}{c^2-b^2}-\frac{1}{b^2}\log\frac{\rho_2}{R}\right\}R$$

together with a doublet $\left(\frac{ab}{c^2-b^2}\right)^3 k$ at a distance ρ_2

The amount of that part of the second image in A which depends on the latter is

$$\tfrac{1}{2}\left\{\left(\frac{ab}{c^2-b^2-c\rho_2}\right)^3+\frac{a^3(c^2-c\rho_2-2b^2)}{2b(c^2-b^2-c\rho_2)^2}\right\}\left(\frac{ab}{c^2-b^2}\right)^3 k$$

and the amount of the part due to the portion of the former at a distance R is

$$-\frac{kb}{2a}\left\{\left(\frac{ab}{c^2-b^2-cR}\right)^3+\frac{a^3(c^2-2b^2-cR)}{2b(c^2-b^2-cR)^2}\right\}\left\{\frac{1}{c^2-b^2}-\frac{1}{b^2}\log\frac{\rho_2}{R}\right\}RdR$$

whence the whole amount due to the former

$$=-\frac{kb}{2ac}\int_0^{\rho_2}\left[\frac{a^3b^3(c^2-b^2)}{(c^2-b^2-cR)^3}-\frac{a^3b(c^2+b^2)}{2(c^2-b^2-cR)^2}+\frac{a^3c^2}{2b(c^2-b^2-cR)}-\frac{a^3}{2b}\right]\left\{\frac{1}{c^2-b^2}-\frac{1}{b^2}\log\frac{\rho_2}{R}\right\}dR$$

$$=-\frac{ka^2b}{4c^2(c^2-b^2)}\left[\frac{b^3(c^2-b^2)}{(c^2-b^2-cR)^2}-\frac{b(c^2+b^2)}{c^2-b^2-cR}-\frac{cR}{b}-\frac{c^2}{b}\log(c^2-b^2-cR)\right]_0^{\rho_2}$$

$$+\frac{ka^2}{4b^2}\int_0^{\rho_2} c\frac{\log\frac{\rho_2}{R}}{c^2-b^2-cR}dR$$

$$+\frac{ka^2}{4bc^2}\int_0^{\rho_2}\log\frac{\rho_2}{R}\frac{d}{dR}\left\{\frac{b^3(c^2-b^2)}{(c^2-b^2-cR)^2}-\frac{b(c^2+b^2)}{c^2-b^2-cR}-\frac{cR}{b}+\frac{bc^2}{c^2-b^2}\right\}dR$$

$$=-\frac{ka^2b^4}{4c^2(c^2-b^2-c\rho_2)^2}+\frac{ka^2}{4(c^2-b^2)}\left\{\frac{2b^2}{c^2-b^2-c\rho_2}+\frac{2b^2-c^2}{b^2.c}\rho_2+\frac{b^2(b^2-2c^2)}{c^2(c^2-b^2)}\right.$$

$$\left.+2\log\frac{c^2-b^2-c\rho_2}{c^2-b^2}\right\}-\frac{ka^2}{4b^2}\int_0^1\frac{\log x}{\frac{c^2-b^2}{c\rho_2}-x}dx$$

Now $c\rho_2=\frac{a^2c^2}{c^2-b^2}$, $=ac\alpha$ say.

Then the above is

$$=-\frac{kb^4}{4c^4}\left(\frac{\alpha}{1-\alpha^2}\right)^2+\frac{k\alpha^2}{4c^2}\left\{\frac{2b^2}{1-\alpha^2}+2(a^2-b^2)-\frac{a^2c^2}{b^2}+\frac{b^4}{c^2}+2(c^2-b^2)\log(1-\alpha^2)\right\}-\frac{ka^2}{4b^2}\int_{0}^{1}\frac{\log x}{\frac{1}{\alpha^2}-x}dx$$

Wherefore the whole amount of the second image in A_1 is $\left(\text{writing } \beta=\frac{ba}{c^2-b^2}\right)$

$$\frac{\mu_2}{\mu_0}=\tfrac{1}{2}\left(\frac{\beta^2}{1-\alpha^2}\right)^3-\tfrac{1}{4}\left(\frac{a\beta^3}{b}+\frac{b^2}{c^2}\right)\left(\frac{\beta}{1-\alpha^2}\right)^2+\tfrac{1}{4}\left(\frac{a^2\beta^2}{b^2}+2\right)\frac{\beta^2}{1-\alpha^2}$$
$$+\tfrac{1}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}\right)\beta^2-\tfrac{1}{2}\beta^2+\frac{a^2\alpha^2}{2c^2}\log(1-\alpha^2)-\frac{a^2}{4b^2}\int_{0}^{1}\frac{\log x}{\frac{1}{\alpha^2}-x}dx \quad . \quad . \quad . \quad . \quad (18)$$

Substituting for $\frac{\mu_2}{\mu_0}$ and $\frac{\mu_1}{\mu_0}$ in (7) we get the part of T depending on v_1^2 correct to the second image in A. Interchanging a and b, the part depending on v_2^2 is found.

In the case of equal spheres

$$\frac{\mu_1}{\mu_0}=\tfrac{1}{2}\left\{\left(\frac{a^2}{c^2-a^2}\right)^3+\frac{a^2(c^2-2a^2)}{2(c^2-a^2)^2}\right\}$$

$$\frac{\mu_2}{\mu_0}=\tfrac{1}{2}P^3-\tfrac{1}{4}\cdot\frac{a^4c^2+(c^2-a^2)^3}{b^2c^2(c^2-a^2)}P^2+\tfrac{1}{4}\left(\frac{a^4}{(c^2-a^2)^2}+2\right)P$$
$$+\frac{a^6}{(c^2-a^2)^2c^2}+\frac{a^4}{2(c^2-a^2)^2}\log(1-\alpha^2)-\tfrac{1}{4}\int_{0}^{1}\frac{\log x}{\frac{1}{\alpha^2}-x}dx$$

16. To find the value of the coefficient of the term in $v_1\, v_2$ we need to find the amounts of the images in B due to the motion of A, and *vice versâ.*

The first image in B of k in A at a distance ρ_1 is $\left(\frac{b}{BP_1}\right)^3 k$ at Q_1 and a negative line doublet thence to B, whose line density is $-\frac{k}{b}\cdot\frac{r}{BP_1}$.

The whole amount is therefore

$$\left(\frac{b}{BP_1}\right)^3 k-\tfrac{1}{2}\left(\frac{b}{BP_1}\right)^3 k=\tfrac{1}{2}\left(\frac{b}{c-\rho_1}\right)^3 k$$

For the first $\rho_1=0$ and $\frac{\nu_1}{\mu_0}=\tfrac{1}{2}\left(\frac{b}{c}\right)^3 k$

To find the amount of the second image in B we start from the first image in A already found. This is, as has been shown, a doublet at $\rho_2=\left(\frac{ab}{c^2-b^2}\right)^3 k$ and a line doublet thence to A, whose line density is

$$\left\{\frac{c}{b.BP_1}\left(\log\frac{\rho_2}{R}-\frac{BQ_1}{c}\right)-\frac{1}{AQ_1}\left(\frac{b}{BP_1}\right)^3\right\}\frac{kR}{a}$$

The amount in B from the former is

$$\tfrac{1}{2}\left\{\frac{ab^2}{(c-\rho_2)(c^2-b^2)}\right\}^3 k$$

and from the second is

$$\tfrac{1}{2}\int_0^{\rho_2}\left(\frac{b}{c-R}\right)^3\left\{\frac{1}{b}\log\frac{\rho_2}{R}-\frac{b}{c^2-b^2}\right\}\frac{kR}{a}dR$$

$$=\frac{kb^2}{4ac}\left\{\frac{\rho_2}{c-\rho_2}+\tfrac{1}{2}\log\frac{c-\rho_2}{c}-\frac{b^2\rho_2^2}{(c^2-b^2)(c-\rho_2)^2}\right\}$$

$$=\frac{kb^2}{4ac}\left\{\frac{c\rho_2}{c^2-b^2}\cdot\frac{c^2-b^2-c\rho_2}{(c-\rho_2)^2}+\tfrac{1}{2}\log\frac{c-\rho_2}{c}\right\}$$

$$\therefore=\frac{kb^2}{4ac}\left\{\frac{a^2}{c^2-b^2}\cdot\frac{(c^2-b^2)^2-a^2c^2}{(c^2-a^2-b^2)^2}+\tfrac{1}{2}\log\left(.1-\frac{a^2}{c^2-b^2}\right)\right\}$$

$$\therefore\frac{\nu_2}{\mu_0}=\tfrac{1}{2}\left\{\frac{ab^2}{c(c^2-a^2-b^2)}\right\}^3+\frac{b^2}{4ac}\{\;\ldots\ldots\;\}$$

So also

$$\frac{\mu'_2}{\nu_0}=\tfrac{1}{2}\left\{\frac{a^2b}{c(c^2-a^2-b^2)}\right\}^3+\frac{a^2}{4bc}\left\{\frac{b^2}{c^2-a^2}\cdot\frac{(c^2-a^2)^2-b^2c^2}{(c^2-a^2-b^2)^2}+\tfrac{1}{2}\log\frac{c^2-a^2-b^2}{c^2-a^2}\right\}$$

Whence from (8)

$$L'=2\pi b^3\left(\frac{\mu'_1+\mu'_2}{\nu_0}\right)+2\pi a^3\left(\frac{\nu_1+\nu_2}{\mu_0}\right)$$

$$=\tfrac{3}{2}M'\left[\left(\frac{ab}{c}\right)^3+\left\{\frac{a^2b^2}{c(c^2-a^2-b^2)^2}\right\}^3-\frac{a^2b^2}{8c}\log\frac{(c^2-a^2-b^2)^2}{(c^2-a^2)(c^2-b^2)}\right.$$
$$\left.+\frac{a^2b^2}{4c(c^2-a^2-b^2)^2}\left\{c^2(a^2+b^2)-2a^2b^2-c^2\left(\frac{b^4}{c^2-a^2}+\frac{a^4}{c^2-b^2}\right)\right\}\right]\quad .\quad .\quad (19)$$

Similarly can be found the coefficient of v^2 when one sphere moves inside another.

Motion in the line of centres.

17. When the two spheres are moving in the line of centres the kinetic energy is given by

$$2T=A_1u_1^2+A_2u_2^2-2Bu_1u_2$$

where

$$A_1=m_1+\tfrac{1}{2}m'_1\left\{1+3Q\left(\frac{1}{q_1}\cdot q\right)\right\}$$
$$B=\tfrac{3}{2}M'Q'(q)$$

and m_1, m_1', M' respectively denote the mass of the sphere (A), the mass of fluid displaced by it, and the mass of fluid in a unit sphere.

It is to be remarked that A_1, A_2, B are functions only of the distance between the spheres, and that therefore $\frac{d}{dx_1}+\frac{d}{dx_2}=0$. Since no forces are supposed to act on the system, both the energy and momentum are constant. Hence

$$\left.\begin{aligned}&2T=\text{constant}\\&\frac{\delta T}{\delta u_1}+\frac{\delta T}{\delta u_2}=\text{constant}=d\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The last equation also follows at once from LAGRANGE'S equation since $\frac{\delta T}{\delta x_1}+\frac{\delta T}{\delta x_2}=0$, and may be written

$$(A_1-B)u_1+(A_2-B)u_2=d$$

We shall transform these equations by referring the motion to the velocity of an arbitrarily chosen point P between the spheres, and the distance between them.

Let P divide the distance (r) in the constant ratio $\frac{\alpha}{1-\alpha}=\frac{\alpha}{\beta}$. Then if x is the distance of P from the origin, u its velocity

$$x_1=x+\alpha r,\quad x_2=x-\beta r$$

and

$$u_1=u+\alpha\dot{r},\quad u_2=u-\beta\dot{r}$$

whence

$$\left.\begin{aligned}&(A_1+A_2-2B)u^2+(A_1\alpha^2+A_2\beta^2+2\alpha\beta B)\dot{r}^2\\&\qquad\qquad+2\{\alpha(A_1-B)-\beta(A_2-B)\}u\dot{r}=2T\\&(A_1+A_2-2B)u+\{\alpha(A_1-B)-\beta(A_2-B)\}\dot{r}=d\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21)$$

which we shall write

$$pu^2+q\dot{r}^2+2lu\dot{r}=2T$$

$$pu+l\dot{r}=d$$

whence

$$(pq-l^2)\dot{r}^2=2Tp-d^2$$

or

$$\dot{r}=\pm\sqrt{\left(\frac{2Tp-d^2}{A_1A_2-B^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

in which we are to take the positive or negative sign according as the spheres are separating or approaching one another. The spheres will move as if they repel or

attract one another *relatively* according as $\frac{d}{dr}\left\{\frac{2\mathrm{T}p-d^2}{\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2}\right\}$ is positive or negative. This condition does not depend on their relative motion at any time, but only on their distance and the ratio of the constant energy to the constant momentum. The above condition may also be expressed, writing $\frac{d^2}{2\mathrm{T}}=k^2$, as the sign of

$$k^2\frac{d}{dr}(\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2)-\left\{(\mathrm{A}_2-\mathrm{B})^2\frac{d\mathrm{A}_1}{dr}+(\mathrm{A}_1-\mathrm{B})^2\frac{d\mathrm{A}_2}{dr}+2(\mathrm{A}_1-\mathrm{B})(\mathrm{A}_2-\mathrm{B})\frac{d\mathrm{B}}{dr}\right\}$$

The last term is positive, for A_1, A_2, B all decrease as r increases. Now k must always be $<p$ since $\dot{r}$ is always real. If we put $k^2=p=\mathrm{A}_1+\mathrm{A}_2-2\mathrm{B}$ in the above, the criterion reduces to the sign of

$$(\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2)\frac{d}{dr}\{\mathrm{A}_1+\mathrm{A}_2-2\mathrm{B}\}$$

i.e., since $\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2$ is always positive to the sign of

$$\frac{d}{dr}(\mathrm{A}_1-\mathrm{B})+\frac{d}{dr}(\mathrm{A}_2-\mathrm{B})$$

Now we are led to conclude from the argument in § 14 that $\frac{d}{dr}(\mathrm{A}_1-\mathrm{B})$ . . . are always positive. Hence when k has its greatest possible value the criterion is positive, much more then is it so for any other value of k. Hence we are led to conclude that whatever be the relation between the momentum and energy the spheres always move so that $\dot{r}$ tends to decrease, whilst in the case of equal spheres, or that in which the radius of one is twice that of the other, we know for certain that such is the case. We cannot prove from this that the spheres move with reference to a *fixed* point as if they repel one another, for it might happen that both the spheres might be accelerated, the extra energy of the motion of the spheres themselves being taken from the fluid motion; or that both are even retarded. We can easily show, however, that both cannot be accelerated if $\dot{r}$ is positive and both move in the same direction, for the distance in this case increases, and therefore so do $\mathrm{A}_1-\mathrm{B}$, $\mathrm{A}_2-\mathrm{B}$, and hence because $(\mathrm{A}_1-\mathrm{B})u_1+(\mathrm{A}_2-\mathrm{B})u_2$ is constant u_1, u_2 cannot both increase. Also if $\dot{r}$ is negative and u_1, u_2 of the same sign the same result holds.

In the case where the spheres are projected so that the momentum is zero

$$\dot{r}^2=\frac{2\mathrm{T}p}{\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2}$$

and the relation between the velocities of projection that this may be the case is given by

$$\frac{u_1}{} = -\frac{A_2 - B}{A_1 - B}$$

When the spheres are equal $u_2 = -u_1$ and the motion is the same as that of a single sphere in a fluid bounded by a plane, and moving perpendicularly to the plane.

For this particular case

$$\dot{r}^2 = \frac{4T}{A+B}$$

or if u denote the velocity relative to the fixed plane $\dot{r} = 2u$, and

$$u^2 = \frac{T}{A+B} = \frac{(A+B)_0}{A+B} u_0^2$$

where $(A+B)_0$, u_0 are the values of $A+B$, and u at the point of projection. If the sphere is projected from contact with the plane

$$\begin{aligned}(A+B)_0 &= m + \tfrac{1}{2}m' + \tfrac{3}{2}m'(\tfrac{7}{8}S_3 - 1 + \tfrac{1}{8}S_3) \\ &= m + \tfrac{1}{2}m' + \cdot 3030853m' \\ &= m + \cdot 803085m'\end{aligned}$$

and at an infinite distance

$$A+B = m + \tfrac{1}{2}m'$$

Hence the ratio of the limiting velocity to the initial velocity is

$$\sqrt{\left\{1 + \cdot 6061707 \frac{1}{2\rho+1}\right\}}$$

where ρ is the density of the sphere.

For densities 0, 1, 10, the values of this ratio are respectively 1·2661, 1·0963, 1·0143. The greatest value is when the density of the sphere is zero, and the least is when $m' = 0$ (no fluid) or $m = \infty$, the ratio then being, as it ought to be, unity.

In the case where the spheres are unequal and projected with no momentum from contact their initial velocities must be opposite and in the ratio of the quantities

$$m_2 + \tfrac{1}{2}m'_2 - \tfrac{3}{2}m'_2 y^3 \{\tfrac{1}{2}D^3 \log_e \Gamma(1+y) + S_3\}$$

and

$$m_1 + \tfrac{1}{2}m'_1 - \tfrac{3}{2}m'_1 x^3 \{\tfrac{1}{2}D^3 \log_e \Gamma(1+x) + S_3\}$$

x and y denoting the quantities $\frac{b}{a+b}, \frac{a}{a+b}$

If $a=2b$, $x=\frac{1}{3}$, $y=\frac{2}{3}$, and we find from LEGENDRE's tables of the Eulerian integrals

$$D^3 \log_{10} \Gamma(1+x) = -\cdot 485$$
$$D^3 \log_{10} \Gamma(1+y) = -\cdot 275$$

and the ratio is

$$\frac{1}{8}\frac{\rho+\cdot 1174}{\rho+\cdot 4642}$$

which when the densities of the spheres and fluid are equal becomes

$$\frac{\cdot 763}{8}=\cdot 0954$$

We find the velocities of the spheres relatively to the fluid by eliminating u between

$$u_1=u+a\dot{r}$$

and

$$pu+l\dot{r}=d$$

whence

$$u_1=\frac{d}{p}+\frac{A_2-B}{p}\dot{r}$$

and

$$u_2=\frac{d}{p}-\frac{A_1-B}{p}\dot{r}$$

Suppose now the same spheres projected with the same initial circumstances except that now the spheres have changed places, and let u'_2, u'_1 be the corresponding velocities at the same distances. Then

$$u'_2=\frac{d}{p}+\frac{A_1-B}{p}\dot{r}$$

since d and $\dot{r}$ do not depend on the question which of the two is foremost.

Now if $a>b$ we see at once from the expressions given for A_1, A_2 in terms of the distances that $A_1>A_2$, and hence that the foremost will be most accelerated when it is the smallest.

If now u_1, u_2 denote the velocities at any moment which we may regard as the velocity of projection

$$k^2=\frac{d^2}{2T}=\frac{\{(A_1-B)u_1+(A_2-B)u_2\}^2}{A_1u_1^2+A_2u_2^2-2Bu_1u_2}$$

Writing ξ for the ratio $\frac{u_1}{u_2}$ the equation to find ξ, in order that k may have a given value, is

$$\xi^2+2\frac{(A_1-B)(A_2-B)+k^2B}{(A_1-B)^2-k^2A}\xi+\frac{(A_2-B)^2-k^2A_2}{(A_1-B)^2-k^2A_1}=0$$

This enables us to find within what limits k must lie, for ξ must have real roots, and therefore

$$\{(A_1-B)(A_2-B)+k^2B\}^2-\{(A_1-B)^2-k^2A_1\}\{(A_2-B)^2-k^2A_2\}>0$$

or

$$k^2(A_1A_2-B^2)(p-k^2)>0$$

Hence k^2 may be any positive quantity less than p. The greatest possible value of this is when the spheres are infinitely distant, and then

$$p=m_1+m_2+\tfrac{1}{2}(m'_1+m'_2)$$

To each value of ξ will correspond two states of motion, the initial velocities in each case being opposite. For example, if ξ is positive, *i.e.*, both velocities in the same direction, the two states will be when (a) is the foremost, and when (b) is the foremost; if ξ be negative, the two states will be, one in which the balls begin to move towards each other, the other in which they begin to move from each other. Thus for every given value of k there are four possible states of motion.

If ever $u_1=0$ then $\xi=0$, and the spheres must be at such a distance that

$$(A_2-B)^2-k^2A_2=0$$

Now, supposing k given, this can only happen if k^2 lies between the greatest and least values of $\frac{(A_2-B)^2}{A_2}$. The least value is when the spheres are in contact, the greatest when they are at an infinite distance, the value then being $m_2+\frac{1}{2}m'_2$.

If $u_2=0$, then k^2 must lie between the greatest and least values of $\frac{(A_1-B)^2}{A_1}$. Now

$$\frac{(A_2-B)^2}{A_2}\gtrless\frac{(A_1-B)^2}{A_1}$$

as

$$(A_1A_2-B^2)(A_2-A_1)\gtrless 0$$

as

$$A_2\gtrless A_1$$

If we suppose $a>b$ then $A_1>A_2$, and calling k_1^2, k_2^2 the least values of the above limits $k_1<k$.

Hence if

$$\sqrt{\frac{d^2}{2T}}<k_1 \text{ or } >m_1+\tfrac{1}{2}m'_1$$

the spheres can neither ever come to rest; if

$$\sqrt{\frac{d^2}{2T}} < k_2 \text{ or } > m_2 + \tfrac{1}{2}m'_2$$

the small sphere can never come to rest.

The effect of the fluid on vibratory motions.

18. Suppose each of the two spheres attracted to a fixed centre of force where the force varies as the distance. Let x_1, x_2 be the distances of the spheres at any time from their respective centres of force measured in the same direction. Then

$$2T = A_1 u_1^2 + A_2 u_2^2 - 2Bu_1 u_2 = C - m_1\mu_1 x_1^2 - m_2\mu_2 x_2^2$$

Also since we neglect squares of small quantities in finding the small vibrations, the equations of motion become

$$A_1\ddot{x}_1 - B\ddot{x}_2 = -m_1\mu_1 x_1$$
$$-B\ddot{x}_1 + A_2\ddot{x}_2 = -m_2\mu_2 x_2$$

and we suppose the spheres so distant, and their motions so small, that we may neglect the small changes in A, B during the motion. The spheres must not be too close, for at contact $\frac{dA_1}{dr}$, &c., are infinite, as was shown in § 14.

Solving the above equations in the usual manner we find

$$x_1 = L_1 \sin(K_1 t + \alpha) + N_1 \sin(K_2 t + \beta)$$
$$x_2 = eL_1 \sin(K_1 t + \alpha) + e'N_1 \sin(K_2 t + \beta)$$

where

$$\left.\begin{matrix} K_1^2 \\ K_2^2 \end{matrix}\right| = \frac{A_1 m_2\mu_2 + A_2\mu_1 m_1 \pm \sqrt{\{(A_1 m_2\mu_2 - A_2 m_1\mu_1)^2 + 4m_1 m_2\mu_1\mu_2 B^2\}}}{2(A_1A_2 - B^2)}$$

$$e = \frac{A_1K_1^2 - m_1\mu_1}{BK_1^2} = \frac{BK_1^2}{A_2K_1^2 - m_2\mu_2} \qquad e' = \frac{A_1K_2^2 - m_1\mu_1}{BK_2^2} = \frac{BK_2^2}{A_2K_2^2 - m_2\mu_2}$$

From this we see that, to the first order of small quantities, the mean position of the spheres is not altered, or to that degree of approximation there is no mean attraction or repulsion.

If we regard the spheres as two pendulums swinging in the fluid, in the same horizontal line, of lengths l_1, l_2, then the motion is given by the above equations if we write

$$\mu_1=\frac{\rho_1-1}{\rho_1}\frac{g}{l_1}\quad \mu_2=\frac{\rho_2-1}{\rho_2}.\frac{g}{l_2}$$

where ρ_1, ρ_2 are the densities of the spheres compared to the fluid.

If in the above we make $m_1=\infty$ we get the case of a forced vibration of period $\frac{2\pi}{\sqrt{\mu_1}}$. In this case

$$N_1=0\quad K_1^2=\mu_1\quad K_2^2=\frac{m_2\mu_2}{A_2}$$

$$x_1=L\sin(\sqrt{\mu_1}t+\alpha)$$

$$x_2=\frac{B\mu_1}{A_2\mu_1-m_2\mu_2}L\sin(\sqrt{\mu_1}t+\alpha)+N\sin\left(\sqrt{\frac{m_2\mu_2}{A_2}}t+\beta\right)$$

If the sphere (b) is set free when (a) is for the moment at rest, and the time be reckoned from this moment

$$x_2=eL\left(\cos\sqrt{\mu_1}t-\cos\sqrt{\frac{\mu_2 m_2}{A_2}}t\right)$$

and the motion of (b) consists of two periodic terms whose amplitude is e times that of (a).

Let now the strength of the centre of force on (b) diminish indefinitely. Then

$$x_2=\frac{B}{A_2}L(\cos\sqrt{\mu_1}t-1)$$

and (b) would oscillate in the same period as (a), without being attracted or repelled towards it except by forces depending on the square of the amplitude of (a). To find, then, whether the action of (a) on (b) is attractive or repulsive we must take account of quantities of the second order of small quantities.

The full equation of motion of (b) is

$$A_2\dot{u}_2-B\dot{u}_1-(\tfrac{1}{2}u_2^2-u_1u_2)\frac{dA_2}{dr}+\tfrac{1}{2}\frac{d}{dr}(A_1-2B)u_1^2=0$$

For a first approximation we have

$$x_2=\frac{B}{A_2}L(\cos\sqrt{\mu_1}t-1)$$

$$u_2=-\frac{BL}{A_2}\sqrt{\mu_1}\sin\sqrt{\mu_1}t$$

Write

$$x_2=\frac{B}{A_2}L(\cos\sqrt{\mu_1}t-1)+z$$

where z is of the order L^2 at least. Substituting for z and neglecting cubes and higher powers of L,

$$A_2\ddot{z}-\frac{dA_2}{dr}dr.\frac{BL\mu_1}{A_2}\cos\sqrt{\mu_1}t+\frac{dB}{dr}dr L\mu_1\cos\sqrt{\mu_1}t$$

$$-\tfrac{1}{2}L^2\mu_1\left(\frac{B^2}{A_2^2}-2\frac{B}{A_2}\right)\frac{dA_2}{dr}\sin^2\sqrt{\mu_1}t.+\tfrac{1}{2}L^2\mu_1\frac{d}{dr}(A_1-2B)\sin^2\sqrt{\mu_1}t=0$$

and

$$dr=x_1-x_2=L\left(1-\frac{B}{A_2}\right)\cos\sqrt{\mu_1}t+\frac{BL}{A_2}$$

Whence the equation takes the form

$$A_2\ddot{z}=f+g\cos\sqrt{\mu_1}t+h\cos 2\sqrt{\mu_1}t$$

where

$$\frac{2f}{L^2\mu_1}=\left(1-\frac{B}{A_2}\right)\left(\frac{B}{A_2}\frac{dA_2}{dr}-\frac{dB}{dr}\right)+\tfrac{1}{2}\frac{B}{A_2}\left(\frac{B}{A_2}-2\right)\frac{dA_2}{dr}-\tfrac{1}{2}\frac{d}{dr}(A_1-2B)$$

$$=-\tfrac{1}{2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

in which last form we may neglect in A_1 the $m_1+\frac{1}{2}m'_1$ as it disappears in the differentiation. Hence the mean action of (a) on (b) is an acceleration towards (a)

$$=-\frac{L^2\mu_1}{4A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

$$=-\left(\frac{\pi L}{T}\right)^2\frac{1}{A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

$$=-\frac{v^2}{A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

if v is the "velocity of mean square" of (a).

If the distance of the spheres is so large that we may neglect twelfth and higher inverse powers of the distance, we need only consider the *first images* or the first terms in A and B. In this case it will be found that the acceleration to (a) is

$$=\frac{18v^2}{2\rho+1}\left(\frac{a}{r}\right)^6\left\{\frac{r^7}{(r^2-b^2)^4}-\frac{3}{2\rho+1}\cdot\frac{1}{r}\right\} \quad \text{. . . . . (23)}$$

To find when there is repulsion

$$\frac{r^8}{(r^2-b^2)^4}<\frac{3}{2\rho+1}$$

or

$$r > \frac{b}{\left\{1-\sqrt[4]{\frac{2\rho+1}{3}}\right\}}$$

which can clearly only happen if $2\rho+1<3$ or the density of the sphere less than the fluid.

In general, then, when the body is denser than the fluid it is attracted. If its density is less than the fluid there will be a critical point (as mentioned by Sir W. THOMSON), beyond which there will be repulsion, and within which it is attractive. This critical distance is given by

$$r = \frac{b}{\sqrt{\left\{1-\sqrt[4]{\frac{2\rho+1}{3}}\right\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

in using which it must be remembered that if r comes out nearly equal to b, the formula fails to give a correct value, as it was obtained on the supposition that the distances were large. It is, however, extremely accurate if we remember that it is true up to inverse powers of the twelfth at least. If the density of the sphere be ·9 the critical distance would be 7·648 times its radius. It may be noticed that while the principal term in the acceleration depends on r^{-7}, if the density be the same as the fluid it depends on r^{-9}.

In the case of a sphere vibrating within another sphere, along the line of centres, the effect of the fluid will be represented by supposing the inertia of the sphere increased by a mass

$$=\tfrac{1}{2}\{1+3Q(q.q_1)\times \text{ mass of fluid displaced by it}$$

where Q has the value given in § 9: provided it is not close to the boundary of the containing sphere, as in that case $\frac{dQ}{dr}$ becomes infinite, and the small motions of the sphere will produce great changes in the value of Q. When its mean position is the centre, $\frac{dQ}{dr}=0$ and Q may be considered constant when we neglect in our equations of motion cubes of small quantities. The value of Q in this case is, as has been already mentioned,

$$\tfrac{1}{2}\cdot\frac{b^3+2a^3}{b^3-a^3}\times \text{ mass of fluid displaced}$$

The foregoing serves to solve the problem of a ball pendulum within a spherical envelope when it is so suspended that its centre lies in the horizontal line through the centre of the envelope. When it oscillates in any other position the value of the coefficient of inertia may be approximated to as in §§ 15, 16.

19. If instead of supposing the sphere (b) free to move we suppose it held fast, and require to find the force necessary to do so, we get a different result from the foregoing. Suppose the sphere (a) moving in any manner, the sphere (b) being for the moment at rest, and suppose a constant force F acting on b.

The equation of motion for B is

$$A\ddot{x}_2-B\ddot{x}_1+\tfrac{1}{2}u_2(2u_1-u_2)\frac{dA_2}{dr}+\tfrac{1}{2}u_1^2\frac{d}{dr}(A_1-2B)=F$$

Suppose now that F is of such a magnitude that u_2 being zero it makes $\ddot{x}_2$ also zero. Then F is the force required to keep (b) at rest at the moment when the motion of (a) is given by u_1, $\ddot{x}_1$. Hence

$$F=-B\ddot{x}_1+\tfrac{1}{2}u_1^2\frac{d}{dr}(A_1-2B)$$

Let $x_1=L\sin Kt$, L being small. Then neglecting cubes of small quantities

$$F=\left(B+\frac{dB}{dr}dr\right)LK^2\sin Kt+\tfrac{1}{2}\frac{d}{dr}(A_1-2B)L^2K^2\cos^2 Kt$$

and

$$dr=x_1=L\sin Kt$$

$$\therefore F=BLK^2\sin Kt+\tfrac{1}{2}L^2K^2\left\{\frac{dB}{dr}+\tfrac{1}{2}\frac{d}{dr}(A_1-2B)\right\}+\lambda\cos 2Kt$$

This is the force at the time t necessary to keep (b) at rest. Hence the mean force is a force $=\tfrac{1}{4}L^2K^2\frac{dA_1}{dr}$ towards (a), which is equal and opposite to the force of (a) on (b). Since $\frac{dA_1}{dr}$ is negative, the action is an attractive one

$$=-\tfrac{1}{4}L^2K^2\frac{dA_1}{dr}$$

$$=-v^2\frac{dA_1}{dr}$$

Taking for A_1 only the first term of Q, which is equivalent to neglecting twelfth and higher inverse powers of r

$$A_1=m_1+\tfrac{1}{2}m'_1\left\{1+3\left(\frac{ab}{r^2-b^2}\right)^3\right\}$$

and the force

$$=9m'_1v^2.\frac{a^3b^3r}{(r^2-b^2)^4}$$

$$=9\frac{v^2}{g}.\frac{a^3b^3r}{(r^2-b^2)^4}\times\text{ weight of fluid displaced by } a \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

For example, for equal spheres at a distance $4a$ (distance between their surfaces $=2a$), the mean square of velocity of (a) being the same as for oxygen at a tempe-

rature of 0° C., $v=1524$ feet per 1″, and the force $=\frac{51\cdot6}{a}\times$ weight of fluid displaced, a being measured in feet. It is clear that while the force decreases indefinitely with the size, the "effective" force increases indefinitely.

If (a) vibrate through a distance $\frac{1}{10}$-inch, 256 times a second, and $a=\frac{1}{4}$-inch, the force is ·01197 weight of water displaced = weight of 12·8 milligrammes.

In the same manner can be found the action of (a) on (b) when a describes any small curve whose plane contains (b).

VALUES of Q, Q′ for equal spheres.

	q_2.	Q.	$\frac{1}{a^3}$ Q′.
1	1	·051800	·150257
1·05	·72985	·028307	·116749
1·1	·6418	·018768	·098312
1·2	·5367	·009531	·073754
1·35	·4431	·004049	·0511
1·5	·3819	·001959	·037142
1·75	·3138	·0007023	·023335
2	·2679	·0002962	·015631
2·5	·2087	·0000723	·008001
3·5	·1459	·0000090	·002895
4·5	·1125	·0000019	·001373

VALUES of Q_1, Q_2, Q′ when $a=2b$ for external spheres.

$\frac{r}{a+b}$	Q_1.	Q_2.	$\frac{1}{a^3}$ Q′.	$\frac{1}{b^3}$ Q′.
1	·0206	·0945	·04452	·35616
1·05	·01228	·04298	·03393	·27144
1·1	·00862	·02572	·02884	·23072
1·25	·003653	·00886	·01918	·15344
1·5	·000719	·00119	·01090	·08720
2	·000186	·00024	·00455	·03643
3	..	..	·000016	·00013

VALUES of Q when $b=2a$ for an internal sphere.

$\frac{r}{a}$.	Q.
0	·142870
·25	·15106
·5	·18046
·75	·25676
1	·61645

XIV. *On the Organization of the Fossil Plants of the Coal-Measures.*—Part X. *Including an Examination of the supposed Radiolarians of the Carboniferous Rocks.*

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[Plates 14–21.]

In 1865 my friend Mr. Edward Wunsch, of Glasgow, made the discovery of some thin carboniferous shales imbedded in volcanic ash at Laggan Bay, in Arran. These beds have already been described by their discoverer,* and their fossil contents referred to by Mr. Binney, Mr. Carruthers, and Sir Charles Lyell. From within a very limited area the bases of more than 13 large erect stems of carboniferous trees have been extracted by Mr. Wunsch, the most important of which he has kindly placed in my hands. In the summer of 1877 we conjointly superintended some quarrymen, who tore up large portions of these strata with the result, I believe, of obtaining a fair knowledge of the nature of these beds and their contents.

The trees certainly stood where they originally grew; most of them consisted of a thin cylinder of the outer bark, which was deeply *fissured* longitudinally but exhibited no true Sigillarian flutings or traces of leaf-scars. The interior was in most cases filled with volcanic ash, but in a few instances by vegetable débris introduced from without; and in one specimen, imbedded in the vegetable mass, are several decorticated Diploxyloid vascular axes of very old stems. These have been referred to as young growths that sprang up within the bark-cylinder;† but such is not the case. Each one is not only decorticated, but is large enough to be the vascular axis of the large tree within which the entire group occurs, and where they are mixed up with fragments of the similar vascular axes of *Stigmaria* and other plants.

The primary question which we endeavoured to determine was the botanical character of these stems, as indicated by the remains of their bark and by the nature of the numerous fragments of twigs, branches, and fruits found in the overlying beds. No one of the trees afforded any evidence of being Sigillarian. The outer surface of each stem exhibited a rough and irregular longitudinal fluting, but this was very different from that characterising a Sigillarian bark; the ridges and furrows were

* 'Geol. Magazine,' 1865, p. 474; 'Trans. Geol. Soc., Glasgow,' vol. ii., p. 98, 1865.

† Lyell's 'Student's Elements of Geology,' second edition, p. 547, 1874.

merely results of the expanding growth of the inner portions of the stem fissuring the outer bark, as is the case with exogenous stems.

I may here express my conviction that many of the trees that have been loosely designated Sigillarian have no claim to be regarded as such. Dr. DAWSON and other palæontologists have recorded the fact that the characteristic leaf-scars and other superficial features of the younger stems and branches disappear towards the base of the older trunks; and this has evidently been the case with the Arran specimens.

Our prolonged researches failed to supply the smallest portion of a true *Sigillaria*. I obtained one fragment of a cast of the outer surface of a bark exhibiting what at the first glance had a Sigillarian appearance. It exhibits strongly-marked ridges, but these proved to be but casts of the unsymmetrical longitudinal fissures just referred to. That such is the case is shown by the positions of the oblong leaf-scars—which are wholly independent of those of the ridges—which could not have been the case in a *Sigillaria*. These leaf-scars are of the Lepidodendroid type.

The most numerous fragments which we met with were long, slender Lepidodendroid twigs, densely clothed with very short, scaly leaves, such twigs being usually about half an inch in diameter; fragments of larger branches were not uncommon, some of these being from two to three inches in diameter, and which were also true *Lepidodendra*, retaining the closely-united, rhomboidal bases of their leaves in union with the outer bark, each rhomboid having a diameter of a quarter of an inch. Mr. BINNEY has figured and described some fruits found by Mr. WUNSCH, all of which are true *Lepidostrobi* furnished with macrospores and microspores; and I am indebted to Dr. YOUNG and to Mr. JOHN YOUNG, of the Glasgow University, for their permission to make a section of a similar *Lepidostrobus* from Laggan Bay, but which unfortunately contained no spores. We have thus a mass of evidence of a positive kind indicating that these stems were true *Lepidodendra* and not *Sigillariæ*—a conclusion which the negative testimony afforded by the entire absence of true Sigillarian fragments from these beds equally sustains. Most of these specimens apparently belonged to one species; only the one fragment of a cast of the bark already referred to appears to have been distinct from the rest.

Fig. 1 is a transverse section of one of the small twigs that abound in the deposit in such numbers as to leave no room for doubting that they are the ultimate branches of the large stems. These twigs are long and slender, generally about half an inch in extreme diameter, including the closely-compressed scale-like leaves. In the centre of the section is the principal vascular bundle, *a*, of which a more enlarged representation is given in fig. 2; it is composed entirely of barred vessels, without any visible intermixture of cellular tissue. The vessels are pretty uniform in size, the largest being about ·005 in diameter*; but at the periphery of the bundle we find a limited number of much smaller vessels. It is from these latter that are derived the numerous foliar vascular bundles (fig. 2, *b*) that cluster round the main axis. The inner portions of

* These measurements are recorded, as in my previous memoirs, in the decimal parts of an inch.

the bark have been destroyed in every one of these twigs that I have examined, but the leaves (fig. 1, *c*) are all in their normal positions, varying in shape according to the portion of the leaf intersected by the section; they present the usual structure of Lepidodendroid leaves, consisting of cellular parenchyma and with the single vascular bundle (fig. 1, *c′*) running through it. The distorted outlines of the sections of these leaves have most probably been due to desiccation; they are chiefly found imbedded in the volcanic ash, which probably fell in a heated condition, producing the shrivelled condition in which most of these leaves are found.

On comparing this section with the similar one of the Burntisland *Lepidodendron*, represented in Plate 41, figs. 2 and 3, of my third memoir, the difference between the two specimens will be seen at a glance. In the Burntisland plant the bundle has a diameter of about ·015. In the Arran specimen it is about ·033. In the former, there is already a small irregular central medullary space from which vessels are absent—nothing of the kind exists in the latter plant. In it the bundle is more compact, the vessels more equal in size, and the structure altogether more symmetrical and robust than in the Fifeshire plant. Fig. 3 represents a section of a branch of larger dimensions; such branches vary from one-and-a-half to three or more inches in diameter. Next to the twigs these are the most common fragments found in the deposit. The bases of the leaves still adhere to the bark, but when these are removed, as they are in a part where the specimen from which the section, fig. 3, had been a little water-worn, they present the aspect of such Lepidodendroids as *L. nothum* of UNGER, *Halonia gracilis*, &c. The central vascular bundle has now expanded into a vascular cylinder, fig. 3, *a*, enclosing a cellular medulla, *d*. The structure of this cylinder is further illustrated in fig. 4, which is enlarged 15 diameters. The cells of the medulla are much disorganized,* but at the junction of this tissue with the inner surface of the vascular cylinder I observe that a few of them are strongly barred—as I have shown to be also the case with many of the medullary cells of *Lepidodendron Selaginoides*.† The diameter of the pith is about ·1, the mean diameter of the vascular cylinder enclosing it being about ·2. Each transverse section of this vascular cylinder has a crenulated outline exactly corresponding with that seen in so many of the Diploxyloid stems described as *Sigillariæ*. The mean size of the vessels composing the cylinder is about ·01. The projecting points of the periphery, fig. 4, *a′*, consist of very small vessels, elongated radially in the sections; these are the orienting numerous foliar vascular bundles, *c*, which evidently detach themselves from the cylinder at first obliquely and then ascend more vertically when passing through the inner bark. Their transverse sections in the latter position, as seen at *c*, are circular. All these vessels are barred.

Returning to the section, fig. 3, we now find, beautifully preserved, much of the middle bark, *e*. It is a delicate parenchyma, the cells of which are very uniform in

* Vertical sections show that these medullary cells are arranged in perpendicular columns as is the case with so many of these Carboniferous Lepidodendroids. See memoir, Part II., Plate 25, fig. 14.

† Memoir, Part II., Tab. 24, fig. 3.

arrangement and size, the latter ranging between ·005 and ·0025. Numerous vascular bundles are seen passing outwards to the leaves, as at *b*. The more external portions of the bark have nearly all disappeared. The little that remains, *f*, binding together the leaves, *c*, *c′*, shows that it possessed the bast-layer, composed of vertically elongated prosenchymatous cells, with an outer investment of parenchyma—the usual structure of this portion of Lepidodendroid and Sigillarian stems. The leaves in this section exhibit nothing worth remarking except that the three *c*, *c*, *c*, have been intersected through their larger transverse diameters, whilst the two alternating ones, *c′*, *c′*, have been crossed close to the point where their bases spring from the bark. Their closely united rhomboidal bases have a diameter of about $\frac{3}{16}$ths of an inch.

The next progressive step brings us to the specimen represented by fig. 5, which is a section of either a small stem or a very large branch found by Mr. WUNSCH. It is represented three-fourths its natural size. We now find the central medulla, *d*, about an inch in diameter, but its cellular tissues have disappeared through decay or mineralisation. At *a* we have the vascular medullary cylinder corresponding exactly, in all but size, with that seen in the specimen, fig. 3. But we now find externally to this cylinder a very thin layer, *i*, of barred vessels arranged in radiating lines. This zone is, of course, the "Cylindre ligneux" of BRONGNIART, and my exogenous zone, the existence of which, according to the French author, *de facto*, transfers the plant possessing it from the Lepidodendroid to the Sigillarian group—or, in other words, from the Cryptogamic to the Gymnospermous divisions of the vegetable world. At *e* we again have the delicate middle parenchyma seen at fig. 3, *e*, but often disturbed by a stalagmitic arrangement of the carbonate of lime, as at *e′*. This inner tissue is less perfectly preserved than in the specimen, fig. 3, but sufficiently so to establish the perfect identity of the two structures. We pass by a somewhat rapid transition from the delicate parenchyma of this middle bark to the much coarser parenchyma of what, in my previous memoir,* I have spoken of as the middle bark, fig. 5, *g*; and which yet more externally, fig. 5, *h*, passes gradually into the radiating lines of prosenchymatous cells of what may be designated the bast-layer. In these two outer portions the foliar bundles are numerous and distinct. Now if any unbiassed student compares the arrangement of these tissues, as shown in this fig. 5, with those seen in many of my previous figures of *Lepidodendra* and *Haloniæ* (*e.g.*, Plate 24, fig. 1, and Plate 26, fig. 13, of memoir, Part II.), and then further compares these with BRONGNIART's own figures of his typical Lepidodendron, as represented in Plate 20, figs. 2, 3, and 4, of vol. 2 of his 'Vegetaux Fossiles,' it will be seen at a glance how complete is the identity of these detailed organizations, and what violence is done in separating them widely asunder merely because nature met additional nutrient wants by an addition to the nutrient machinery. In fig. 5 of this memoir we see the beginnings of this process, which in the larger stems underwent a much further extension. Fig. 6 represents a transverse section of one of the vascular axes already referred

* See memoir, Part II., Plate 24, fig. 1, *h*.

to three-fourths the natural size. At *d* we have the medulla; the inner, non-exogenous ring is seen at *a*; and at *i* we have the, now very much thickened, vascular exogenous zone. It is unnecessary to enter into a detailed description of these structures, since they correspond in the closest manner with similar ones described in my previous memoirs—especially with those of the Burntisland plants. I would only remark that we notice in their inner or medullary vascular cylinder the same remarkable proof of the increase which has taken place in the number of the medullary vessels, *a*, as the stem increased in size, as I have already referred to in previous memoirs. I can only conclude that these additions to this vascular zone must have been made by a conversion into vessels of the cells of the central or medullary parenchyma by a centripetal process of development. If this has been the case, this centripetal growth of the medullary vascular ring affords another feature of resemblance to the vascular bundles of the living Lycopods, of which I believe it to be the true homologue. Fig. 6A represents a portion of a photograph of a section similar, but further enlarged, to fig. 6, for which I am indebted to Mr. WUNSCH. It shows the exact relations of size and number of vessels which the medullary and exogenous zones bear to each other. All the vessels in these stems exhibit with great distinctness the innumerable vertical parallel lines of lignine (fig. 4*, *b*), connecting the transverse bars (fig. 4*, *a*) which I described and figured in my memoir, "On the Structure and Affinities of some Exogenous Stems from the Coal Measures," ('Monthly Micros. Journal,' Aug. 1, 1869, p. 71, plate 20, fig. 10). Before examining the condition in which the bark of the large Arran stems is found, we may advantageously recall some observations made by Dr. DAWSON, of Montreal. In his 'Report on the Fossil Plants of the Lower Carboniferous and Millstone Grit Formations of Canada,' that observer remarks: "In older stems (of Lepidodendron) three modes of growth are observed. In some species the expansion of the bark obliterates the leaf-bases and causes the leaf-scars to appear separated by wide spaces of more or less wrinkled bark, which at length becomes longitudinally furrowed and simulates the ribbed character of *Sigillaria*."* This description accurately represents the condition of the exteriors of the larger Arran stems. All that remains of their bark is a cylinder, usually about two inches thick, which consists almost entirely of the prosenchymatous or bast-layer—the cells of which are always seen, in transverse sections, arranged in radiating lines, and which is interposed in younger stems between the coarse outer-bark parenchyma, fig. 5, *g*,—and the somewhat similar outermost or sub-epidermal parenchyma. As I have already pointed out, the surface parenchyma has entirely disappeared from these Arran stems. Since the true leaf-bases and their subjacent leaf-scars belong solely to the outer surface of this parenchyma, it follows that when it was thrown off through the pressure of internal growth, combined with atmospheric agencies, all definite traces of the outlines of these leaf-scars would disappear. Such a change brings the bast-layer to the surface where it forms a protective or "healing" tissue corresponding teleologically to the corky layer of ordinary exogens.

* *Loc. cit.*, p. 41.

This has been the case with our Arran plants, in most of which this bast-layer has increased with the general growth of the stem to a thickness of fully two inches. As in the stems referred to by Dr. DAWSON, deep longitudinal furrows exist at its outer surface,—but these furrows are, as I have already shown, perfectly distinct from those of the true *Sigillariæ*. They are but the fissures occasioned by the increase in the diameter of the stem, due to the growth of its more internal tissues. One fact is rendered further obvious from these specimens, viz., that this bast tissue increases in thickness along with the increase in the diameter of the other tissues forming the stem, but my specimens throw no additional light upon the physiological question discussed in my previous memoir (Part IX., p. 355), viz.: Whether the additions to the thickness of this bast layer are due to a plane of genetic activity on its external or its internal surface. Tangential sections of this bast layer exhibit no peculiarities in the arrangement of its prosenchymatous cells.

I have already stated that I have obtained one cast or impression of what appears to have been the outer surface of the bark of one of these large stems. In it the leaf-scars are fairly preserved. They are elongated vertically, as shown in fig. 7, which represents the form of one of these scars three-fourths the natural size. There is a rounded central depression, *a*, *in the cast*, marking a corresponding elevation in the original bark, above and below which the cicatrix is prolonged upwards and downwards into a somewhat acuminate form; a slightly elevated sharp ridge, *b*, in the cast, marking a corresponding vertical groove in the original cicatrix. These cicatrices are arranged in the usual oblique rows seen in *Lepidodendron*, the centre of each scar being about five-eighths of an inch apart from its next diagonally-disposed neighbour. The rhomboidal and closely contiguous leaf-bases seen in the smaller branches are here separated widely apart from each other, besides being enlarged longitudinally. This fragment, however, may belong to a species distinct from the more abundant Arran examples.

As already stated, the only fruits that have hitherto been found in the Arran beds are true *Lepidostrobi*. Of these Mr. BINNEY has figured three* which he regards as distinct species, and all of which obviously contained, when living, both macrospores and microspores. In two of the specimens both kinds of spores are preserved, and in the third, which is only the basal part of the cone, the macrospores which it displays imply that microspores occupied its apex. The specimen already referred to, for which I am indebted to Professor YOUNG and Mr. JOHN YOUNG, of Glasgow, is a true *Lepidostrobus*, but all its sporangia are empty. Of such fruits as *Trigonocarpum* and other forms, such as large and matured Gymnospermous trees might be expected to produce, not a trace has been discovered. We are thus led to the conclusion that these Arran trees are essentially Lepidodendroid. The large collection of specimens accumulated by Mr. WUNSCH during a series of years was examined with the utmost care, to see if

* 'Observations on the Structure of some Fossil Plants found in the Carboniferous Strata,' Part II., plate xi. (Palæontographical Society.)

the smallest Sigillarian fragment could be discovered in it, but in vain. I found *Lepidodendra* in abundance of various sizes, and *Stigmariæ* in sufficient numbers; but no one specimen existed indicating the possibility that these trees might have been *Sigillariæ*.

Ulodendron.

In my second memoir (Phil. Trans., 1872) I described the only specimen of *Ulodendron* hitherto met with in which the organization was preserved. I have received a second specimen from Mr. D'ARCY W. THOMPSON, of Edinburgh, which, though less perfect, is confirmatory of the truly Lepidodendroid character of these stems, and their conformity to the type of *Lepidodendron Harcourtii*. In that memoir, in speaking of the bilateral series of scars which characterise the genus, I observed: "It seems probable that these scars sustained objects which were chiefly developed from the epidermal layer, and whose bases rested upon the outer bark; they certainly were not roots or branches, and I incline to the belief that they were organs of fructification" (*loc. cit.*, p. 210). My young friend Mr. THOMPSON has recently collected a fine series of these *Ulodendra* from the Edinburgh coalfield, some of which conclusively settle this question. I am indebted to him for a fine specimen, in which the branch bifurcates in the usual Lepidodendroid manner. The large characteristic scars are seen, as usual, on both the inner and outer surfaces of these branches. The inner series is curious, since one of them is located at the exact point of bifurcation of the branch—a position from which no ærial root could possibly be given off. The scars average about 0·9 in diameter. Mr. THOMPSON has also obtained two branches, each of which exhibits the usual rows of scars, *but in each specimen one scar supports an actual cone in situ.* The actual area of attachment of each cone to its branch is a very circumscribed one, being wholly different from that of a secondary to a primary branch. On the other hand, the diameter of the entire base of the cone is very considerable—corresponding, in fact, with that of the peripheral margin of each characteristic Ulodendroid scar. It is now easy to understand these scars. We constantly find the greater part of their surfaces covered with the modified bases of leaves. The cone obviously originated as a small lateral bud, but its development into a branch was arrested, leading to the formation of a strobilus. As this expanded, though the actual attachment of its central axis to its parent branch was a very circumscribed one, the entire base of the cone became broad, and as a result of its active growth it pressed down the leaves of the branch over an area corresponding to the diameter of that base. The very large scars seen on many specimens bear no relation to the size of the cone, but have obviously increased in size through the continued growth of the stem after the deciduous strobili had fallen.

Spores.

In my last memoir (Phil. Trans., Part I., 1878, Plate 23, figs. 59–64) I figured a series of remarkable macrospores, which occur abundantly in some portions of the

Halifax beds, associated with a still greater profusion of microspores, which latter I concluded belonged to the same species of strobilus as the macrospores. Since that memoir was written we have found the strobilus in a section made for me by Mr. EARNSHAW, of Oldham. We had looked for a strobilus of considerable size, like those common in the carboniferous strata, whereas it is remarkable for its small dimensions; but that it has been fully grown is shown by the perfect development of its macrospores. The specimen is much crushed, as is seen in fig. 8, which represents it enlarged nearly 12 diameters. At *a* we find the usual central vasculo-cellular axis. At *b*, *b* are clusters of macrospores, several of them being enclosed in their sporangial envelopes. The upper part of the strobilus, *c*, has been cut off obliquely, owing to some accidental curvature in its form; but the entire absence of macrospores from the portion of it which remains shows the distinctiveness of its character from the lower macrospore-bearing part. At *d* we have two sporangia full of microspores, possibly disturbed by the pressure from their original positions.* The surrounding matrix is full of microspores. The entire length of this section is ·33. The strobilus was doubtless a little longer when perfect at its upper extremity; but when every allowance is made for the missing portion it remains a diminutive organism, reminding us more of the dwarfed fruits of many living *Selaginellæ* than of the larger *Lepidostrobi* which are so abundant in the upper Coal-measures.

In Plate 23, figs. 64 and 64*, of my last memoir I figured some anomalous peduncular organs connected with the macrospores of this strobilus, which were wholly inexplicable to me when that memoir was written. Further research has cleared up the mystery. Fig. 9 represents a cluster of four macrospores, which demonstrate that these pedunculate appendages are but collapsed portions of the spore-wall, due probably to the destruction of their contained protoplasm, and consequent arrested growth. This is especially obvious in the two macrospores, *a* and *b*. Fig. 10 represents a tangential section of a detached bract of a different *Lepidostrobus*, with its sporangium on its upper surface filled with microspores. The slightly-distorted bract is seen at *a*, composed of strongly-defined cells at its upper portion, *a′*, and of a more delicate parenchyma inferiorly. In this strongly-marked differentiation of the upper and lower tissues of the bract the strobilus differs from the Burntisland one.† The structure of the sporangium-wall, *b*, is identical with that of the similar one surrounding the macrospores.‡ The mode of attachment of this cellular investment to the

* At the same time it must not be forgotten that whilst in the greater number of the *Lycopodiaceæ* the macrospores occupy the lower extremity of the strobilus, and the microspores its apical portion, there are several, such as *Selaginella Martensii*, in which the macro- and micro-sporangia are intermingled without any regularity in their arrangement. I have found this to be partially the case with the *Lepidostrobus* from Burntisland. In the macrospores of *S. Martensii* the exosporium is furnished with long radiating spines, which, as BURMEISTER has shown, become brittle and are often broken off. I have frequently found the similar appendages of the Halifax macrospores broken into detached fragments.

† Memoir, Part III., Plate 44, fig. 25.

‡ Memoir, Part IX., Plate 23, fig. 64.

bract shows it to be, as is also the case in the Burntisland species, an extension of the epidermal layer of the bract. But in this new species I find what I have not previously observed in these sporangia: an apparently distinct inner membrane, *c*, investing the mass of microspores. The inferior keel, *d*, of the bract is also much shorter than the elongated one of the Burntisland plant. The entire maximum diameter of the sporangium, including its bract, is only ·05, corresponding in this respect with the proportions of the crushed strobilus, fig. 8.

The relations between the structure of these primeval spores and sporangia and that of recent ones are not devoid of interest. In my last memoir I showed that each macrospore consisted of a very thick dark-coloured outer layer, and a very thin inner one. I think I cannot err in regarding the former as the representative of the exosporium and the latter as the endosporium of the spores of living Lycopods. The macrospore of the recent *Selaginella Martensii* possesses a similar thick dark-brown exosporium. I have already identified the cells contained within the endosporium of the fossil species with the endospermic cells of the recent ones. I have searched, but thus far in vain, for any representative of the prothallus which many living macrospore develop in addition to the endospore. The structure of the sporangium also requires to be observed. In most living Lycopods the sporangium-wall possesses, in its early state, three distinct cellular layers—an outermost epidermal one, which is merely the uplifted epiderm of the parent plant, and two subjacent layers of chlorophyll-bearing cells. As I have already explained, the wall of the fossil sporangium, fig. 10, consists of *two* layers. The outer one, *b*, is obviously identical with the epidermal layer of living forms. This is made clear equally by its structure and by the way in which it terminates inferiorly on the surface of the bract, *a*. The inner membrane, *c*, *appears* to be structureless; but when we remember that in recent sporangia the three layers eventually become reduced to two as the spores ripen, through changes that affect the two inner or chlorophyll-bearing layers solely, I shall probably not be far wrong in regarding the layer, fig. 10, *c*, as the representative of these layers.

In Plate 22, figs. 38–57, of memoir IX. I gave representations of a new strobilus with spores of one kind and of a peculiar character; but in the only specimens of that strobilus which I then possessed the structure of the central vascular axis of this fruit was very imperfectly represented by a crushed mass of vessels. I have more recently received from Mr. SPENCER, of Halifax, transverse sections of a second specimen of this strobilus, in which the central structures of the axis are well preserved. The *outer* bark (fig. 11, *d*) in these sections corresponds with that of fig. 53, *b*, of the previous memoir, except that the cells of its inner portion (fig. 11, *d'*) are more delicate, in comparison with the outermost ones, than in the older specimen; the bracts (fig. 11, *e*), also composed of numerous much-elongated cells, are longer and do not exhibit any proof of their having been so much expanded at their free extremities in a *lateral* direction as fig. 56 of the previous memoir showed them to have been *vertically*. In

the interior of several of these bracts I observe a slender vascular bundle composed of two or three minute barred vessels.

Fig. 12 represents the cellulo-vascular axis of fig. 11; in its centre, *a*, is the cylindrical médulla, composed of parenchymatous cells. It has a maximum diameter of about ·01. Surrounding this is a vascular zone, *b*, composed of barred vessels, not arranged in radiating lines, and remarkable for the approximate uniformity of their size. This cylinder, which has a mean diameter of about ·034, exhibits several indentations, *c*, in its outer margin as if characterised by a tendency to break up into separate wedges. I am unable to say whether this is a normal feature or whether it is a result of partial dessication, but the same feature presents itself in both my sections. The individual vessels range from ·0016 to ·001 in diameter, the peripheral ones being the smallest. Surrounding this vascular axis is a sheath, *d*, of very delicate parenchymatous cells, radial prolongations of which, *d'*, *d'*, composed of similar but more elongated cells, proceed outwards. Though I have detected no vascular bundles on these cellular prolongations, I have no doubt that they accompanied such bundles in their passage outwards to reach the sporangiferous bracts. It will be observed that the specimens nowhere exhibit traces of the orientation of these bract-bundles, neither do we see any traces of transverse sections of them grouped around the vascular axis such as constitute so striking a feature of the corresponding sections of true *Lepidostrobi.* Otherwise the general features of these sections correspond with those previously figured in possessing a generally Lycopodiaceous aspect. A very large unoccupied space separates the inner bark (fig. 11, *c*) and the more robust outer cortical zone *d*, which space was doubtless originally occupied by a delicate middle parenchymatous layer. The disappearance of this layer, leaving a thin inner zone like fig. 11, *d*, surrounding the vascular axis, is a common feature of the Lepidodendroid axes both in stems and fruits. At the same time the structure of this newly-discovered vascular axis agrees with the features previously described in giving to this strobilus a very distinctive individuality. This strobilus may be named *Lepidostrobus insignis.*

Fig. 12A represents a transverse section of an object from the Halifax deposit, for which I am indebted to Mr. SPENCER; its mean diameter is about ·033. It is evidently a section of the upper extremity of either a fruit or a young shoot with pentamerous phyllomes alternating in contiguous verticils. The only plants with which we are at present acquainted having verticilate phyllomes are *Calamites Sphenophyllum, Annularia, Asterophyllites,* and their immediate allies; but I have seen nothing amongst them that corresponds with this organism. The nearest approach to it is perhaps the little fruit figured in my last memoir (Plate 25, fig. 103), but even in this case the resemblance is but remote. In the fruit *Calamostachys Binneyana* the phyllomes are also arranged in alternating verticils; but in them we find an hexamerous arrangement, *six* fertile bracts alternating with twelve sterile ones.

Since the publication of my fifth memoir I have obtained some additional illustrations of the structure of *Calamostachys Binneyana.*

Fig. 13 is a fine longitudinal section of this strobilus from Halifax, for which I am indebted to Mr. BINNS. The vascular axis is seen at *a*, composed of barred vessels at its inner portion, but at each of the three prominent angles of this bundle there is a cluster of smaller spiral vessels. These clusters mark the points from which, as I shall show directly, the vascular bundles are given off to the bracts. The cortical portion, *b*, the sterile bracts, *c*, and the fertile ones, *d*, have already been described; but the section well illustrates the oblong cells which constitute the uppermost layer of each sterile bract as distinguished from the coarser parenchyma of its inferior portion. The sporangia, some few of which contain spores, appear at *e*; but that which is of the most importance to my present purpose is the existence of some symmetrically-disposed apertures, *d′*, *d″*, in the bark, through which apertures the vascular bundles of the bracts have originally passed—those marked *d′* going to the fertile bracts and consequently being six in number in each verticil, whilst those indicated by *d″* have been double that number, and destined for the corresponding number of sterile bracts. Fig. 14 is a beautiful transverse section through the sterile bractigerous disk, enlarged 17 diameters. At *a* is the vascular axis; at *b* the *outer* cortical parenchyma, the inner layer being almost entirely wanting; at *c*, *c′* we have the tissues forming the bractigerous disk, chiefly composed of coarse cellular parenchyma, but at *c* we find the cells elongated radially,* going to the free vertical bracts already described by Mr. BINNEY, Mr. CARRUTHERS, and myself. Running along the centre of these lines of prosenchymatous cells I find in each of two or three of the bracts the bundle of small spiral vessels pursuing its way across the disk to reach the free marginal bract. At *f*, *f* we have transverse sections of the vertical portions of the bracts belonging to the next inferior verticil, whilst the extreme tips of those of the next lower verticil but one appear at *g*, *g*.

Fig. 15 is a transverse section of the *Calamostachys* made through the plane of the fertile sporangiophores. It is the only specimen I have yet seen in which the cortical structures are preserved in their entirety. The central vascular axis, *a*, is closely surrounded by a dense cellular layer, *b*; this passes into a more open and delicate cellular tissue, *b′*, in which there are large lacunæ—but which latter are probably due to partial desiccation, since they are irregularly distributed. Bounding this externally is the outermost cortical layer preserved in all specimens of this fruit, and of which the sporangiophores, *d*, are mainly prolongations. At *d′* we have part of the peripheral peltate expansion of the sporangiophore, *d*.

But the points of this section which are specially important relate to the central vascular axis with its dense cellular investment; the form of its transverse section is here unmistakably that of a triangle with each of the angles truncated at its extremity. We further see at each of *two* of these truncated angles two small vacant spaces, which, like the similar spaces, *d′*, *d″*, in the outer bark of fig. 13, doubtless

* These lines of elongated prosenchymatous cells only occupy the upper surface of the disk until they reach the upper extremities of the free bracts, *f*, *g*, which portions are chiefly composed of such cells.

transmitted vascular bundles to the sporangiophores. In the present instance the apertures are in the innermost layer of the bark, and indicate the points at which vascular bundles emerged from the central vascular axis. The third angle very distinctly exhibits *one* such lacuna, and it is easy to discover in the specimen itself the position of the second one. We have here a triquetrous axis, each truncated angle of which gave off two vascular bundles. There being but six sporangiophores, one bundle obviously went direct to each sporangiophore; but in the sterile verticils there were 12 bracts, and the specimen fig. 13 affords the evidence that in these verticils a corresponding number of 12 vascular bundles must have emerged from the outer bark, the result of dichotomous division of the six primary ones.

In my memoir, Part V., p. 65, I endeavoured to demonstrate, so far as the less perfect specimens then in my cabinet enabled me to do so, the triquetrous character of the axis of this fruit and its consequent affinity with the similar axes of *Sphenophyllum* and *Asterophyllites.* The specimens now described fully confirm the conclusions then arrived at—viz.: that *Calamostachys Binneyana* had not the slightest relationship with the *Calamites,* but that it had strong affinities with *Asterophyllites* and *Sphenophyllum.* A comparison of the positions of the symmetrically-disposed pairs of apertures in each angle of my fig. 14 with that of the vascular bundles of the transverse section of *Sphenophyllum* figured by M. RENAULT* will demonstrate how close is the resemblance between the two; and I have no doubt that a similar section made through a perfect example of one of the sterile verticils would exhibit exactly identical results with the cortical portion of the same figure—*i.e.*, it would show the division of the six bundles into 12, exactly corresponding with the number of the lines of elongated cells seen in my fig. 14. Unfortunately, the fine section represented in the latter figure has lost the cortical layer through which the inner portions of these bundles had to pass; but they are distinctly seen, as I have already pointed out, in some of the 12 radial lines of elongated cells, *c*, belonging to the disk represented in that figure.

In fig. 38 of my memoir, Part V., I represented a section of the central vascular axis of *Calamostachys Binneyana,* in which there appeared to be an exogenous growth investing the primary vascular bundle. In describing this axis I pointed out (*loc. cit.*, p. 61) that whilst its centre consisted of barred vessels grouped in the usual way, its periphery was composed of similar vessels but arranged in radiating laminæ, the inner extremities of which exhibited a decided tendency to curve inwards towards three points in the periphery of the primary vascular bundle. At the time when this specimen was described it was the only one which I had seen exhibiting this peculiar structure, and its peripheral margin being imperfect I was unable to determine what its normal outline had been. I have more recently received from Mr. SPENCER another, and fortunately more perfect, specimen of the same form of strobilus, the axis of which —surrounded by a portion of its outer cortical parenchyma—is represented in fig. 16.

* "Nouvelles recherches sur la structure de *Sphenophyllum*, et sur leurs affinités Botaniques," 'Annales des Sciences Naturelles,' 6e série, Bot., tom. 4, plate 7, fig. 3, *i*.

The central portion of the vascular axis, *a*, is composed, as previously described, of the ordinary group of barred vessels. The exogenously developed layer is produced into three prominent lobes, *b*, *b*, *b*. The component radiating laminæ of each lobe are more or less curved, their concavities being directed towards the centre of each lobe. The result of this arrangement is to exhibit in a new manner the tendency towards a triquetrous arrangement which I have already shown to be a characteristic feature of some of these strobili whenever we obtain them in anything approaching to a perfect condition.

In the sporangia of this specimen, which is from the Halifax bed, we find beautiful examples of numerous mother-cells, each of which exhibits three, but doubtless contains four daughter-cells. These latter appear to me to be the true spores since they correspond, so far as size is concerned, with most of those seen in my numerous other sections of this fruit. A second specimen from Halifax, for which I am indebted to Mr. BINNS, exhibits this condition of the spores yet more perfectly. Fig. 17 represents one of the mother-cells of this example, enlarged 400 diameters. Its three contained daughter-cells are perfect in their outlines; those in Mr. SPENCER'S section are more shrivelled and ruptured. In none have we a trace of elaters. In the slide containing Mr. BINNS' specimen is an obliquely transverse section of another strobilus of which the axis has had the same structure as that represented in fig. 16. It displays four of its sporangia, all of which are filled with parenchyma, in which no spores or daughter-cells are yet visible. The sporangium walls, fig. 18, *a*, are strongly defined, and the parenchyma, *b*, entirely fills the cavity of each sporangium, differing in this respect from most of the other examples which I have seen. This absence of daughter-cells from each of the parenchymatous cells obviously suggests that we here have the strobilus in a very young state; yet we find that, in its axis, the peripheral radiating vascular laminæ seen in fig. 16, *b*, are already present. This fact seems to show that we have at least two species of these fruits, distinguished from each other by these differences in the structure of the central axis, but identical in every other respect; both being equally characterised by a tendency on the part of the vascular axis to assume more or less distinctly the triquetrous form.

Ferns.

The Halifax beds have furnished two new forms of fern-stems or petioles. Of the first of these, of which a transverse section is represented in fig. 23A, I have received sections both from Mr. SPENCER, of Halifax, and from Mr. EARNSHAW, of Oldham. In its essential features this agrees with many of the other *Rachiopterides* which I have already described. Its chief characteristic resides in the form of its very large oval vascular bundle which consists of a dense mass of thick-walled barred vessels. Its maximum diameter is about ·11. I propose to distinguish it by the provisional name of *Rachiopteris robusta*.

For specimens of a second and more remarkable *Rachiopteride* I am indebted to both Mr. BINNS and Mr. SPENCER. They are from the Halifax beds. Fig. 19 represents a transverse section of Mr. BINNS' specimen enlarged 15 diameters, and fig. 20 represents the central portion of fig. 19, enlarged 46 diameters. The outer cortex, *a*, is identical in all respects with the corresponding tissue in the other Rachiopterides that I have described, consisting of a thick-walled cellular prosenchyma, the peripheral cells of which are much smaller than those of its inner portion. Within this is a more delicate thin-walled parenchyma, *b*, the transition from which to the investing prosenchyma is somewhat abrupt. This inner tissue is frequently wanting in these *Rachiopteride* sections, it having apparently disappeared through decay, prior to the mineralisation of these stems. The yet more internal structures are best illustrated by the enlarged fig. 20, in which *b* again represents the parenchyma of the middle cortex. Within this we find a narrow zone, of inner cortex *c*, the cells of which exhibit a decided tendency to arrange themselves in regular radiating lines. This tissue passes into a still more internal series of cells, *d*, which fill up the interspaces and round off the outlines of the vascular bundle. Between the two series of cells, *c* and *d*, a dark, ill-defined, but nevertheless distinct line of demarcation, *h*, surrounds the entire central vasculo-cellular axis. The vessels, *e*, *f*, are arranged in this plant as in the genus *Zygopteris* of authors. The central ones, *e*, are extremely large, some of them having a diameter of ·01. These are virtually arranged in two rows, which separate at each end of the bundle, where the vessels rapidly diminish in size, and recombine to form the two vascular loops, *f*, *f*, enclosing a small mass of parenchyma, *g*, *g*, located at each extremity of the double row of large vessels. Many of the vessels of these peripheral loops, *f*, are not more than ·0012 in diameter, but, like the larger ones, they are thick-walled. The tissues *d* and *g* probably represent liber structures.

All the larger vessels of this bundle are densely crowded with tylose-cells, constituting the second example which I have met with of this tissue existing in carboniferous plants. I described a previous one in the case of the fern-like *Rachiopteris corrugata* in my memoir, Part VIII.*

Fig. 21 represents the central part of a transverse section of a similar stem to fig. 19, but made so obliquely as to be almost a longitudinal one. From it we learn that all the cells *c* and *d* enclosed within the middle cortical layer *b*, *b*, excepting *g*, are long, narrow, and have square ends. We also see that the large vessels, *e*, and the small ones, *f*, are alike barred, the former being filled with tylose as in the transverse section.

Fig. 22 represents the transverse section of another stem, for which I am indebted to Mr. BINNS. It corresponds with fig. 20 in almost every essential respect, but the large vessels, *e*, exhibit no traces of tylose. The cells of the innermost portions, *c*, of the cortical layer exhibit less tendency to arrange themselves in radiating lines than they do in fig. 20, and the dark line, *h*, separating that layer from the

* Phil. Trans., Vol. 167, Part I., p. 214, Plate 6, figs. 15 and 16.

endophlœum, *d*, is still more strongly marked than in the former specimen. I cannot doubt for a moment the specific identity of figs. 20 and 22. The presence of the tylose, therefore, would seem to be an accidental phenomenon, and not a specific feature of the former specimen. As for the dark line, *h*, I think I cannot be wrong in regarding it as identical with a similar boundary-line or bundle-sheath seen in many recent ferns, and which occurs in a form closely resembling that of fig. 20 in the petioles of *Woodwardia orientalis*. So far as its central vascular bundle is concerned, this plant somewhat resembles the *Zygopteris elliptica* of M. RENAULT;* but it differs in the relative proportions of its transverse terminal arcs, which are very much longer and less robust in *E. elliptica* than in my species. The various cellular layers, *b*, *c*, *d*, and *g*, of the latter not being preserved in M. RENAULT's specimen, makes a more detailed comparison of the two plants impossible. I propose to distinguish my plant by the name of *Rachiopteris insignis*.

Fig. 23 represents a lateral bundle passing outwards through the cellular layer, *b*, of fig. 21. In its centre we obviously have the central bar, *e*, of fig. 20; but it is impossible to say which of the surrounding structures are vascular and which cellular, or to determine whether this is the bundle of a secondary petiole, or of a root, but it is most probably the former.

Conceptacles.

In my last memoir I described some remarkable reproductive structures under the generic names of *Sporocarpon* and *Oidospora*. Having obtained additional examples of these objects, I am now able to throw further light upon them, and also to add to their number. I described one of these under the name of *Sporocarpon elegans* (*loc. cit.*, p. 348, Plate 23, figs. 67, 68, 69, 69A), and a second I designated *S. compactum* (*loc. cit.*, p. 349, Plate 24, fig. 76A). Specimens recently discovered suggest the possibility that these apparently distinct species may be but different stages in the development of the same organism; the latter being the younger, and the former the more matured states. It is also possible that the minute objects which I designated *Oidosporæ*, may be *very* young forms of the same, though at present I have not sufficient evidence to prove that such is the case.

Fig. 24 is a transverse section through the centre of a very perfect specimen of the *Sporocarpon elegans*. Its radially disposed, hour-glass-shaped cells, *a*, are arranged in the most symmetrical manner. Some of the cells are prolonged into long, radiating, unicellular hairs, *b*, *b*, whilst in others, *c*, *c*, these hairs have been broken off near the periphery of the organism. This example has further satisfied me that in the specimens ordinarily met with, most of these hour-glass cells are open at their peripheral extremities, and that all of them were once more or less prolonged into hollow hairs, the free portions of which have in most instances been broken off. I have already referred to the brittleness of many of these cellulose hairs, as illustrated by those clothing the

* 'Annales des Sciences Naturelles,' 5e série, Bot., tom. 12, plate 7, fig. 10.

exterior of the macrospores of *Selaginella Martensii*. In my previous memoir (*loc. cit.*, p. 347) I was unable to discover any evidence that smaller cells occupied the spaces, *d*, intermediate between the hour-glass cells, hence I was disposed to believe that the constricted portions of those latter were surrounded by a crypt-like cavity, whose ramifications were co-extensive with the entire sphere. But on bringing one of the new oil-immersion lenses made by KLEISS, of Jena, to bear upon an oblique section I discovered that this was an error.

Fig. 28 represents a portion of the section in question enlarged 320 diameters. At *a* we have the inner extremities of the cells forming the continuous boundary of the central cavity of the organism. At *b* are the constricted portions of the same cells rising up like a series of hollow pillars from their closely-united flattened bases. At *c* these cells again expand, and form by their conjunction a second or outer continuous tissue, whilst their constricted portions, *b′*, now appear descending from them like so many funnels; at *d*, *d*, we have some of these cells prolonged into hollow tapering hairs. But what is most significant in this section is the series of extremely delicate lines, *e*, which proceed from one constricted cell to its next neighbour, and which in several instances, as at *e′*, are seen to be double. The discovery of these lines, which unquestionably represent the two walls of extremely thin-walled, contiguous cells, demonstrates that the lozenge-shaped interspaces (*d*, of fig. 24) are really occupied by a delicate parenchyma.*

Fig. 25 is a section through a slightly crushed specimen exhibiting a transverse section of the sporocarpal wall at *a*, and a tangential one of the outer surface of the same wall at *c*. In the interior of the organism is a structureless membrane containing a small number of relatively large cells. I had already figured a similar membrane, but devoid of cells, contained in Plate 23, fig. 67, of my previous memoir, and also another in fig. 69A of the same Plate. In the latter case the membrane is filled with numerous small parenchymatous cells, varying from ·001 to ·0015 in diameter. In the specimen now described these cells are nearly double that size, the largest being ·0025 in diameter.

Figs. 26 and 27 represent two specimens intersected more tangentially. The former of these is especially important, because whilst it displays the peripheral ends of the

* Since writing the above description I have discovered the specimen represented in fig. 57. It is a tangential section of the wall of the above fruit which displays the outer surface of the organism, where the cells, *a*, are arranged with great regularity and approximate uniformity of size. At *b*, *b*, we have the inner bases of four hairs, which are merely outward prolongations of some of the cells, *a*. Opposite *c* the section has penetrated a little more deeply into the structure, bringing into view a lower, optical section of the tissues forming the centre of the fruit-wall. At *d*, *d*, we have optical sections of the constricted portions of the hour-glass cells, and at *e*, *e*, we have the radiating walls of five or six cells which surround these central constricted portions. These are here seen so distinctly as to place the existence of a central layer of smaller cells, occupying the lozenge-shaped space *d* of fig. 24, beyond all doubt. In fig. 57 it is the outer conjoined extremities of the hour-glass cells that are in focus. The radiating walls of the central cells, *d*, are seen *through* and *below* the funnel-shaped contracting walls of the surface areolations under a low magnifier.

cells arranged with the regularity of ordinary parenchyma, it further shows the prolonged hairs to be much more numerous than in any of the specimens previously described—indeed, in several portions of the section they are developed from every cell.

Fig. 29 represents an almost superficial tangential section of the *Sporocarpon cellulosum* of my previous memoir. The numerous component cells constituting its wall are of nearly uniform size, and arranged with parenchymatous regularity. The dark zone marks the boundary of the central cavity into which we look through the small central orifice, where the section has sliced off the most prominent part of the sphere. On the opposite side of the slide the section has passed nearly through the centre of this cavity, the maximum diameter of which corresponds with the outer boundary of the dark zone. On employing a one-eighth oil-immersion lens, we discover that the free or peripheral extremities of most of the cells are slightly prolonged into small, dark-coloured mammillæ, as represented in fig. 30.

Fig. 31 is a transverse section of an important specimen, since it seems to indicate a connexion between figs. 29 and 24. It reveals the cells of the boundary wall in every stage of transition from the form seen in fig. 29, to those elongated into the radiating hairs of fig. 24. Thus at *a* the terminal mammilla of fig. 30 is becoming slightly elongated, at *b* it is yet more drawn out, and at *c* it has nearly assumed the full dimensions of the hairs of figs. 24, 25, and 26.

Like fig. 25, the interior of fig. 31 is filled with large cells, averaging about ·0025 in diameter.

This fine series of illustrations suggests the possibility that these objects are cellular spheres, the cells of which were in the first instance short, and compactly grouped; but that as their growth advanced the peripheral extremities of many of them became prolonged into hollow and extremely brittle hairs, and that as this growth progressed further, a small number of large cells appear in the inner cavity, and are gradually developed into a large number of small ones, like those which occupy the interior of fig. 69A of my ninth memoir. On applying the oil-immersion one-eighth objective to the latter specimen, I discovered that many of its contained cells displayed the features represented in figs. 32, 33, and 34. The protaplasms of 32 and 34 have subdivided to form four daughter-cells, whilst fig. 33 contains two such cells. Whether these are the ultimate spores, or whether they are destined to be the mother-cells of yet further developments, it is impossible to say; but seeing that in the only specimen in which these spore-like objects have been found the sphere-wall has attained the fullest development with which we are acquainted, whilst its subdivided cells are no longer free but are reduced to the state of a somewhat compact parenchyma, I think we have strong reasons for inferring that they are destined to be the true spores of the organism.

Of the specimens just described I am indebted to Mr. BINNS for the slides containing figs. 24 and 29, to Mr. EARNSHAW for 25, 27, and 30, and to Mr. SPENCER for 26.

Two of the slides from Halifax, for which I am indebted to Mr. SPENCER, contain

clusters of conceptacles of a perfectly distinct type. Each conceptacle has a diameter of about ·016 to ·012. As seen in figs. 35 and 36, their form is that of rounded spheres, having a very thick outer investing layer, *a*, and a delicate, structureless inner one, *b*. In fig. 35 this inner membrane is intersected through its centre, but in fig. 36 the section has passed through it tangentially, illustrating its spherical form. In the latter instance it contains a number of very minute granular bodies that look like spores. When examined under low powers, even with the quarter-inch objective, the outer layer, *a*, appears to be composed of a mass of small, irregular parenchymatous cells, but the oil-immersion (one-eighth) revealed the structure shown in fig. 37. It consists of a dense mass of branching and interlacing tubes of varying diameters, the interstices of which appear to be filled up with a structureless substance. At numerous points, as at *a*, *a*, in the figure, the section has cut through these branches transversely. In other places, as at *b*, *b*, we have the meshes of the tubular network. In a few specimens I find two inner circular cavities of this form enclosed within a common peripheral tubular investing layer *a*; but there is usually but one, and their ordinarily detached condition and comparatively uniform contour suggests that they have been closely grouped but independent organisms, and not the broken up portions of a common mass. Figs. 35 and 36 are enlarged 162, and fig. 37 750 diameters. I propose for them the name of *Sporocarpon pachyderma.*

What these objects are is not easy to determine. It is impossible to overlook the resemblance between the branching tubules of the outer investment, *a*, and fungoid hyphæ; but I know of no fungoid reproductive structures that in any way resemble them in their entirety.

Fig. 38 represents the only example I have seen of what appears to be a transverse section of another conceptacle which exhibits very distinct features. It is contained in a slide of the Halifax material, for which I am indebted to Mr. SPENCER. Its maximum diameter, including the radial prolongations, is about ·016. Its peripheral layer consists of a thick investment of parenchyma, *a*, which is prolonged into six unequal obtuse rays. Within this tissue is a central spherical cavity, *b*, within which again is a very thin structureless membrane, the tangentially intersected margin of which is seen at *c*. That the interior cavity is a spherical one is beyond doubt. Whether the six parenchymatous radii were the only ones which this object possessed, or whether similar ones existed perpendicular to the plane of the section, I am unable to say. I propose the name of *Sporocarpon asteroides* for this object.

Fig. 39 is a transverse section of an organism from Halifax, for which also I am indebted to Mr. SPENCER. Being the only specimen I have seen of this type I am unable to determine whether, in its perfect condition, it was a spherical or a cylindrical body. Its central cavity has a mean diameter of about ·055. This is surrounded by a ring of dark, carbonaceous, compressed fragments, *a*, which I have no doubt represent compressed parenchyma, apparently the innermost portions of the layer, *b*. This is a layer bounded externally by an undulating outline and consists of a very regular form

of parenchyma, the cells of which appear as if slightly thickened at their angles. Under a low power these dark angular points are the only portions of the cells that are visible. The outermost of these cells have a mean diameter of ·0022, but they become smaller as they approach the inner boundary of the tissue where many of them have less than half that size. The outermost layer of this organism is very peculiar. I have already observed that the periphery of the middle layer is an undulating one. Each of its peripheral projections sustains a cluster of very large thin-walled cells, *d*, *d*, most of which are prolonged radially. Each interval between these projecting cell-clusters is occupied by a single row of very strongly-marked, thick-walled cells, *c*, which appear to be modifications of the inner parenchyma, but whose entire cell-walls are thickened to form a protective layer at the depressed points of the surface of the organism. I find no trace whatever of any epidermal or other layer external to the large cells, *d*, and the obviously protective character of those marked *c* makes it clear that this is not an organism torn from its surroundings, but that we have substantially its true peripheral outline. There are some features of resemblance between the layer, *b*, and the cells, *a*, of fig. 38, hence it is not impossible that they may ultimately prove to belong to the same, or at least to allied plants. I propose to designate fig. 39 by the provisional name of *Sporocarpon ornatum.* One common feature characterises the whole of the objects which I have included in the provisional genus *Sporocarpon*, viz.: they exhibit no trace of having possessed any peduncular appendage wherewith to be attached to their parent plants.

Wide diversity of opinion has long existed between Mr. CARRUTHERS and myself respecting the next specimens to be described. At the meeting of the British Association for the Advancement of Science held at Bristol, Mr. CARRUTHERS described some small objects from the lower Coal-measures of Lancashire, to which he gave the name of *Traquaria*, and which he believed to be carboniferous *Radiolarians.* An abstract of his communication appeared in the Report of the Association for 1872.* Having examined specimens of these objects in my own cabinet in 1874, I ventured to doubt their Radiolarian character, and suggested† that they bore more resemblance to some Cryptogamic spores than to the marine siliceous *Protozoa* with which Mr. CARRUTHERS associated them. The study of a fine series of these objects so far confirmed my convictions that I presented a communication to the British Association at its Dublin meeting held last summer, in which I assigned what appeared to me sufficient reasons for regarding them as reproductive organisms belonging to some unidentified Cryptogamic plants. I next forwarded the more important of my specimens to our highest authority on the subject of the Radiolarians, viz.: Professor HÆCKEL, of Jena, who, in addition to his own careful study of them, kindly invited his colleague, Professor STRASBURGER, to examine them along with him. Both these distinguished biologists have arrived at the same conclusion as myself respecting them, viz.: that they are

* Trans. of Sections, p. 126.

† Phil. Trans., 1874, Memoir, Part V., p. 56.

vegetable and not animal structures. The *Traquaria* is a spherical organism with a thin structureless investing layer or capsule-wall prolonged into numerous radiating tubular appendages, which for convenience may be designated spines. The mean diameter of the central sphere in eight specimens is ·01333, the maximum being ·02, and the minimum ·01. The interior of the sphere is occupied, in several examples, by cells, enclosed within one or more inner membranes, and the entire organism, including its spines, *appears* as if it had been invested by some plastic substance.* Since these several parts of the organism exist in various forms in different examples, the most convenient mode of describing them will be to examine the more characteristic individual specimens in detail.

Fig. 40 represents the smallest specimen I have met with. Though crushed, it is valuable, since it shows that, at this stage of its development, both the capsule-wall, *a*, and the spines, *b*, were free from brittleness, having been flexed by the pressure to which they have been subjected, but without breaking. The plastic (?) investment is seen at *e*, as a faintly granular element devoid of any very definite outline. At *b'*, a portion of the outer capsule-wall has got displaced, revealing sections of three spines, and the entire extent of this tissue exhibits numerous circles and parts of circles shortly to be explained. Fig. 41 represents a fine example of which the central sphere, *a*, has been intersected a little on one side of its maximum diameter, the latter being shown at the circumference, *a'*, of the dark ring. The outer capsule-wall is thus seen obliquely and exhibits numerous small rings. The spines, *b*, are here more rigid than in fig. 40. They have yielded to pressure, but at *b'*, *b'*, *b'* they have done so less readily than in fig. 40—hence they have been thrown, at the yielding points, into more angular forms than seen in the curved ones of that figure. Each spine is broadest at its base, and is muricated externally throughout its entire length, but these muricated projections become more sharp and prominent as we pass from the basal to the free extremity of each spine. The murications are arranged in irregular verticils and ultimately develop into branching tubes, but in the example under consideration they have not reached this stage of growth, displaying as yet little more than sharp projecting points. One or two of the spines, *b''*, exhibit a disposition to branch at their free extremities; the more faintly shaded ones represent some seen out of focus. The plastic investment is seen at *e*, not only surrounding the central sphere but extending to the extremities of the spines.

In fig. 42 the densely clustered spines are imperfectly preserved. The outer capsule-wall, *a*, now appears as an extremely thin tissue. At *f* we find it separating from a second thin, structureless layer of membrane, whilst at *f'* we have another spherical membrane enclosing a mass of detached cells, *g*, the diameters of which range from ·0008 to ·00125. Most of these cells display their outer cell-walls, with the contracted primordial utricle (?) free in the interior of each cell. Fig. 45 represents a transverse section of an example in which the spines, *b*, have become more rigid, and, in addition,

* The "spongy substance" of Mr. CARRUTHERS.

their murications, *b*, have now become tubular. Fig. 47, which represents one of the spines of fig. 46, exhibits this condition more distinctly. The murications on the surface, *a*, have had their extremities broken off, consequently we can look through the circular apertures thus left, into the interior of the hollow spine. We further see, at *a'*, that these murications are now developing into tubular secondary extensions of the primary cavity of the spine: a feature to which further attention will be drawn. Within the capsule-wall, fig. 45, *a*, we again find the inner membrane, *f*, distended by large cells, *g*. These latter now touch one another, though their mutual pressure has not been sufficient to interfere with their spherical form. These cells now average ·0025 in diameter. The plastic investment reappears as before at *e*.

Fig. 46 is a section of a crushed specimen in which both the capsule-wall, *a*, and the spines, *b*, now become brittle, are broken up into innumerable sharply-defined fragments, *b'*. These brittle spines all display the tubular form of murication represented in fig. 47. But the most curious feature of this section is furnished by its inner layers of membrane. We have a structureless one at *f*, splitting into two layers at *f'*, the innermost of the two laminæ uniting with a yet more internal one, *f''*. This latter is covered with numerous closely-grouped circles. A more enlarged representation of a portion of this innermost membrane is given at fig. 48. The rings are now seen to be circular prominences, *f*, the summit, *f'*, of each of which is a little depressed and circumscribed by a sharply-defined circular groove, *f''*.

Figs. 43, 44, and 49 throw light upon each other. Fig. 43 represents the surface of a portion of the capsule-wall, as seen under a comparatively low power, exhibiting the bases of numerous spines. When the microscope is so focussed that we obtain an optical section at a plane corresponding to the *external* surface of the fragment, we merely see the transverse sections of the hollow cylindrical spines, as at *b*, *b*, but on increasing the magnifying power and bringing a more internal surface of the fragment into focus we obtain the effects seen in fig. 44, which represents the upper left-hand portion of fig. 43, enlarged 650 diameters. We now see that at their bases, *a'*, *a'*, the cavities of several spines *appear to* open into one another. But this arrangement is explained by the segment of a transverse section of a *Traquaria*, fig. 49. We here observe that not only are the lower extremities of the spines enlarged, as already described, but that they frequently spread out (fig. 49, *a''*, *a'''*) like the roots of a tree, covering areas many times wider than the maximum diameter of the spine. Fig 49 only exhibits such of these outspread roots as run in the plane of the section; but at *a*, *a*, *a*, we have transverse sections of other similar roots which spread out at right angles to that plane, and which interlace and often appear to anastomose with other similar ones. It is this ramification of the bases of the tubular spines that occasions the numerous irregular circular and semi-circular areolations visible in the capsule-wall in such sections as figs. 40, *a*, and 41, *a*, *a'*. A considerable portion of the inner surface of fig. 43 is covered with a very irregular and ill-defined reticulation.* In fig. 49 the free ends of the spines subdivide into large branches.

* Explained in some supplementary observations on p. 533.

On applying KLEISS' oil-immersion lens to the specimen illustrated by fig. 44, new phenomena were brought to light. I had already obtained some faint glimpses of two or three delicate threads springing from some of the pointed murications of the spines in such specimens as fig. 41, and which threads appeared to lose themselves in the investing plastic substance. In the specimen now under consideration we find that such threads are but the commencements of a very complicated system which permeates the plastic substance in every direction. I was long inclined to believe that these branching threads anastomosed to form a regular network. I noticed that whenever two of them met there was a small but very decided apparent thickening of the tissue, visible even when the threads themselves were almost invisible; but later observations each have led me to conclude that such is not the case, but that they merely start from each pointed mucro as at fig. 44, *a*, *a*, *a*, and spread through the plastic substance by a succession of dichotomous divisions. It is scarcely necessary to say that the structures, *a*, *a*, *a*, of fig. 44, represent the free portions of spines that have been intersected as they pass through the plastic element a little above the outer surface of the capsule-wall, but similar threads are given off *from the upper surfaces* of the branching tubes, *a′*, *a′*.

I have already pointed out that in the more matured spines, as in fig. 47, *a′*, we find the mucronate projections of specimens like fig. 44 also converted into branching tubes. This condition is well represented by fig. 50, where three of the spines are seen intersected transversely, whilst a fragment of a fourth appears in its longitudinal aspect. All four spines demonstrate that the mucronate projections, with their divergent threads of fig. 44, are here replaced by freely branching tubes which radiate in every direction. I have examined the section from which this figure is taken to see if I could discover any anastomoses between the separate sets of branching tubules; but I have failed to do so. In all the cases where such anastomoses appeared to exist, it became manifest that separate branches merely overlaid one another.

I think there can be no doubt that the branching threads of fig. 44 are identical with the branching tubules, *b*, *b*, of fig. 50. If so we must assume that the former is their undeveloped condition, whilst in the latter they have not only attained, but have even passed their maturity. Fig 50 represents them in the most perfect form in which I have yet found them; but it is obvious that the branches, *b*, *b*, of that figure are but the truncated bases of what were at one period much more extended ramifications. Wherever we see into these truncated branches, we find open mouths of thin-walled cylinders, as at *b′*, *b′*.

It appears to me most probable that the external capsule-wall indicated in all the specimens figured by the letter *a*, is a cellulose exosporal membrane, which has been prolonged into numerous radiating branching tubes, the secondary branches of which very closely resemble those figured by VAN TIEGHEM and LE MONNIER in their 'Recherches sur les Mucorinées;'* but having their ramifications more multiplied and

* 'Annales des Sciences Naturelles,' 5e série, Bot., tom. 17, plate 20, figs. 12 and 13.

extended than is the case in the conjugating cells of the *Phycomyces nitens* described by the French biologists.

That these dark-coloured, hollow, thin-walled branching tubes are as different as possible from the transparent and colourless siliceous spines of the *Radiolarians* is too obvious to require further remark. Soft and flexible in their young state, they became brittle only when more matured, a condition to which I have referred in an earlier part of this memoir as not uncommon amongst the macrospores of such recent Lycopods, as *Selaginella Martensii.* The cells seen in figs. 42 and 45 are wholly undistinguishable from similar endospermic cells seen in the Lycopodiaceous macrospores figured in Plate 23 of my last memoir, Part IX. Hence I adhere to my previously expressed conviction that the *Traquairæ* are really vegetable organisms, and that there are strong grounds for supposing them to be Cryptogamic macrospores. Professor STRASBURGER suggests that their nearest allies will possibly be found in those of Azolla and other Rhizocarpous genera.

In my last memoir I gave small figures (*loc. cit.*, Plate 23, figs. 72, 73, and 74) of three small bodies, respecting which I observed: "It is impossible to overlook the striking resemblance of these little objects to the fossil Xanthidia of the chalk flints, and to the zygospores of some of the Desmideæ." When these words were penned I was not certain that these objects might not prove to be young states of some of the numerous spores with which the Halifax rock abounds. Since then I have obtained numerous additional examples of these objects in sections for which I am indebted to Messrs. SPENCER and EARNSHAW, and find their characteristic features to be so constant that I cannot doubt their being matured organisms, whatever may be their botanical nature. That they were all more or less spherical, with radial appendages distributed over their entire periphery, is certain.

Fig. 51 represents one of these objects in their most common aspect. The diameter of the central disk is about ·0014. The radiating arms are of somewhat variable length. These arms always branch more or less peripherally as represented by the further enlarged fig. 52. It is not always easy to trace their exact ramifications owing to imperfections in their mineralisation, but in those which are well preserved there are usually two or three primary divisions, the extremities of which are further subdivided.

Fig. 53 represents another of these objects, the extreme diameter of the disk of which, exclusive of the radiating arms, is ·0018. The arms in this example are rather shorter than in the last one. The section has here passed tangentially through the uppermost portion of the disk—hence we see at *a* the bases of arms springing at regular intervals from what remains of its convex surface. Fig. 56 is a less highly magnified figure of a much larger specimen, the disk of which has a diameter of ·006. It is chiefly interesting from the fact that it unmistakably exhibits an inner structureless membrane, *b*, devoid of all radial extensions. I have found faint evidences of the existence of such a membrane in several of my specimens, leaving no room for doubting

that it is an integral part of the organism. Fig. 55 is also one of the larger specimens, its disk having also a diameter of ·006. Each of its radial arms terminates in an unbranched, semi-clavate extremity, but I doubt if this is normal. It appears rather to be a result of imperfect preservation. The real importance of this specimen is seen in the large and very distinct cells, *c*, located within the cavity of the disk. Other examples have afforded similar, though less strongly-marked indications that such cells belong to the organism, and were doubtless developed, as in other spores, within the interior of the membrane, *b*, of fig. 56.

I have met with several examples of the apparently distinct form represented by fig. 54, and as all the sections exhibit the same aspect, having a pentagonal or hexagonal disk, with a diameter of about ·0018, and five or six long, slender arms, the length of each arm being about equal to the entire diameter of the central sphere, this object is probably distinct from the other species described. I would therefore designate figs. 51, 53, 55, and 56 *Zygosporites brevipes*, and fig. 54, *Z. longipes*.

I have given to these objects the name of *Zygosporites*, without meaning that they are actually zygospores, though there is much reason for adopting such a conclusion. M. CHARLES BRONGNIART, of Paris, informs me that he has found similar objects amongst the discoveries of M. GRAND-'EURY, at St. Étienne, and that he refers them to living types of Desmideæ. I am not prepared to advance thus far. Had true Desmideæ existed in the carboniferous strata, I see no reason why their extremely characteristic bilateral cells should not have been preserved as readily as other cellular tissues which my cabinet contains. Having many scores of examples of these *Zygosporites*, the plants to which they belonged must have been common in the locality in which they grew; but I have not discovered the slightest trace of a Desmid in these deposits.

Figs. 58 and 59 represent sections of an organism of which I have met with several examples in two slides from Halifax, for which I am indebted to Messrs. SPENCER and BINNS. Fig. 58 is a transverse section through the centre of one of these objects, enlarged 1260 diameters. Its actual length is ·0042. Fig. 59 is a more tangential section of another specimen about ·0041 in length. In both specimens the wall of the organism exhibits numerous points at which it is projected outwardly into small, hollow prominences, and which appear to have subdivided extremities like the radial arms of the *Zygosporites*. Indeed, these objects only appear to me to differ from *Zygosporites* in their oblong form, and in the smaller size and greater numbers of their peripheral appendages. Such being the case they may be named *Zygosporites* (?) *oblongus*.

In my memoir, Part VIII., I described and figured (Plate 9, fig. 44 and 46, *g*, *g*) sections of branches of *Dadoxylons*, in which pairs of vascular bundles passed outwards through the woody zone. These specimens left me under the impression that these paired bundles proceeded to leaves, and not to branches. Fig. 60 represents part of a transverse section of a branch obtained by Mr. SPENCER from the marine Ganister

bed near Halifax, from which bed, as is also the case at Oldham, I have obtained my principal specimens of these Gymnospermous stems. They have obviously been drifted fragments. The pith, *a*, consists as usual of large-celled parenchyma. The prosenchymatous woody tissues, *c*, *c*, *c*, are arranged in their normal manner. The pith is prolonged outwards at *b* and *b*; its coarse, parenchymatous cells rapidly developing into radially elongated prosenchymatous ones, as the two medullary outgrowths proceed outwards. In the early part of their course these outgrowths do not appear to be accompanied by any of the woody fibres, *c*, through which they pass; but more externally, as at *c′*, there are clear indications that many of these fibres are deflected outwards in the same direction as the pith-cells. This specimen leaves no doubt in my mind that these pairs of fibro-cellular bundles are sometimes destined to supply branches, which must have sprung from the stem in pairs. Since no such dual arrangement of either leaves or branches is seen in M. GRAND-'EURY's fine examples of *Cordaites*, which plants that author regards as identical with the *Dadoxylons*, this want of harmony between his specimens and mine strengthens my conviction already expressed elsewhere,* that our English *Dadoxylons* cannot, as yet, be identified with the French examples of *Cordaites*.

That pairs of vascular bundles given off from *small* twigs may have proceeded to leaves, as suggested in my eighth memoir (*loc. cit.*, p. 231), is possible, and does not militate against my explanation of the morphology of the specimen just described, since the primary orientation of its branches must have been from the axils of corresponding leaves.

In the memoir, Part VIII., I described the organization of the seed to which I gave the name of *Lagenostoma ovoides*. In the specimens of that seed which were obtained from the Oldham deposits, the outermost layer of the testa was converted into an almost structureless carbonised substance (*loc. cit.*, Plate 10, figs. 60, 62, and 71 A). In figs. 65 and 66 of that memoir there is also observable a thin layer of tissue external to the layer, *e*, or what I have designated "the canopy," or folded tent-like prosenchymatous membrane which encloses the lagenostome, or pollen-chamber, whilst in fig. 69, *a′*, *a′*, I showed that the testa split into two layers, the inner one of which is I believe identical with those seen in figs. 65 and 66.

Mr. SPENCER has sent me a slightly oblique longitudinal section of one of these seeds from the Halifax bed, which throws additional light upon its structure. Fig. 61 represents this specimen enlarged 18 diameters, in which *a* is the outer layer of the testa, *b*, *b′* portions of the canopy, *c′* the wall of the lagenostome, and *g* the embryosac.

Fig. 62 represents a portion of the outer layer of the testa, 61, *a*, enlarged 200 diameters. It consists wholly of sclerenchymatous cells, of which the central cavities are nearly obliterated owing to the thickness of the ligneous deposits lining their cell-walls. A regular superficial layer of nearly cubical cells, *a″*, constitutes the external

* 'Nature,' June 21, 1877, p 138.

surface of the testa, whilst the rest of its substance is made up of others, *a*, less regular in size and form. Within this portion of the seed the layer, fig. 61, *a'*, is seen on both sides of the seed, intervening between the sclerenchyma just described, and the prosenchymatous folds of the canopy, and which layer obviously corresponds with the similar one shown in figs. 65 and 66 of my memoir, Part VIII., and probably also with the layer *a'* of fig. 69 of the same memoir. Fig. 63 represents a portion of this tissue as seen in fig. 61. It consists of extremely delicate prosenchymatous, barred or spiral cells, such as are seen in so many living seeds. When writing my previous description of *Lagenostoma ovoides*, I was not aware of the extreme distinctness of this layer as a differentiated portion of the testa. I presume it may be regarded as the endotesta, though the exact identification of these subdivisions of the testa in recent and fossil Gymnospermic seeds is necessarily difficult and somewhat uncertain.

Fig. 64 is a vertical section through the shorter axis of *Cardiocarpon anomalum*. The memoir, Part VIII., fig. 119, showed the aspects of this seed when cut through in the plane of its maximum diameter. The present figure exhibits the appearance of the same seed when intersected in the plane vertical to that of the above figure. We find the exotesta at *a*—the delicate prosenchymatous endotesta at *b*; the prolonged micropylar canal at *d*; the Chalaza, with the prolonged funiculus at *i*, *i'*, and what in the previous memoir I have designated the perispermic membrane at *g*.

At a very early stage of my researches my attention was arrested by the circumstance that the fragments of wood and bark found in the calcareous modules, both of the Oldham and Halifax districts, were frequently drilled through by numerous circular canals. It soon became obvious to me that these passages had been produced by Zylophagous animals. Similar borings have been described by Dr. DAWSON in his Triassic *Dadoxylons* from Prince Edward Island,* and still later by M. CHARLES BRONGNIART from some of the French carboniferous strata.†

Fig. 65 represents a specimen of some prosenchymatous bark, which has been perforated by animals of diameters varying from about ·0066 to ·0011. The creatures have not merely pushed the prosenchymatous cells aside, but have eaten their way through them. I was long perplexed by the occurrence of many specimens like that represented in fig. 66. I found numerous groups of small, round, or oval spore-like bodies, like those seen at *a*, *b*. They usually occurred in clusters, those composing each cluster being generally of very uniform size. Under high powers they exhibited a somewhat granulated structure. At length the truth dawned upon me that these were the copros of vegetable feeders—probably the same as those that had drilled the round holes in fig. 66. I noticed that these objects were invariably lodged in cavities from which the tissues had been extracted. Thus in both, 66 *a* and *b*, the cellular parenchymas, *a'* and *b'*, have been eaten away, and the copros, *a* and *b*,

* 'Report on the Geological Structure and Mineral Resources of Prince Edward Island,' by J. W. DAWSON, LL.D., assisted by B. T. HARRINGTON, B.A., 1871, plate 3, fig. 27.

† 'Annales de la Société Entomologique de France.' Séance du 12 Avril, 1876.

have been deposited in the cavity left as the result of the manducatory efforts of the animals. Their variation in size is obviously also a result of corresponding variations in the Zylophagi, whilst their uniform granular structure is explained by their origin. These copros are exactly like those of the phytophagous larvæ of recent insects; but beyond this probable association of them with insect forms, I discover no grounds for arriving at a more definite identification.

In 1874, Count CASTRACANE announced to the Academia del Nuovi Lincei, at Rome, that after subjecting pulverised coal to a process of combustion and afterwards to the action of certain chemical reagents, he not only found the siliceous frustules of Diatoms in the residual ash, but that many of these Diatoms were of well-known living species. The coals thus operated upon were derived from near Liverpool, from Newcastle, and from the French coal-field of St. Étienne. This discovery, if real, possessed an obvious importance. Hence its verification or the reverse became very desirable. My colleague, Professor ROSCOE, kindly allowed Mr. SMITH, one of his able assistants, to subject numerous specimens of coal to Count CASTRACANE's process. The coals thus experimented upon were the following Yorkshire and Lancashire ones :—

Bradford Better bed.
Worsley Binns.
„ Roof of Cannel.
„ Cannel.
„ „ Base of bed.
„ Fourfoot. Top of seam.
„ Black.
„ Dow.
„ „ Bottom of seam.
„ „ Top of seam.
„ Brassey.
„ „ Roof of coal.
„ Trencherbone. Top layer.
„ „ Middle coal.
„ White.
„ „ Roof of coal.
„ Yard.
Middleton (Yorkshire) Main.
„ „ Settle Black.
„ „ Adwal. Top of Cannel.
„ „ Main. Top of coal.
Australian coal. Three samples.

The result of these investigations was to obtain a series of preparations of coal-ash

of very diversified character, but in no one example did I discover the smallest fragment of a Diatom.

Mr. F. KITTON, of Norwich, informs me that he examined samples of Welsh, Durham, and Newcastle coals, as well as others from "Inland" collieries, and from Scotland. Like myself, he could find no trace of Diatoms. The Rev. E. O'MEARA,* of Hazlehatch, near Dublin, states that he examined specimens from the Whitehaven coal-field. He says in a letter: "The result was that in all cases not the slightest trace of Diatomaceous forms was found; and if any had been present I have no doubt they could not have escaped my observation." The same correspondent also informs me that the Rev. GEORGE DAVIDSON, of Logie Coldstone, Aberdeen, a gentleman highly competent to conduct such investigations, also examined a series of coals with the same negative results. Under these circumstances I can only conclude that Count CASTRACANE has been mistaken as to the source of his Diatoms.

Calcisphæra.

It only remains for me to examine a group of objects which may have no claim to be noticed in this series of memoirs, since it is quite possible that they may ultimately prove to be animal and not vegetable forms; but having already inquired into one supposed *Radiolarian* with the result of relegating it to the vegetable kingdom, it may not be undesirable to examine some other carboniferous organisms, for which a *Radiolarian* rank is also claimed.

Attention has already been directed to these objects by Professor JUDD. In the discussion that succeeded the reading of a memoir on siliceous sponges by Mr. SOLLAS, before the Geological Society of London, Professor JUDD is reported† to have "referred to the discovery of *Radiolarians* in carboniferous rocks near Chester, and stated that, on dissolving portions of the rock that show the *Radiolarian* structure, the latter entirely disappears, but at the same time the rock itself furnishes small crystals of quartz. This seemed to be confirmatory of Mr. SOLLAS's statements"—*i.e.*, that siliceous organisms imbedded in calcareous rocks might have their siliceous elements replaced by carbonate of lime.

I am indebted to Mr. SIDDALL, of Chester, for specimens of the limestone in question, which comes from Rhydymwyn, near Mold, in Flintshire. It is a very fine-grained limestone of a light brown colour, containing vast numbers of the minute objects referred to by Professor JUDD.

It is impossible to obtain these organisms free from their investing matrix, hence they can only be examined either in thin sections of the limestone as transparent objects, or on polished flat surfaces as opaque ones. The differences which they exhibit, according to the method of viewing them, throw some light upon their morphology.

* Now unhappily lost to science, May 6th, 1880.

† 'Quarterly Journal of the Geological Society of London,' May, 1877, p. 835.

The most indisputable feature which they present is that they are all hollow spheres, most of which are furnished with varying forms of peripheral appendages. The true sphere-wall is always *darker* than either the investing matrix or the contents of the spherical cavities when examined by transmitted, and *lighter* when seen by reflected, light. In these respects the conditions are identical with those presented by the shells of *Foraminifera* seen in the same matrix. The differences seen in various parts of each object are of material value in enabling us to distinguish between primitive organic elements and secondary infiltrated ones. The former appear to be always opaque, and to exhibit structural organic features. The latter are always translucent and crystalline. Differences are further observable according to whether a very thin section is taken out of the centre of a sphere, or whether a sphere is merely cut into two equal or unequal halves.

The discrimination of species, when we only know the objects through sections of them, is always difficult and sometimes impossible. At the same time it is often desirable that we should have provisional names whereby to recognise certain typical forms. Hence I venture to follow the plan adopted in the case of the *Foraminifera*, in which latter the purport of the names, generic as well as specific, is understood to have no reference to real genetic distinctions. I propose for the objects under consideration the generic name of *Calcisphæra*, as not involving any premature hypothesis respecting their nature.

Fig. 70 represents the inner portion of a hemisphere of *C. lævis*, viewed as an opaque object. I select this for our first consideration, because it exhibits these organisms in their simplest form. Its maximum diameter is about ·006, whilst the thickness of the sphere-wall is about ·00058. I can detect no trace of structure in the sphere-wall, neither has it any peripheral appendages. It is simply a smooth sphere—with a thick sphere-wall and an equally smooth internal spherical cavity—the latter portion being occupied by a crystalline calcic carbonate, which has obviously reached the cavity as a solution that filtered through the permeable sphere-wall.

Fig. 79, *Calcisphæra cancellata*.—This is rather a rare form. The drawing represents a thin equatorial section viewed by transmitted light. The central sphere cavity is filled with infiltrated crystalline calcic carbonate. The sphere-wall is now not only double, but the inner and outer layers enclose between them numerous small cubical compartments separated by radiating partitions. The compartments are filled, like the central sphere-cavity, with infiltrated translucent calcic carbonate. This object is of the same size as fig. 70.

Fig. 67, *Calcisphæra fimbriata*, is also of the same maximum size as fig. 70, though, like that object, we find it of very variable dimensions, the smallest specimen being not more than an eighth part of the diameter of the larger ones. The central sphere-cavity as seen by transmitted light is filled with crystalline infiltrated calcic carbonate. This is surrounded by a dark sphere-wall, which is obviously not homogeneous, but has rather the appearance of being composed of radiating fibres. I suspect that this

is a modification of the condition seen in fig. 79, only the radiating partitions are much more numerous, and consequently the compartments are very much smaller. But externally to this sphere-wall we now have a second investing layer, which is semi-translucent by transmitted light, and in which the existence of numerous opaque radiating lines is sufficiently obvious. I presume that this is a second sphere-wall, constructed like the inner one, but that, in it, the primary calcareous radiating partitions have been extremely thin, whilst the larger, long, narrow cavities which they enclose having been hollow, are again filled by calcic carbonate, hence the greater translucency of this outer sphere-layer as compared with the inner one.

Fig. 69, *Calcisphæra hexagonata.*—This form is not very uncommon, though more so than the variety last described. The central cavity is again filled with crystalline calcic carbonate, and the dark, double-inner sphere-wall is now more clearly defined than in fig. 67. The space between the two layers of which this sphere-wall consists, is again occupied by radiating opaque partitions separated by translucent lines. Its distinctive feature is seen in the outline of the outermost investing layer. So far as structure is concerned it differs in no respect from the same layer in fig. 67—save that it is somewhat thicker—but it has a perfectly hexagonal peripheral outline, the sides of the hexagon being almost geometrically equal in size; occasionally they exhibit a slight degree of convexity. The specimen figured is rather larger than fig. 67, having a diameter of about ·0066.

Fig. 68, *Calcisphæra Sol.*—This form exhibits a general resemblance to fig. 67, only both its opaque, inner sphere-wall, and its outer translucent layer are thinner in proportion to the entire diameter of the organism than in that species. Its distinctive feature, however, is found in the outermost sphere-wall, which is prolonged into numerous elongated pointed radii, arranged with a considerable degree of regularity both as regards size and position. These radii are somewhat translucent, like the investing layer of which they appear to be extensions. I think it would not be difficult, by persevering search, to find specimens linking this form to that of fig. 67.

Figs. 71 to 77 appear to represent a series of modifications of fig. 70, inasmuch as in them the sphere-wall appears to be single and homogeneous, but much thinner than in that example, and the surface is drawn out into a series of tubercules and spines of very variable number, length, and acuteness. In many, the section of the sphere is rounded, as in fig. 71. In others it is pentagonal as in figs. 72 and 73, whilst in 76 it becomes trigonal. But I find so many connecting links between these varieties that I propose to unite all this series under the name of *Calcisphæra spinosa.* So far as I can discover, the entire series differs from figs. 67, 68, and 69, in the simpler structure of the sphere-wall.

Fig. 78 is a representation of an example in which, as also in fig. 75, the section has only cut off a small tangential slice from one side of the sphere, the remainder of the hemisphere being seen through the somewhat translucent matrix in which it is embedded. It is larger than the other specimens figured, the diameter of its sphere-

cavity being ·01. Its superficial protuberances are not drawn out into acute spines, but are short, obtuse tubercles, which are irregularly distributed over the surface of the hemisphere. Fig. 75 exhibits a combination of these short tubercles with elongated spines in the same individual, hence this specimen may be regarded as a largely modified example of the group which I have designated *C. spinosa.*

The only additional example of the Welsh series to which I would call attention is that represented by fig. 80, but as seen under a one-sixth objective. It is enlarged 190 diameters. It may belong to the type of *S. spinosa,* but that is not certain. Its importance is found in the minute but obvious foramination of its sphere-wall, a condition that readily explains the surface structure of such examples as fig. 67.

Such are the objects which Professor JUDD believes to be Carboniferous *Radiolarians* —a conclusion which neither I nor my experienced friend HENRY BRADY, F.R.S., are able to accept. In support of this determination I would call attention to some specimens, myriads of which constitute almost the mass of a "Corniferous limestone" from the Devonian beds of Kelly's Island, U.S.A., for specimens of which I am indebted to Mr. BRADY, and which we both believe to be closely related to the Welsh organisms. These objects have also been spherical bodies, having a diameter of from ·05 to ·04. Like the Welsh specimens, they are more opaque than the mean of the surrounding matrix, when viewed by transmitted light, and more brightly white when examined by reflected light. The limestone consists almost entirely of perfect examples and fragments of these objects, the intervals between these being chiefly occupied by a translucent crystalline carbonate of lime. Each organism has been a hollow sphere. The sphere-wall has been much thicker in proportion to its entire diameter than is the case among the Welsh specimens. Externally, the transverse section of each sphere presents an undulating outline, due to the intersection of prominences and ridges that characterise its surface. Sometimes these projections surround the entire section; but more frequently, as is the case with fig. 81, they are absent from limited portions of the periphery. Occasionally these ridges may be seen pursuing an oblique direction like the bands crossing the nucules of a *Chara.* The central cavity is always occupied by crystalline infiltrated carbonate of lime. Though the sphere-wall often exhibits a granular texture, I discover a radiating structure in a sufficient number of the specimens to convince me that, in this respect, they have closely resembled some of the Welsh objects. Since I cannot learn that this American form has received a name, it may be designated *Calcisphæra robusta.* Whilst I am thoroughly satisfied that these objects are *not Radiolarians,* it is not easy to say what they are. Like the *Traquariæ,* they are altogether different from all known *Radiolarians.* The *C. robusta* constitutes the chief part of the material of the Corniferous limestone, which appears to have been as much indebted to them for its calcareous matter as the chalk is to the *Foraminifera.* Hence in this case any idea of a substitution of calcareous matter for silica is out of the question. They are, and obviously have always been, calcareous organisms, and the Welsh examples present many

features leading to the same conclusion. The structure of the latter forms is different from that of any existing *Radiolarians.* Instead of possessing the fenestrated skeleton which would allow the calcareous ooze, constituting the matrix, to penetrate freely into the interior of each sphere-cavity, all such intruded material has been absolutely excluded from those cavities. The interior of each is occupied, not by amorphous matter like the investing matrix, but by crystalline calcic carbonate, a solution of which had obviously filtered through the porous sphere-wall, and crystallized within the interior of a closed cavity. In this respect the conditions of the objects correspond exactly with those of the carboniferous *Foraminifera,* which I find associated with them; I am convinced that no known *Radiolarian* would exhibit similar conditions. Then their form is more like that of the *Foraminiferous Orbulina* than any *Radiolarian.* Occasionally chance sections like figs. 73 and 76 remind us of a Dictyoca; but other specimens, such as figs. 74 and 77, show that these apparently regular symmetrical forms and arrangements of the spines are not characteristic of the organisms. Even in fig. 76 the second unsymmetrical spine to the right hand of the figure is fatal to the idea of the object being a Dictyoca. It is obvious that these organisms were closed spheres with a sphere-wall so nearly solid as to exclude all inorganic matter save such crystalloids as were in a state of solution, and which, consequently, were capable of reaching the sphere-cavity by infiltration.

But a further difficulty stands in the way of our regarding these objects as *Radiolarians.* Unable generally to accept Mr. SOLLAS's hypothesis of the replacement of silica by calcareous matter, I am still less able to do so in the case of the objects under consideration. I have already said that such a hypothesis is wholly inapplicable to the *C. robusta* (fig. 81), and it appears to me equally so to the other forms. Mr. H. BRADY has arrived at the same conclusion. But anxious to obtain the opinion of some of our leading chemists on this point, I showed my specimens to Professors ROSCOE and SCHORLEMMER, and they both express their inability to understand how such a substitution could take place. I presume that on the subject of organic chemistry no higher authority than Professor SCHORLEMMER could be appealed to. After examining my specimens, he writes to me as follows: "I don't know what *morphological* evidence you may possess rendering it probable that the minute calcareous objects in limestone that you showed me were originally siliceous animals or organisms; but I should require such evidence to be overwhelmingly strong before I should accept such a conclusion. I know of no agency by which siliceous structures could be converted into calcareous ones, by mineralogical substitution, under the condition in which these organisms exist, embedded in a calcareous matrix. The fact that the silica was of animal origin does not appear to me to render the possibility of such a substitution more probable."* I have already shown that such morphological evidence as Professor SCHORLEMMER demands is non-existant—hence I am impelled towards the conclusion that these *Calcisphæræ* were calcareous, and not siliceous organisms, and consequently were not

* *In litera.* Jan. 22, 1879.

Radiolarians. Unfortunately, such a negative conclusion is more readily arrived at than a positive one, telling what these objects really were. They are wholly unlike any living organisms that we are acquainted with. Their spherical form suggests the possibility that they may have been the tests of some extinct form of *Protozoa.* The porous tissue seen in fig. 80 gives some support to the idea of their having been Foraminiferous—and the radiating structure seen in so many of the transverse sections is quite compatible with the same idea. On the other hand it is not impossible that they may have had some affinity with the recent Rhabdoliths and Coccoliths, though this does not seem very probable. The only other possibility that suggests itself is that they may be reproductive capsules of some marine form of vegetation, but no facts yet discovered afford any definite support to this hypothesis. Mr. BRADY informs me that in one instance he found indications of spore-like objects in the spherical cavity, but the whole of the thousands which I have examined were entirely devoid of such elements. This fact is suggestive of their having been filled with some material incapable of fossilisation—*e.g.*, of sarcodic protoplasm, pointing to a Protozoan nature.

I have once more to acknowledge the assistance I have received from Mr. SPENCER and Mr. BINNS, of Halifax, from Mr. EARNSHAW, of Oldham, and especially from Mr. WUNSCH, of Glasgow, to whom we owe the discovery of the magnificent carboniferous forest of Laggan Bay, and whose invaluable aid demands my warmest thanks. Mr. SIDDALL, of Chester, has laid me under obligation for the Welsh limestones, and I am indebted to Mr. HENRY BRADY for calling my attention to the corniferous limestone of Kelly's Island.

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PLATE 14.

LYCOPODIACEOUS PLANTS.

Arran Lepidodendron.

Fig. 1. Transverse section of one of the youngest twigs. *a.* Central vascular bundle. *c.* Leaves. *c'.* Foliar vascular bundle. Enlarged 16 diameters.

Fig. 2. Central vascular bundle of fig. 1. *a'.* Small vessels at the periphery of the axis, from which the foliar vascular bundles (*b*) arise. Enlarged 37 diameters.

Fig. 3. Transverse section of a larger branch. *a.* Central vascular bundle expanded into a medullary cylinder. *b.* Foliar vascular bundles. *c.* Leaves. *d.* Cellular medulla. *e.* Delicate parenchyma of middle bark. *f.* Remnant of the bast-layer of the outer bark. Enlarged 3 diameters.

Fig. 4*. Portion of a vessel of the vascular cylinder. *a.* Transverse Lignine bars. *b.* Longitudinal threads of Lignine.

Fig. 4. Central vascular axis of fig. 3 enlarged 14 diameters. *a*. Vascular medullary cylinder. *a'*. Vessels going off radially to form the free foliar bundles, *c*. *d*. Cellular medulla.

Fig. 5. Transverse section of a still larger branch. *a*. Vascular medullary cylinder. *d*. Cellular medulla. *e*. Middle bark corresponding to *e* of fig. 3. *g*. Inner portion of prosenchymatous bark. *h*. Bast-layer of outer bark. *i*. Exogenous layer of vascular axis. Three-fourths the natural size.

Fig. 6. Transverse section of the vasculo-medullary axis of a stem of large size. *a*. Vascular medullary cylinder. *d*. Medulla. *i*. Enlarged exogenous vascular cylinder. Natural size.

Fig. 6A. Segment of a section like fig. 6 enlarged 3 diameters.

Fig. 7. One of the rhomboidal scars of a cast of the outer bark-surface of a large stem. Three-fourths the natural size.

PLATE 15.

Halifax Lycopodiaceæ.

Fig. 8. Small crushed *Lepidostrobus* from Halifax enlarged nearly 12 diameters. *a*. Vasculo-cellular axis. *b*. Clusters of macrospores. *c*. Upper part of the strobilus. *d*. Clusters of microspores in their sporangia.

Fig. 9. Sporangial cluster of four macrospores, showing that the "peduncular appendages" are but collapsed portions of the spores. Enlarged 34 diameters.

Fig. 10. Tangential section of a sporangiferous bract from Halifax, with its sporangium filled with microspores. *a*. Bract. *b*. Outer sporangium-wall. *c*. Inner membrane of the sporangium. Enlarged 45 diameters.

Fig. 11. Transverse section of the central axis of the strobilus figured in Plate 22, fig. 53, of memoir, Part IX. *a*. Medulla. *b*. Vascular cylinder. *c*. Innermost cortical parenchyma. *d*. Outer cortical parenchyma. *e*. Bracts. *f*. Sporangia. Enlarged 12 diameters.

Fig. 12. Central axis of fig. 11, enlarged 45 diameters. *a*. Medulla. *b*. Vascular cylinder. *c*. Radial breaks in the continuity of the cylinder. *d*. Innermost cortical layer. *d'*. Radial prolongations of *d*, apparently accompanying vascular bundles proceeding outwards to the bracts.

Fig. 12A. Transverse section of a pentamerous fruit (?) from Halifax.

Calamostachys Binneyana.

Fig. 13. Longitudinal section of a specimen of *Calamostachys Binneyana* from Halifax. *a*. Vascular axis. *b*. Cortical layer of the axis. *c*. Sterile bracts. *d*. Fertile bracts or sporangiophores. *d'*, *d''*. Empty passages

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Fig. 14. Transverse section through a sterile bractigerous disk of a *Calamostachys Binneyana.* *a.* Vascular axis. *b.* Cortical layer of the axis. *c.* Elongated cells passing horizontally outwards to the sterile bracts. *c'.* Parenchyma of the bractigerous disk. *e.* Sporangia. *f.* Transverse sections of the apices of the bracts of the next inferior sterile verticil. *g.* Tips of the next inferior sterile bracts. Enlarged 16 diameters.

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PLATE 16.

Fig. 16. Transverse section of the vascular axis of a strobilus of *Calamostachys Binneyana* (?). *a.* Vascular axis. *b, b.* Radially disposed additions to the central vascular bundle, disposed in three radiating groups. *c.* Middle cortical layer. Enlarged 46 diameters.

PLATE 15.

Fig. 17. A mother-cell with three daughter-cells (spores ?) from a sporangium of *Calamostachys Binneyana.* Enlarged 405 diameters.

PLATE 16.

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PLATE 16.

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PLATE 17.

Cryptogamic Reproductive Organs.

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Fig. 26. A yet more tangential section of a *Sporocarpon elegans* displaying the symmetrically disposed, outer ends of the cells at its peripheral surface, and passing obliquely into the internal cavity of the organism at the interior of the section. Enlarged 93 diameters.

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Fig. 28. Oblique section through a part of the cellular-wall of *Sporocarpon elegans.* *a.* Its inner surface formed of the expanded centripetal extremities of the hour-glass cells. *b.* The inner portions of the constricted parts of the same cells. *c.* The outer expanded portions of these cells. *d.* The free portions of the cells prolonged into hairs. *e*, *e'*. Delicate cell-walls of cells forming the central layer, and connecting the hour-glass constricted cells. Enlarged 320 diameters.

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Traquaria (CARRUTHERS).

PLATE 19.

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PLATE 18.

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Fig. 44. Part of fig. 43. Enlarged 660 diameters. *a.* Small mucronate projections of the exterior of the tubular spines, giving off branching and interlacing tubular (?) threads, which spread through the investing substance, fig. 41, *e.* *a'.* Coalescing portions of several tubular branches.

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Zygosporites.

Fig. 51. *Zygosporites brevipes.* Enlarged 637 diameters.
Fig. 52. Single ray of *Zygosporites brevipes.* Enlarged 2258 diameters.
Fig. 53. *Zygosporites brevipes.* Enlarged 850 diameters.
Fig. 54. *Zygosporites longipes.* Enlarged 450 diameters.
Fig. 55. *Zygosporites brevipes*, containing cells in its interior.
Fig. 56. *Zygosporites brevipes*, exhibiting an inner membrane.

PLATE 20.

Fig. 57. A tangential section of *Sporocarpon elegans.* *a.* Cells seen at the surface of the sphere. *b.* Peripheral prolongation of the cells into hairs. *d.* Constricted portions of the hour-glass cells. *e.* Similar constricted portions, as seen under a lens of high penetrating power, and showing the radiating walls of the middle layer of cells (fig. 28, *e*), filling up the intervals between the constricted portions of the hour-glass cells.
Fig. 58. Transverse section of *Zygosporites* (?) *oblongus.* Enlarged 1260 diameters.
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PLATE 20.

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Calcisphæræ. The supposed Radiolarians of the Welsh Carboniferous Limestones.

Fig. 67. *Calcisphæra fimbriata*, seen as a transparent object by transmitted light.
Fig. 68. *Calcisphæra Sol*, seen as a transparent object by transmitted light.
Fig. 69. *Calcisphæra hexagonata*, seen as a transparent object by transmitted light.
Fig. 70. *Calcisphæra lævis*, seen as a transparent object by reflected light.
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Fig. 79. *Calcisphæra cancellata*, seen as an opaque object by reflected light.
Fig. 80. *Calcisphæra spinosa* (?) seen as an opaque object by reflected light.
Fig. 81. *Calcisphæra robusta*, from the corniferous limestone of Kelly's Island, U.S.A.

All the above objects are enlarged 112 diameters, except fig. 80, which is enlarged 180, and fig. 81 is magnified 33 diameters.

Supplementary Observations.

(Added August 12, 1879.)

I have obtained some important additional information respecting some of the organisms described in the preceding memoir since its communication to the Society, especially in reference to the supposed *Radiolarians*, hitherto known as *Traquariæ*.

I am indebted to Mr. Cash, of Halifax, for some valuable specimens from his cabinet, originally prepared by Mr. Binns. The most important are a series of sections of a crushed *Lepidostrobus* in all of which *Traquariæ* occur, under such conditions as leave no doubt that they are the macrospores of a Lycopodiaceous plant. The structure of a transverse section of the axis of this fruit is represented in fig. 82, enlarged 16 diameters. Its vascular centre, *a*, is a nearly solid cylinder of vessels, in the middle of which are what appear to be a very small number of cells representing the medulla. The entire cylinder has a diameter of ·05. The vessels are uniform in size, except at the extreme periphery where they are very small, as is usual with the Lepidodendroid cones. Surrounding this vascular axis we have numerous small *cellular* cylinders, *b*, each one of which contained a foliar vascular bundle supplying the bracts or sporangiophores of the cone; the vessels composing these bundles have disappeared. The similar disappearance of the inner and middle cortical layers of the axis of the cone has left these foliar cylinders isolated. Both this isolation of the individual cylinders, the mode in which they are clustered round the vascular axis, and the disappearance of the vessels of each foliar bundle are features identical with what we see in nearly all

our sections of *Lepidostrobi*. The detached masses of prosenchyma, *c*, *c*, are portions of the outermost cortical layer of the axis of the strobilus, which are being prolonged radially into the usual Lepidostroboid sporangiferous bracts. Extensions of these bracts radiate, in a more or less fragmentary form, to the circumference of the specimen, which even in its imperfect state indicates a cone having a diameter of fully an inch. Interspersed amongst these bracts or sporangiophores are the usual sporangia, the wall of each of which displays the structure so common amongst these carboniferous Cryptogams, viz.:—a single series of cells elongated vertically to the surface of the sporangium, and having their two extremities flattened, so that the two surfaces of the sporangium wall exhibit the ordinary aspect of tubular thick-walled parenchyma (fig. 83), whilst vertical sections present the aspect seen in fig. 84 or that of cylindrical parenchyma.

One of these sporangia is shown in fig. 85, *a*, enlarged 16 diameters, and contains three of the Traquarian macrospores. At *a′* are fragments of two other contiguous sporangia. Throughout the greater part of its extent the sporangium wall, *a*, exhibits the appearance of fig. 84, but here and there its flexures have caused it to be intersected obliquely, as at *a″*, where it resembles fig. 83. Two of the macrospores are intersected nearly through their centres, the third one more tangentially, hence its apparent smaller size. The specimen from which this and other similar sections were prepared not only places the vegetable nature of these *Traquariæ* beyond the possibility of doubt, as well as demonstrates the fact that they are Lycopodiaceous in character, but from the excellent preservation of the macrospores, throws further light upon their structure.

That the specimens with tuberculated but unbranched spines (represented by figs. 40, 41, and 42, of the earlier portion of this memoir) are immature, whilst those represented by figs. 45 and 46 are more matured examples, is now clear. I think there is no doubt but that in the young state there was a distinct outer exosporium and an inner endosporium. At an early period the exosporium became differentiated into two layers. Of these, the inner one (represented by *f* in fig. 42) retained its structureless, spherical form, being undistinguishable from the endosporium, fig. 42 *f′*, in all points save in its more external position. The outer layer of this exosporium, fig. 42, *a*, on the other hand, underwent a development into a system of ramifying tubes, the complexness of which exceeds what I had observed to be sufficiently remarkable when the earlier part of this memoir was written. There is now no doubt that the minute projections, *a*, from the radiating spines of fig. 44 with the delicate branching threads which spring from those projections, are the early conditions of the branching tubes seen in fig. 50. The almost invisible threads expand into a series of tubular dichotomous branches.

Fig. 86 represents a portion of a tangential section of one of the macrospores from Mr. CASH's strobilus. At *a* we find what I assume to be the inner structureless layer of the differentiated exosporium, now very distinctly separated from its outer tubulated one; *b*, *d* represent the transversely intersected bases of 11 of the branching tubes, *a*, *b*, of the figs. 40–49 of the memoir. We now see that in addition to

the *shorter* radiating branches given off from the entire length of the tube, there is a special *basal* series, fig. 86, *c*, much longer and less freely supplied with secondary ramifications than is the case with the upper ones. These basal branches appear to be similar to the upper ones in their general features and origin, those radiating from each central tube forming a system independent of the corresponding ones given off by its neighbours. They interlace most freely, enclosing the endosporium in a perfect network of superficial ramifications, but I have not been able to detect a single example in which they anastomose with those of the surrounding similar systems of tubes.

We obtain further light on this subject from fig. 87, which represents a portion of a macrospore from the same strobilus, but which has been intersected vertically. The drawing exhibits the bases of three of the tubular spines, *b*, *b'*, and *b'''*, the two former being the principal ones in focus. The spines have been cut through longitudinally and tangentially, so that we look into their interior, which is very large in comparison with the thickness of the enclosing tube-wall. The spine, *b'''*, lies deeper in the section, which has only sliced off a little of its base. At *c* we have one of the large basal branches of *b*. We now see that it gives off short, thick, lateral branches, *d*, in every direction, downwards as well as upwards. These branches subdivide by repeated dichotomisations, each of the secondary and subsequent branches being very short. Hence each of these secondary branch-systems constitutes a dense tuft of hollow tubes, whose repeated and peculiar ramifications remind us of the characteristic branching of a tuft of the well-known sea-weed, *Chondrus crispus*, only in the case before us the ultimate subdivisions are so fine that we fail to trace their individual outlines where they interlace with those of the neighbouring tufts. This condition probably explains the nature of the network shown in fig. 43, and referred to on page 513 of the memoir. The lateral branches, *e*, given off from each spine become rapidly shorter as we ascend, but in all those above the basal series, *c*, we find the ternary clusters of branches to be more numerous and more closely crowded together than in these basal ones; in other respects they present no differences. There is in them the same succession of curvilinear dichotomisations as before. The consequence is that the primary spines and their secondary branches are closely invested by a dense interlacing network of these ramifications. In the earlier part of the memoir I suggested the probability that these tubes had been invested by a plastic substance, in which the ultimate ramifications of the tubes distributed themselves. I am now satisfied that whatever may have been the case with the immature macrospores the matured ones were not so invested. What gave that appearance was merely the imperfectly preservation of the minute extremities of these ramifying tubules. We have at *c''*, in fig. 5, the intersected portions of other branches corresponding to *c*, but emanating from other spines.

Whilst these portions of the organism are now interpreted without difficulty, other features of the structure become less easy of explanation. Proceeding upon the

supposition that each of these *Traquariæ* was a spore, and as such, primarily a single cell, it was easy to regard the branching tubes as outward extensions of the spherical exosporium, originally the outer cell-wall of the organism. The development of the remarkable ramifications of the sporocarp of the Fungoid *Phycomyces nitens*, described by Professor BORNET,* appeared to furnish an illustration of the mode in which these radiating and branching extensions had been formed, but in the latter example the central stem of each branch-system opens at its base by a wide communication with the central cell cavity of which its tubules are but extensions. When describing fig. 43 (p. 513) I pointed out that the specimen there delineated appeared to favour a similar explanation, but those now described seem to tell a different story. It will be seen that the bases of each of the three principal stems, b, b', and b''', are *closed* and not *open*. I have examined every available specimen of *Traquaria* in reference to this point with the utmost care, but they all seem to lead to the same conclusion, viz.: that in their matured state each of these branching spines is closed at its base and thus assumes the form of a separate, unicellular, branching trichome; that they are really trichomes is, however, too improbable to be accepted as true. Hence the only conclusion at which I can arrive at present is that they were primarily mere extensions of the exosporium, but that when this exospore became differentiated into two layers and the outer one developed its remarkable system of branching tubes, the latter organs became separated from one another and each had its basal aperture closed in by the contraction of that part of its exosporial wall; the inner layer of the differentiated exosporium meanwhile retaining its primary simplicity as a thin spherical cell-wall.

Another feature of interest in these objects is the condition of the cells contained in the interior of several of the macrospores. The modification of the endosporal cells seen in some of the recent *Selaginellæ*, especially the large cells which PFEFFER not only compares with the endospermic cells of *Angiosperms*, but even designates by the same name, invest the study of these ancient forms with importance. I have now met with nine examples of these Traquarian macrospores containing cells. Some of these are already represented in figs. 42 and 45.

I find several modifications in the cellular contents of these spores, which are probably significant. In one to which I will refer as A (fig. 45), the endospermic cavity is filled with comparatively large, thin-walled cells, which compress each other so closely as almost to constitute a loosish parenchyma, their individual diameters being about ·0036; none of these cells display any definite organised contents. In specimen B, for which I am indebted to Mr. SPENCER, of Halifax, the endosperm consists of similar cells, though of smaller size, having a diameter of from ·0011 to ·0015. Most of these are like those of specimen A, though less closely packed, and about six of them contain each a single small, round, dark-coloured cell, having a diameter of about ·00036. A third specimen, C, which I obtained from Mr. BINNS, of Halifax, is filled with similar thin-walled transparent cells (diameter ·0014, ·0015), but now *every* cell contains one

* 'Annales des Sciences Naturelles,' 5^{e} série, tom. 17, pl. 20, figs. 7–13.

of the smaller, dark-coloured cells, the diameters of these latter varying from about ·0011 to ·0009. In specimen D in the same slide as B, I find many of the thin-walled cells (diameter ·0013 to ·0015). Some of these are empty, and have lost some of the turgid, rotund form that usually characterises them. Others are more rounded, and each one contains, in its interior, one of the smaller, dark-coloured cells (diameter ·0009 to ·0007). Intermingled with these larger cells are many *free* examples of the smaller ones, which I expect have been liberated from the interiors of the empty larger ones. Specimen E only contains a few of the large, thin-walled cells (diameter ·0015, ·0014). Some of these are empty, two of them contain each a cell, with a diameter of about ·0009, and in one other was a single example of the smaller cells (·00015). Along with these is a very large number of the smaller cells in a free state. This specimen is in the cabinet of Mr. CASH. Specimen F is that of which portions are represented in fig. 87, and some of the endosporal cells of which are represented in fig. 88, enlarged 320 diameters. At *a* is a portion of the enclosing endosporal membrane. At *b* are some of the large thin-walled cells (diameter ·0013, ·0014). Each of these contains an inner cell, having a diameter of from ·0009 to ·0008, whilst each of these inner cells again contains, usually adhering to one side of it, a somewhat irregular dark mass of rather variable size. Of these larger cells my section of the spore contains about a score. But along with them we have 300 or 400 of the smaller, dark-coloured cells, *c*, of somewhat smaller size than those already described, their diameter ranging from ·00058 to ·0004. These are aggregated into a loose irregular central group, detached from the endosporal membrane, and condensed at its innermost portion into an opaque, somewhat defined mass. The true nature of the apparent central membrane *f″* of fig. 46, is now clear enough. It consists of an aggregated layer of these small cells, of which the mean diameter is about ·00055. Why they should have assumed so much the aspect of a continuous membrane as they have done is not so clear.

From the above description I think we may conclude that the large turgid cells, *b*, belong to the earlier stage in the development of this macrospore, and that the smaller and far more numerous ones, *c*, belong to a later stage. The degree of advancement in the development of these endosporal structures does not appear to correspond exactly with that of the respective exosporal tissues; still there is sufficient of an approximation to such a correspondence to sustain my general conclusion that the specimens A, B, C, and D are all immature spores, whilst E, F, and the crushed specimen fig. 46, are highly matured ones. At the same time, A is undoubtedly a more advanced growth, so far as the exosporium is concerned, than B, C, and D; but much less so than any of the remaining three, in which the small dark-coloured cells are so abundant. The question now arises, What are these dark objects? That they are cells is certain. In many of them the true cell-wall is sufficiently obvious, though in most it fits so closely upon its cell contents, as to be almost invisible. I have now no doubt that in fig. 46 the outer cell-walls are in contact, and that the transparent ring surrounding each central circular body represents the space between that cell-wall and its contents.

In the sections of the strobilus many of the sporangia contain a quantity of disorganised fragments, which appear to me to be the remains of microspores. Of course this Traquarian macrospore now merges its specific individuality in the strobilus of which it forms so characteristic a feature. But as it is desirable that the well-known name should be retained in connexion with it, I propose for the entire cone the name of *Lepidostrobus Traquaria.*

In the previous part of the memoir (p. 510, fig. 38) I described, under the name of *Sporocarpon asteroides*, a structure which appeared to me to be a spherical reproductive organ, and not a mere section of some cylindrical body. Mr. SPENCER, of Halifax, has obtained several additional examples of this organism, which demonstrate the correctness of my previous conclusion. They vary much in the size, shape, and number of their radial appendages, but in the peculiar features of their regular parenchymatous tissue, and in the perfectly spherical form of their central cavity, they agree with the example already figured. The one now represented (fig. 89) displays a second membrane (a) within the clearly-defined spherical cavity; this membrane encloses an opaque, spherical mass (b). Similar conditions are seen in several of the other *Sporocarpons* which I have already described.

I am indebted to Mr. GEORGE WILD, of the Bardsley Collieries, Ashton-under-Lyne, for the fine specimen of a new Zygopteroid form of fern-stem or petiole, of which a transverse section is given in fig. 90, enlarged nearly 6 diameters. Mr. WILD found the specimen in a shale heap, containing the usual marine Ganister fossils (*Goniatites, Aviculopectens,* &c.), and which had come from the roof of the "Bullion" coal near Burnley. The maximum diameter of the slightly ovoid section is three quarters of an inch, but we certainly have not the entire bark, which has lost some of its external portions, though how much I cannot ascertain. Its present outermost layer of large cells corresponds pretty closely with that seen in *Rachiopteris bibractiensis,* immediately below the prosenchymatous layer which constitutes its peripheral portion; as a corresponding prosenchyma forms the periphery of the allied *R. Lacattii,* and, though in a less degree, of the *R. elegans,* described in this memoir, it is most probable that a similar layer invested the coarse outer parenchyma of the plant under consideration.

The vascular axis, a, approaches nearer, in its general contour, to that of *R. bibractiensis,* than to that of *R. Lacattii,* especially in the trim neatness of its vessels, and in the perfect parallelism of the two sides of the central bar, a, which in *R. Lacattii* are oppositely convex. But the two transverse bars, a', a', differ from the similar ones in *R. bibractiensis,* in the absence of the very distinct peripheral bands of vessels somewhat imperfectly shown at a'', in fig. 49 of my sixth memoir. In the plant now described, the outer margin of each of the transverse bars, a', a', is occupied, as in *R. Lacattii,* by a series of vessels much smaller than those constituting the rest of the axis, and from which the foliar (?) bundles have arisen. Surrounding this vascular axis is a thin layer of parenchymatous cells, b, which I expect has originally formed

the symmetrical boundary or endoderm, enclosing a sheath of endophlœum that has invested the axis, but which has disappeared. At *c*, *c*, we have a pair of vascular bundles passing upwards and outwards, also enclosed in a supposed phlöem boundary, and at *d*, *d*, is a second similar pair which have entered the coarse parenchyma of the outer bark, but which are immediately surrounded by a parenchymatous zone, *e*, of a much more delicate texture. At *f*, on the opposite side of the section, we have a single bundle passing outwards, which from its symmetrical form and central position seems to correspond with the united bundles of each of the two opposite pairs. Here again the bundle, *f*, is surrounded by thin-walled parenchyma.

The greater portion of the tissues of the middle and inner bark have disappeared. Here and there we obtain faint glimpses of them, indicating that they consisted, as is usually the case in these ferns, of delicate, thin-walled parenchyma.

On turning to vertical sections of this plant, we find that they agree with similar ones of the other species of the Zygopteroid group of fern petioles. Fig. 91 exhibits a vertical section through the central part, *a*, of the vascular axis, enlarged 56 diameters. Its vessels, *a*, are now seen to be of the most perfect scalariform type that I have hitherto met with amongst these carboniferous ferns. Immediately external to this vascular axis is a thin investment of three or four layers of very long and narrow, thin-walled, square-ended cells, *b*, *b*, the extreme delicacy of which makes them almost invisible save under high magnifying powers. Vertical sections through the transverse bars, *a'*, of the vascular axis, show that whilst the greater number of its vessels are of the same type as those of the single central bar, *d*, the outermost ones, some of which become detached to form the foliar (?) bundles, *c*, *d*, and *f*, approach more closely to spiral vessels. The foliar bundles themselves consist wholly of vessels of the spiral type, enclosed in a distinct investment of delicate, oblong, square-ended cells. The outer bark, *h*, is seen to consist in the vertical section, of vertically disposed rows of cubical parenchymatous cells of large size and coarse texture.

The orientation of the supposed foliar bundles in symmetrical pairs is a curious feature in several of these Zygopteroid forms. I have already described its occurrence in *Rachiopteris duplex* and *R. Lacattii*. The peculiar form which the transverse section of the vascular bundle of the specimen now described exhibits, so closely resembles that of two severally inverted Greek capital letters, that the stem may be provisionally recognised as *Rachiopteris di-upsilon*.

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PLATE 21.

Fig. 82. Transverse section of the central axis of a *Lepidostrobus* from Halifax. Enlarged 16 diameters. *a*. Central vascular cylinder. *b*. Cellular investments of the vascular bundles going to the bracts. *c*, *c*. Portions

of the prosenchymatous tissues of the bracts or sporangiophores of the cone.

Fig. 83. Superficial aspect of the cells of the sporangium-wall.

Fig. 84. Vertical section of a portion of a sporangium-wall.

Fig. 85. A sporange, *a*, *a''*, containing three *Traquariæ*, two of which are intersected equatorially, and the smaller one tangentially. Enlarged 16 diameters. *a'*. Fragments of two other sporangia.

Fig. 86. Portion of a tangential section of one of the Traquarian macrospores of the strobilus. Enlarged 225 diameters. *a*. Inner structureless layer of the exosporium. *b*, *d*. Transversely intersected bases of 11 of the radiating tubular spines. *c*. Long radiating tubes given off from the base of each spine.

Fig. 87. Portion of a transverse section of the outer or tubular layer of the exosporium of a macrospore. Enlarged 640 diameters. *b*, *b'*, *b'''*. The basal portions of three radiating tubular spines, with their lateral branches. *c*. Large basal tubular branch of the spine, *b*, corresponding to *c*, *c*, of fig. 86. *c'*. A similar tube of the spine, *b'*, *b''*. *d*. Secondary tufts composed of dichotomously branching tubes. *e*, *e*. Secondary tubular branches of the upper portions of the primary spines.

Fig. 88. Portion of the contents of the endosporium of fig. 87. Enlarged 320 diameters. *a*. Portion of the endosporial membrane. *b*. Large thin-walled cells. *c*. Smaller dark-coloured cells.

Fig. 89. Transverse section of a specimen of *Sporocarpon asteroides*. Enlarged 18 diameters. *a*. Inner free membranous capsule. *b*. Condensed central substance.

Fig. 90. Transverse section of a petiole of *Rachiopteris di-upsilon*. Enlarged nearly 6 diameters. *a*, *a'*. Vascular axis. *b*. Thin layer of small parenchymatous cells. *c*, *c*. A pair of vascular foliar (?) bundles. *d*, *d*. A second, yet more external pair of similar bundles. *f*. An undivided symmetrical bundle, apparently similar to *c* and *d*. *h*. Coarse, large-celled parenchyma in vertical columns.

Fig. 91. Vertical section across the middle of the central vascular bundle, *a*, of fig. 90. Enlarged 55 diameters. *a*. Scalariform vessels. *b*, *b*. Thin investment of narrow, oblong cells.

XV. *On the Relation between the Diurnal Range of Magnetic Declination and Horizontal Force, as observed at the Royal Observatory, Greenwich, during the years* 1841 *to* 1877, *and the Period of Solar Spot Frequency.*

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[PLATES 22–24.]

THE equipment of the Magnetical and Meteorological Department of the Royal Observatory at Greenwich, established and organised by the present Astronomer Royal, Sir GEORGE B. AIRY, K.C.B., was generally complete at the latter part of the year 1840, since which time observations have been continuously made. Until the end of the year 1847 these consisted of eye readings of the various instruments, taken at intervals of two hours. But Mr. CHARLES BROOKE having at this time arranged a practical system of photographic registration, continuous records of the indications of the instruments have been, since the beginning of the year 1848, by this means obtained.* These records form a sure basis on which to found any magnetic inquiry.

The magnetic elements which have been the subject of observation are, firstly, absolute determinations of magnetic declination, of the horizontal component of the earth's magnetic force, and of magnetic dip or inclination; and, secondly, the continuous variations to which the declination and the horizontal and vertical components of the earth's force are subject. The absolute measures are important in combination with similar measures made at other places for determination of the general magnetic condition of the earth, and of the slow changes occurring therein. But the smaller variations of shorter period, as observed at any one place, also throw great light on

* For information in reference to this subject, see papers by Mr. BROOKE in the Philosophical Transactions for the years 1847, 1850, and 1852; and also the Addendum to the introduction to the 'Greenwich Magnetical and Meteorological Observations' for 1847.

the general laws and phenomena of magnetic action. Thus the variations of declination, and of the horizontal and vertical components of the magnetic force, each include a well-marked diurnal period, analogous in some degree to that of atmospheric temperature; and, speaking generally, the range of the diurnal variation at Greenwich is, in each case, greater in summer than in winter. In addition, however, to the annual inequality in the magnetic diurnal range there appears to be yet another, also of marked character but of longer period, one which resembles in its features the apparently well established eleven year sun-spot period, and which it is the object of the present paper to evolve.

This is not by any means the first time that the relation alluded to has been discussed. The investigations of General Sir E. SABINE, Professor BALFOUR STEWART, and Mr. BROUN in our own country, and those of Professor LAMONT and Dr. WOLF among foreign workers, will immediately occur to all who may be in any way acquainted with the literature of the subject. But it appeared to me that the long series of Greenwich observations might be applied as a valuable independent test of the accuracy of the generally received relation. For (as regards the results made use of in this paper) the observations at Greenwich have been throughout made on the same general plan, and with the same instruments;* the results are therefore well adapted for use in an inquiry of the kind, and the conclusion arrived at is one in which it is reasonable to suppose that considerable confidence may be placed.

To proceed now with the subject. The indications of the declination magnet are not directly affected by temperature, but those of the horizontal force magnet are so affected. The correction applicable in the case of the latter instrument has been at various times determined by different processes with fair general accordance of results, and no error of importance is likely to have been by this cause introduced. Moreover, as regards the present inquiry, since the effect of any such small error would simply be to only slightly raise or depress parts of the horizontal force curve as figured in the diagrams, the general deductions of this paper would in no degree be affected. The indications of vertical force are for the present object not very manageable; several different instruments have been employed during the period under discussion, and the results present some anomalies which are possibly in part instrumental. Our attention at present is therefore confined to a discussion of the inequalities of declination and horizontal force.

The mean diurnal range of declination in each individual month is taken to represent (relatively to other months) the magnetic energy of the month. And similarly for

* It is true that a new magnet (precisely similar in dimensions to the old one) was brought into use at the beginning of the year 1865 for photographic registration of the variations of declination; but its indications are compared four times daily with those of the old magnet (still used for determination of the absolute declination), so that the complete correspondence of the whole series of observations is thereby assured. For horizontal force the same identical magnet was used throughout.

horizontal force. Two series of numbers are thus formed, such series being treated independently.

By the mean diurnal range of declination or horizontal force is to be understood a number formed as follows :—Means of the indications at each separate hour being taken through a month, the difference between the greatest and least amongst these mean values is the monthly mean diurnal range. It should be stated that in the formation of these means, days of great magnetic disturbance were rejected, and also certain other days on which there prevailed a lesser but considerable amount of disturbance (not, however, defined by any strict numerical rule, but estimated according to a general standard formed in the examination of many thousands of photographs). The numbers, both for declination and horizontal force (those referring to the latter being corrected for temperature), for the years from 1841 to 1877 inclusive, are given in the following table. Until the end of the year 1847 the numbers were obtained from two-hourly values, and may therefore be a little small as compared with those for the remaining years which depend on hourly values, but no correction has on this account been applied. The two sets of numbers are taken as forming one uniform series.

TABLE I.—Monthly mean diurnal range of declination and horizontal force as deduced from observations made at the Royal Observatory, Greenwich. (The declination is expressed in minutes of arc; for horizontal force the unit is ·0001 of the whole horizontal force.)

Year.	Declination.												Horizontal Force.											
	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1841	8′·0	10′·8	10′·6	12′·2	11′·1	12′·3	9′·6	11′·1	9′·2	8′·3	6′·8	6′·1	21	14	17	28	26	31	32	26	25	18	17	10
1842	6·5	8·9	10·1	10·2	11·3	9·1	10·4	10·3	10·2	9·6	6·5	5·4	6	8	17	28	26	20	25	27	25	19	7	8
1843	6·5	7·3	8·3	11·4	10·7	12·0	10·6	10·9	11·1	8·3	5·4	5·6	12	7	11	18	28	29	26	28	23	23	12	5
1844	5·7	6·8	9·9	10·6	9·8	10·7	10·1	10·8	9·0	9·0	6·7	5·1	11	13	20	29	29	29	28	30	23	18	9	13
1845	5·8	7·6	8·7	12·4	10·6	9·0	11·8	14·3	10·5	7·9	6·5	6·8	12	11	21	29	30	28	25	30	25	16	11	12
1846	6·3	7·1	11·5	12·2	13·9	11·6	11·7	9·8	9·4	8·7	7·3	5·9	7	8	16	28	33	34	37	39	27	20	14	13
1847	8·4	6·9	10·7	11·4	12·0	11·0	12·7	15·1	13·1	11·5	10·5	8·8	11	9	20	31	31	28	33	26	26	22	18	12
1848	10·7	11·2	13·1	13·5	13·3	14·5	14·5	14·5	12·5	12·0	9·4	7·4	16	23	24	32	30	29	42	35	30	24	15	14
1849	9·0	10·6	13·9	14·0	13·3	14·3	13·4	11·4	12·2	10·3	8·3	5·9	17	21	24	36	34	31	33	21	21	18	14	12
1850	8·0	10·5	11·7	12·3	12·5	13·2	12·4	13·0	13·1	10·2	7·3	5·1	13	20	22	29	26	18	26	24	26	25	21	18
1851	6·5	6·8	8·9	11·3	10·8	10·2	11·2	11·6	10·3	9·1	7·1	6·1	18	20	19	28	27	25	21	17	21	17	11	11
1852	6·2	7·1	10·2	12·8	10·3	11·3	9·2	10·0	10·2	9·5	7·4	6·7	18	20	22	28	27	24	24	20	21	23	18	15
1853	6·0	6·8	8·9	8·9	8·5	10·3	10·1	8·5	8·0	9·1	5·9	5·7	17	17	20	20	24	31	27	20	20	19	16	20
1854	7·6	9·5	9·1	10·8	10·7	9·5	9·2	10·1	8·5	7·4	4·9	4·7	16	19	15	16	20	20	20	14	18	15	15	12
1855	5·9	7·1	9·3	10·3	8·2	8·0	7·4	9·7	8·5	7·9	6·3	4·4	12	12	16	19	19	15	15	19	15	17	15	15
1856	3·1	5·1	5·8	8·1	7·9	8·6	8·5	9·8	8·6	6·5	5·5	4·7	14	14	17	21	16	23	16	20	26	14	11	12
1857	5·7	6·0	7·3	8·3	8·6	5·7	4·7	6·5	8·1	7·1	5·9	5·5	18	15	14	16	17	24	23	20	18	17	19	19
1858	5·6	7·4	10·3	11·8	10·3	8·5	11·2	11·1	11·8	11·5	6·8	6·2	21	26	28	26	25	34	30	18	26	27	23	20
1859	6·2	9·5	11·6	16·4	14·7	13·5	11·5	14·4	12·7	11·2	6·9	6·1	24	29	29	35	24	25	28	25	38	27	26	22
1860	5·7	7·4	15·5	13·0	12·2	14·5	13·4	14·9	11·4	10·8	8·1	7·0	25	24	30	33	31	35	32	34	31	25	20	16
1861	7·5	9·9	12·6	14·6	11·9	12·3	10·9	12·6	10·4	8·8	7·4	7·7	28	26	33	35	23	22	23	23	24	20	18	19
1862	7·5	8·0	8·8	11·3	9·9	12·1	8·2	7·8	9·2	8·7	5·8	4·3	12	13	20	29	28	29	32	35	26	22	12	12
1863	9·1	8·3	10·2	11·0	10·8	10·5	9·3	9·6	10·4	9·5	8·1	7·6	12	13	25	38	38	33	27	26	18	24	10	11
1864	7·8	9·0	10·0	11·3	10·8	10·5	9·6	9·9	10·3	9·2	7·6	6·1	11	11	21	29	31	29	26	27	19	21	11	9
1865	6·5	9·7	9·8	11·6	10·8	10·5	9·9	10·1	10·2	8·9	7·1	4·7	9	9	17	19	25	25	25	27	21	18	12	7
1866	6·4	9·0	8·3	11·1	10·2	10·2	9·8	8·9	7·8	8·1	7·4	4·7	8	9	15	24	21	21	23	21	19	20	11	6
1867	4·9	6·5	8·2	10·1	9·1	9·2	9·3	9·9	8·7	7·8	6·3	4·2	9	9	15	23	25	24	20	21	19	17	10	8
1868	5·7	7·4	9·5	12·4	9·3	10·0	10·7	11·7	9·2	8·5	6·9	5·9	7	9	18	27	22	25	24	24	19	18	11	8
1869	6·6	8·2	10·0	12·2	11·9	13·3	13·2	12·1	11·5	9·0	7·7	5·6	12	14	21	28	30	38	36	29	28	19	15	10
1870	7·4	8·9	13·4	16·0	15·9	14·7	16·1	15·6	13·5	12·1	10·1	6·6	13	22	29	37	40	44	38	40	37	28	22	14
1871	8·0	10·1	13·5	17·7	14·2	15·6	14·2	16·0	12·7	11·5	9·3	7·6	15	18	28	46	40	37	38	34	26	23	19	14
1872	8·9	9·4	12·9	15·0	13·2	14·5	13·2	14·7	13·4	10·1	8·8	7·8	15	16	22	33	34	37	32	30	26	26	19	7
1873	9·6	9·3	13·0	14·7	11·2	10·8	11·4	11·9	10·4	8·7	6·8	5·9	13	15	24	33	28	28	33	25	21	18	11	7
1874	7·4	7·6	10·0	12·0	10·8	10·4	10·9	11·3	9·9	7·1	7·2	4·2	9	15	18	25	27	26	24	22	21	18	9	5
1875	4·3	5·3	9·0	11·1	9·9	9·3	8·2	9·7	7·9	6·8	5·2	4·3	6	6	12	21	20	21	22	18	16	12	9	6
1876	5·0	4·9	7·5	9·2	8·6	9·7	10·2	10·1	7·9	6·6	5·4	4·3	7	8	11	18	19	19	21	20	16	11	11	4
1877	4·4	4·7	6·7	8·8	7·3	9·2	9·4	9·7	7·8	6·1	4·3	3·3	5	6	11	18	20	19	19	17	16	13	8	5
Means.	6·77	8·02	10·24	11·96	11·00	11·10	10·76	11·84	10·26	9·01	7·05	5·78	13·5	14·8	20·1	27·4	26·9	27·3	27·2	25·2	23·2	19·8	14·3	11·6
General mean.	9′·44												20·9											

The observations were partly interrupted during the year 1864 in consequence of the construction of the "magnetic basement," to be alluded to hereafter. The values for 1864, inserted in the preceding table, both for declination and horizontal force, are inferred values, as also are those for horizontal force for January 1847, July 1861, and January 1865. All other numbers (excepting those for the years 1865, 1866, and 1867, which have only recently been deduced from the photographs) may be verified by reference to the several annual volumes of 'Greenwich Magnetical and Meteorological Observations,' 1841 to 1847, to the 'Results of Magnetical and Meteorological Observations' for 1859 and 1867, and to those for the several years commencing with 1868. The numbers for the years 1841 to 1847 will, in some cases, slightly differ from those to be obtained from the several printed volumes, because, in the formation of the magnetic abstracts, until the year 1847, no separation of days of unusual magnetic disturbance was made. And commencing with the year 1868 the numbers for horizontal force, as given by the yearly volumes, require correction for temperature: the correction is, however, very small, and has been here duly applied.

The increase of the numbers in the summer months in both elements is, in the preceding table, plainly apparent. But in order to estimate progressive change, a number for each month is required which shall be free of annual inequality, and such number has been formed as follows. Taking, for example, the month of July, the new number—suppose for declination—is equal to

$$\tfrac{1}{24}[\text{Number for January (preceding)}+\text{Number for January (following)}]$$
$$+\tfrac{1}{12}(\text{Number for February}+\text{Number for March} \ldots\ldots+\text{Number for December})$$

and it represents an annual mean applying to the year whose centre is the middle of July. And similarly for each individual month. The process, which assumes the months to be equal in length, is equivalent to taking the means of each twelve consecutive monthly numbers, and again taking the means of each two consecutive numbers. Thus is obtained, both for declination and horizontal force, a set of numbers practically free of annual inequality. Throughout this discussion the effect of lunar inequalities, as being presumably small, is neglected. The new numbers are contained in the following table.

Table II.—Annual means of the monthly mean diurnal range of declination and horizontal force as deduced from observations made at the Royal Observatory, Greenwich. (The declination is expressed in minutes of arc; for horizontal force the unit is ·00001 of the whole horizontal force. The number standing under January represents the annual mean for the year of which the middle point is January 15, and similarly for other months.)

Year.	Declination.												Horizontal Force.											
	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
	′	′	′	′	′	′	′	′	′	′	′	′												
1841	..	..	..	..	..	..	9·62	9·47	9·38	9·27	9·19	9·07	..	..	..	..	..	..	215	206	203	203	203	199
1842	8·97	8·97	8·97	9·07	9·12	9·07	9·04	8·98	8·83	8·81	8·83	8·93	191	189	189	190	186	181	183	185	182	175	172	176
1843	9·06	9·09	9·16	9·13	9·04	9·00	8·97	8·92	8·97	9·00	8·93	8·83	180	181	181	182	185	186	185	187	193	201	206	207
1844	8·77	8·73	8·65	8·58	8·68	8·70	8·69	8·72	8·71	8·73	8·84	8·30	207	209	210	208	205	207	210	210	210	210	210	210
1845	8·81	9·02	9·23	9·24	9·19	9·26	9·34	9·35	9·44	9·55	9·68	9·92	209	208	208	208	208	209	206	203	200	197	198	202
1846	10·03	9·83	9·61	9·59	9·66	9·65	9·71	9·78	9·74	9·68	9·56	9·46	209	218	222	225	228	230	232	234	236	239	239	236
1847	9·47	9·74	10·11	10·38	10·63	10·89	11·10	11·38	11·66	11·84	11·99	12·18	232	225	219	219	222	223	225	233	240	242	242	242
1848	12·41	12·46	12·41	12·41	12·38	12·27	12·15	12·05	12·06	12·11	12·13	12·13	245	254	259	262	261	261	262	262	261	262	266	268
1849	12·07	11·89	11·76	11·67	11·56	11·44	11·34	11·30	11·20	11·03	10·93	10·86	265	256	246	240	237	236	233	231	230	226	220	211
1850	10·77	10·79	10·89	10·93	10·88	10·81	10·71	10·50	10·23	10·07	9·95	9·76	203	201	205	210	215	221	225	228	226	225	225	228
1851	9·58	9·47	9·30	9·14	9·08	9·12	9·14	9·15	9·21	9·33	9·37	9·40	229	224	219	213	206	199	196	196	197	198	198	198
1852	9·36	9·21	9·13	9·15	9·18	9·22	9·23	9·21	9·15	8·93	8·69	8·57	199	201	202	205	210	215	216	215	213	208	204	205
1853	8·58	8·55	8·39	8·28	8·21	8·10	8·12	8·30	8·43	8·52	8·68	8·74	210	211	210	208	206	207	209	209	208	204	201	195
1854	8·67	8·70	8·79	8·73	8·62	8·54	8·48	8·26	8·17	8·18	8·07	7·91	187	132	178	176	174	170	165	160	158	160	160	158
1855	7·77	7·67	7·66	7·68	7·76	7·81	7·68	7·47	7·24	6·99	6·86	6·88	154	154	155	154	155	156	158	160	161	162	162	164
1856	6·95	6·99	7·01	6·95	6·86	6·83	6·96	7·11	7·20	7·27	7·32	7·22	168	169	174	177	174	171	172	174	173	170	168	169
1857	6·94	6·65	6·48	6·49	6·53	6·58	6·62	6·67	6·85	7·12	7·34	7·53	172	175	172	170	174	180	185	190	201	211	218	226
1858	7·91	8·37	8·73	9·06	9·28	9·35	9·40	9·52	9·65	9·90	10·27	10·67	233	235	238	245	251	253	255	257	259	263	266	262
1859	10·89	11·04	11·22	11·23	11·23	11·23	11·21	11·09	11·17	11·19	10·94	10·88	257	260	268	272	274	276	277	275	274	273	275	283
1860	11·01	11·10	11·08	11·00	11·03	11·12	11·23	11·42	11·39	11·34	11·40	11·29	288	294	295	291	287	283	281	283	285	287	285	276
1861	11·09	10·90	10·76	10·68	10·52	10·52	10·55	10·47	10·23	9·93	9·72	9·63	267	259	251	246	243	244	238	226	215	207	207	212
1862	9·51	9·19	8·94	8·88	8·82	8·61	8·53	8·62	8·68	8·72	8·76	8·73	219	228	233	235	233	228	225	225	227	233	241	247
1863	8·70	8·82	8·95	9·03	9·17	9·39	9·48	9·45	9·48	9·48	9·49	9·49	246	240	233	231	231	230	229	227	225	220	218	208
1864	9·50	9·53	9·54	9·52	9·49	9·40	9·29	9·26	9·28	9·29	9·30	9·30	206	206	207	206	205	205	203	202	199	193	187	183
1865	9·32	9·33	9·34	9·32	9·29	9·21	9·14	9·12	9·02	8·94	8·89	8·85	180	180	181	180	180	179	178	177	177	178	178	175
1866	8·84	8·78	8·63	8·50	8·47	8·49	8·43	8·26	8·16	8·11	8·02	7·93	173	169	166	166	166	165	165	166	166	165	167	170
1867	7·87	7·89	7·98	8·00	7·94	7·87	7·88	7·96	8·04	8·20	8·30	8·34	170	168	168	167	165	166	166	165	166	169	170	169
1868	8·48	8·57	8·66	8·72	8·77	8·86	8·97	9·04	9·09	9·11	9·21	9·46	171	174	175	175	176	177	179	183	186	188	192	200
1869	9·69	9·82	9·93	10·04	10·10	10·13	10·14	10·21	10·37	10·68	11·00	11·22	211	218	224	228	230	232	234	238	244	251	267	266
1870	11·40	11·68	11·90	12·12	12·34	12·48	12·55	12·62	12·68	12·76	12·75	12·72	269	275	283	290	297	302	304	303	301	305	308	305
1871	12·68	12·62	12·60	12·54	12·48	12·49	12·57	12·58	12·52	12·39	12·23	12·14	303	300	293	286	283	282	282	281	277	270	262	259
1872	12·06	11·97	11·93	11·91	11·83	11·82	11·86	11·87	11·88	11·87	11·78	11·54	257	253	251	252	253	250	247	245	246	247	244	238
1873	11·81	11·12	10·87	10·69	10·55	10·38	10·22	10·06	9·86	9·62	9·49	9·46	235	233	229	223	217	213	212	210	207	202	198	197
1874	9·43	9·37	9·33	9·24	9·19	9·14	8·93	8·72	8·57	8·49	8·43	8·33	192	187	186	186	185	183	181	176	170	166	161	156
1875	8·18	8·00	7·85	7·75	7·66	7·58	7·61	7·62	7·55	7·40	7·28	7·23	153	151	147	143	140	140	141	142	143	141	140	138
1876	7·83	7·43	7·45	7·44	7·44	7·45	7·42	7·39	7·35	7·30	7·25	7·19	137	138	138	138	138	138	137	135	134	134	135	135
1877	7·14	7·09	7·08	7·04	6·98	6·89	..	..	..	..	..	..	134	132	131	132	131	130	..	..	..	..	..	..

A very slight examination of the numbers contained in the foregoing table is sufficient to show the existence of distinct epochs of minimum and maximum, the successive epochs of minimum and the successive epochs of maximum being separated by an interval of ten or eleven years, that between minimum and maximum being about four years, and that between maximum and minimum about seven years.

The numbers of the last table were now employed to form the two curves of magnetic diurnal range given on Plate 22.

The numbers for horizontal force indicate variation of northerly force, and are expressed in parts of the whole horizontal force: those for declination imply westerly force equivalent in terms of horizontal force to "horizontal force × sine of number for declination." Or the sine of the number for declination represents the westerly force in terms of the horizontal force. The scales for declination and horizontal force on Plate 22 are therefore so arranged that each minute of arc of declination is represented by ·0003 of horizontal force, and so on in proportion. By imagining the scales to be extended downwards to their zeros, which will be found to coincide, the comparative magnitudes of the diurnal ranges, as well as their variations of magnitude, are more clearly perceived.

One other matter, of no particular significance, may perhaps be mentioned, which is that no account is taken of the slow change in the absolute magnitude of the horizontal force: the effect of neglecting it is simply that the two magnetic curves are, in the later years, slightly depressed, as compared with the earlier years, but without affecting the relation of the curves, each one to the other.

The upper curve on Plate 22, indicating sun-spot frequency, is formed by laying down corresponding numbers taken from the table given by Dr. RUDOLF WOLF in his 'Astronomische Mittheilungen,' No. 42. They are identical with those contained in the table included in Dr. WOLF's "Mémoire sur la Période commune à la Fréquence des Taches Solaires et à la Variation de la Déclinaison Magnétique" ('Memoirs of the Royal Astronomical Society,' vol. xliii., page 199). Dr. WOLF's monthly numbers of relative sun-spot frequency, as determined directly from observation, are given for the years under consideration in other parts of the 'Astronomische Mittheilungen.' But for the purpose of smoothing their accidental irregularities he treats them (so forming the numbers above indicated) precisely as the numbers in our Table I. have been treated, in order to eliminate annual inequality. The magnetic curves and the sun-spot curve are thus strictly comparable. Dr. WOLF's smoothed table terminates with the month of June 1876. The monthly numbers for the succeeding year, to June 1877, are taken from the 'Astronomische Mittheilungen,' No. 46. In laying down the sun-spot numbers on Plate 22, one minute of arc of declination is taken as corresponding to 20·0 in sun-spot number.

An examination of Plate 22 shows immediately the remarkable correspondence between the three curves. It will be noticed that the magnetic curves, in the earlier years, show more sinuosities than in the later years. Now, until the year 1863 the

instruments were situated in the original "upper magnet room" of the Magnetical and Meteorological Observatory—a room subject to the ordinary changes of temperature of an above-ground apartment. But on completion of the new magnetic basement, excavated under the old magnetic building in the year 1864, the instruments were moved to this new room, in which the diurnal range of temperature is, on the average, less than 1°. The advantage of the latter location of the instruments is evident; it has undoubtedly tended to give greater smoothness to the results. That those of the horizontal force instrument should show improvement was likely, but that the improvement should extend also to the results for declination seems to indicate that some general causes of disturbance, other than the direct action of temperature, are avoided by the location in a room kept at a more uniform temperature.

On further comparing together the curves, the general flatness of the 1860 maximum in all three curves, and the opposite sharpness of the 1870 maximum, are noteworthy; the maximum of 1848 occupies in this respect an intermediate position. The rise from the epoch of minimum to that of maximum appears to be particularly rapid. And in all three cases of descent from the epoch of maximum to that of minimum there occurs a greater or less check in the fall of the curve, and sometimes even a second small rise. The near coincidence in the check in the rise in 1869, and in the fall in 1872, shown in both cases in all three curves, seems also remarkable.

If we select the extreme points of the curves we obtain the following epochs of minima and maxima.

TABLE III.—Epochs of minima and maxima of extreme points of the curves.

Phase.	Epoch. Declination.	Epoch. Horizontal Force.	Epoch. Mean Magnetic Effect.	Sun Spot Epoch.	Excess above Sun-Spot Epoch. Declination.	Excess above Sun-Spot Epoch. Horizontal Force.	Excess above Sun-Spot Epoch. Mean Magnetic Effect.
Minimum	1844·3	1842·9	1843·60	1843·5	+0·8	−0·6	+0·10
Maximum	1848·1	1849·0	1848·55	1848·1	0·0	+0·9	+0·45
Minimum	1857·2	1855·1	1856·15	1856·0	+1·2	−0·9	+0·15
Maximum	1860·6	1860·2	1860·40	1860·1	+0·5	+0·1	+0·30
Minimum	1867·5	1867·6	1867·55	1867·2	+0·3	+0·4	+0·35
Maximum	1870·8	1870·9	1870·85	1870·6	+0·2	+0·3	+0·25
Mean Excess at Epoch of Minimum .					+0·77	−0·37	+0·20
Mean Excess at Epoch of Maximum .					+0·23	+0·43	+0·33
General Mean Excess					+0·50	+0·03	+0·27

If we take differences between the successive epochs of minimum and maximum of the mean magnetic effect we obtain the intervals

$$4^{y}\!\cdot95 \qquad 7^{y}\!\cdot60 \qquad 4^{y}\!\cdot25 \qquad 7^{y}\!\cdot15 \qquad 3^{y}\!\cdot30$$

From the sun-spot epochs the intervals are

$$\underset{(1)}{4\overset{y}{\cdot}60} \qquad \underset{(2)}{7\overset{y}{\cdot}90} \qquad \underset{(3)}{4\overset{y}{\cdot}10} \qquad \underset{(4)}{7\overset{y}{\cdot}10} \qquad \underset{(5)}{3\overset{y}{\cdot}40}$$

The order of magnitude of the shorter intervals (minimum to maximum) is 5, 3, 1, for both sets of numbers; the order for the longer intervals (maximum to minimum) is 4, 2, also for both sets of numbers. This shows how complete is the relation between the two phenomena.

The mean of the three intervals from minimum to maximum of magnetic effect is $4\overset{y}{\cdot}17$, and of sun-spot frequency is $4\overset{y}{\cdot}03$; the mean of the two intervals from maximum to minimum of magnetic effect is $7\overset{y}{\cdot}38$, and of sun-spot frequency is $7\overset{y}{\cdot}50$. Whole period of magnetic effect $11\overset{y}{\cdot}55$, of sun-spot frequency $11\overset{y}{\cdot}53$.

It is to be remarked that the extreme points of curves having small irregularities, such as are seen on the diagram, do not quite fairly represent the actual epochs of minimum and maximum. The numbers contained in Table II. (and also WOLF's corresponding numbers) were therefore smoothed, by numerical process, as seemed necessary, and epochs of minima and maxima again selected, with the following result.

TABLE IV.—Epochs of minima and maxima of extreme points of the curve numbers after being smoothed.

Phase.	Epoch.			Sun-Spot Epoch.	Excess above Sun-Spot Epoch.		
	Declination.	Horizontal Force.	Mean Magnetic Effect.		Declination.	Horizontal Force.	Mean Magnetic Effect.
Minimum .	1844·4	1842·9	1843·65	1843·7	$+\overset{y}{0}\cdot7$	$-\overset{y}{0}\cdot8$	$-\overset{y}{0}\cdot05$
Maximum .	1848·2	1848·7	1848·45	1848·2	0·0	+0·5	+0·25
Minimum .	1857·0	1855·3	1856·15	1856·0	+1·0	−0·7	+0·15
Maximum .	1860·4	1860·3	1860·35	1860·2	+0·2	+0·1	+0·15
Minimum .	1867·2	1867·0	1867·10	1867·2	0·0	−0·2	−0·10
Maximum .	1871·0	1870·8	1870·90	1870·7	+0·3	+0·1	+0·20
Mean Excess at Epoch of Minimum					+0·57	−0·57	0·00
Mean Excess at Epoch of Maximum					+0·17	+0·23	+0·20
General Mean Excess					+0·37	−0·17	+0·10

Taking differences between the magnetic epochs of minimum and maximum, as before, we obtain the intervals

$$4\overset{y}{\cdot}80 \qquad 7\overset{y}{\cdot}70 \qquad 4\overset{y}{\cdot}20 \qquad 6\overset{y}{\cdot}75 \qquad 3\overset{y}{\cdot}80$$

the sun-spot epoch intervals being

$$\underset{(1)}{4\overset{y}{\cdot}50} \qquad \underset{(2)}{7\cdot80} \qquad \underset{(3)}{4\overset{y}{\cdot}20} \qquad \underset{(4)}{7\overset{y}{\cdot}00} \qquad \underset{(5)}{3\overset{y}{\cdot}50}$$

The order of magnitude of the shorter intervals (minimum to maximum) and of the longer intervals (maximum to minimum) is the same for both sets of numbers, as before.

The mean, as before, of the three intervals from minimum to maximum of magnetic effect is $4^{\text{y}}{\cdot}27$, and of sun-spot frequency is $4^{\text{y}}{\cdot}07$; the mean of the two intervals from maximum to minimum of magnetic effect is $7^{\text{y}}{\cdot}23$, and of sun-spot frequency is $7^{\text{y}}{\cdot}40$. Whole period of magnetic effect $11^{\text{y}}{\cdot}50$, of sun-spot frequency $11^{\text{y}}{\cdot}47$.

These results generally agree closely with those deduced from Table III.*

It will be noticed (Tables III. and IV.) that in two instances in which the declination epoch of minimum was retarded, that of horizontal force was accelerated, giving a mean magnetic epoch according well with the sun-spot epoch.

It has already been pointed out how closely the intervals between successive magnetic epochs agree with those between the corresponding sun-spot epochs, notwithstanding the difference in magnitude of the different intervals. As related to this it may be here further mentioned that if we add together the successive values of the numbers immediately following Table IV., to form complete periods, we get

For magnetic periods	$12^{\text{y}}{\cdot}50$	$11^{\text{y}}{\cdot}90$	$10^{\text{y}}{\cdot}95$	$10^{\text{y}}{\cdot}55$
For sun-spot periods	12·30	12·00	11·20	10·50

These numbers, whilst showing that the duration of the period has, for several periods, been steadily decreasing, exhibit in a yet more striking manner the correspondence between the two phenomena.

One other circumstance may be mentioned, which is that according to the numbers of Tables III. and IV. (last column), the epochs of magnetic minimum and maximum appear, on the whole, to follow slightly the corresponding solar epochs. Further allusion will, however, be made to this point.

The general circumstance that the diurnal ranges of magnetic elements are subject to an eleven year period, concomitant with that of sun-spot frequency, being thus, by the comparison of the smoothed curves of these phenomena, considered to be sufficiently well established, it seems now desirable to ascertain whether, by comparison of the actual monthly indications, the more fitful changes of the phenomena in any way also correspond. Before proceeding to explain how this has been done it is necessary to premise that whilst (as has been previously mentioned) the magnetic diurnal ranges are subject to an inequality of annual period, of considerable amount and large in comparison with the other changes to which they are subject, the solar spot energy

* The mean periods deduced from Tables III. and IV. are exhibited simply for the purpose of showing the accordance between the mean magnetic period and the mean sun-spot period *as given by the series of observations discussed in the present paper*, without at all implying any correction of the generally received mean value of the sun-spot period, or indeed stipulating for any definite length of period.

(so far as examination of the sun-spot numbers shows) has, as might be expected, no corresponding annual period.* The numbers in Table VI. do indeed yield a small inequality of irregular character (see the means at the foot of the table), but it is, if not wholly accidental, of very small magnitude as compared with the general changes shown by the sun-spot numbers, and is not further considered here. The magnetic diurnal ranges must therefore be now treated in such a way as shall eliminate their average annual inequalities without destroying or reducing their other fluctuations which are the proper subject of comparison with the fluctuations of the sun-spot numbers. To proceed now with a description of the process used. We have at the foot of Table I. the means of the whole of the values standing in each vertical column of the table; also the general mean both for declination and horizontal force. And it will be seen, in the case of either element, that the differences between the general mean and the several mean monthly values give corrections, applicable severally to all the numbers in each of the twelve columns of the table. The corrections to the declination values of Table I. so found are, for each month respectively, as follow: For January +2′·7, February +1′·4, March −0′·8, April −2′·5, May −1′·6, June −1′·7, July −1′·3, August −1′·9, September −0′·8, October +0′·4, November +2′·4, and December +3′·7. The corresponding corrections to the horizontal force values of Table I. for each month respectively are: For January +7, February +6, March +1, April −7, May −6, June −6, July −6, August −4, September −2, October +1, November +7, and December +9. By application of these series of corrections to the values of Table I. the average annual inequality of each element is removed, whilst the accidental variations remain. The numbers found in the way described are given in the next table.

* The annual inequality of magnetic diurnal range varies with locality. For instance, at Hobarton, in latitude 43° south, the annual inequalities, as compared with Greenwich, are reversed, the diurnal ranges being greatest in our winter, and least in our summer. The sun-spot variation, an independent cosmical phenomenon, can have no relation with the constant part of the annual inequality which depends on local geographical position.

TABLE V.—Monthly mean diurnal range of declination and horizontal force as deduced rom servations m e at the Royal Observatory, Greenwich, but cleared of average annual inequality. (The declination expressed in inut of arc; or horizontal force the unit is ·0001 of the whole horizontal force.)

Declination.

Year.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1841	10′·7	12′·2	9′·8	9′·7	9′·5	10′·6	8′·3	9′·2	8′·4	8′·7	9′·2	9′·8
1842	9·2	10·3	9·3	7·7	9·7	7·4	9·1	8·4	9·4	10·0	8·9	9·1
1843	9·2	8·7	7·5	8·9	9·1	10·3	9·3	9·0	10·3	8·7	7·8	9·3
1844	8·4	8·2	9·1	8·1	8·2	9·0	8·3	8·9	8·2	9·4	9·1	8·8
1845	8·5	9·0	7·9	9·9	9·0	7·3	10·5	12·4	9·7	8·3	8·9	10·5
1846	9·0	8·5	10·7	9·7	12·3	9·9	10·4	7·9	8·6	9·1	9·7	9·6
1847	11·1	8·3	9·9	8·9	10·4	9·3	11·4	13·2	12·3	11·9	12·9	12·5
1848	13·4	12·6	12·3	11·0	11·7	12·8	13·2	12·6	11·7	12·4	11·8	11·1
1849	11·7	12·0	13·1	11·5	11·7	12·6	12·1	9·5	11·4	10·7	10·7	9·6
1850	10·7	11·9	10·9	9·8	10·9	11·5	11·1	11·1	12·3	10·6	9·7	8·3
1851	9·2	8·2	8·1	8·8	9·2	8·5	9·9	9·7	9·5	9·5	9·5	9·8
1852	8·9	8·5	9·4	10·3	8·7	9·6	7·9	8·1	9·4	9·9	9·8	10·4
1853	8·7	8·2	8·1	6·4	6·9	8·6	8·8	6·6	7·2	9·5	8·3	9·4
1854	10·3	10·9	8·3	8·3	9·1	7·8	7·9	8·2	7·7	7·8	7·3	8·4
1855	8·6	8·5	8·5	8·3	6·6	6·3	6·1	7·8	7·7	8·3	8·7	8·1
1856	5·8	6·5	5·0	5·6	6·3	6·9	7·2	7·9	7·8	6·9	7·9	8·4
1857	8·4	7·4	6·5	5·8	7·0	4·0	3·4	4·6	7·3	7·5	8·3	9·2
1858	8·3	8·8	9·5	9·3	8·7	6·8	9·9	9·2	11·0	11·9	9·2	9·9
1859	8·9	10·9	10·8	13·9	13·1	11·8	10·2	12·5	11·9	11·6	9·3	9·8
1860	8·4	8·8	14·7	10·5	10·6	12·8	12·1	13·0	10·6	11·2	10·5	10·7
1861	10·2	11·3	11·8	12·1	10·3	10·6	9·6	10·7	9·6	9·2	9·8	11·4
1862	10·2	9·4	8·0	8·8	8·3	10·4	6·9	5·9	8·4	9·1	8·2	8·0
1863	11·8	9·7	9·4	8·5	9·2	8·8	8·0	7·7	9·6	9·9	10·5	11·3
1864	10·5	10·4	9·2	8·8	9·2	8·8	8·3	8·0	9·5	9·6	10·0	9·8
1865	9·2	11·1	9·0	9·1	9·2	8·8	8·6	8·2	9·4	9·3	9·5	8·4
1866	9·1	10·4	7·5	8·6	8·6	8·5	8·5	7·0	7·0	8·5	9·8	8·4
1867	7·6	7·9	7·4	7·6	7·5	7·5	8·0	8·0	7·9	8·2	8·7	7·9
1868	8·4	8·8	8·7	9·9	7·7	8·3	9·4	9·8	8·4	8·9	9·3	9·6
1869	9·3	9·6	9·2	9·7	10·3	11·6	11·9	10·2	10·7	9·4	10·1	9·3
1870	10·1	10·3	12·6	13·5	14·3	13·0	14·8	13·7	12·7	12·5	12·5	10·3
1871	10·7	11·5	12·7	15·2	12·6	13·9	12·9	14·1	11·9	11·9	11·7	11·3
1872	11·6	10·8	12·1	12·5	11·6	12·8	11·9	12·8	12·6	10·5	11·2	11·5
1873	12·3	10·7	12·2	12·2	9·6	9·1	10·1	10·0	9·6	9·1	9·2	9·6
1874	10·1	9·0	9·2	9·5	9·2	8·7	9·6	9·4	9·1	7·5	9·6	7·9
1875	7·0	6·7	8·2	8·6	8·8	7·6	6·9	7·8	7·1	7·2	7·6	8·0
1876	7·7	6·8	6·7	6·7	7·0	8·0	8·9	8·2	7·1	7·0	7·8	8·0
1877	7·1	6·1	5·9	6·8	6·2	7·5	8·1	7·8	7·0	6·5	6·7	7·0

Horizontal Force.

Year.	Jan.	Feb.	Mar.	April	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1841	28	20	18	21	20	25	26	22	23	19	24	19
1842	13	14	18	21	20	14	19	23	23	20	14	17
1843	19	13	12	11	22	23	20	24	21	24	19	14
1844	18	19	21	22	23	23	22	26	21	19	16	22
1845	19	17	22	22	24	22	19	26	23	17	18	21
1846	14	14	17	21	27	28	31	35	25	21	21	22
1847	18	15	21	24	25	22	27	22	24	23	25	21
1848	23	29	25	25	24	23	36	31	28	25	22	23
1849	24	27	25	29	28	25	27	17	19	19	21	21
1850	20	26	23	22	20	12	20	20	24	26	28	27
1851	25	26	20	21	21	19	15	13	19	18	18	20
1852	25	26	23	21	21	18	18	16	19	24	25	24
1853	24	23	21	13	18	25	21	16	18	20	23	29
1854	23	25	16	9	14	14	14	10	16	16	22	21
1855	19	18	17	12	13	9	9	15	13	18	22	24
1856	21	20	18	14	10	17	10	16	24	15	18	21
1857	25	21	15	9	11	18	17	16	16	18	26	28
1858	28	32	29	19	19	28	24	14	24	28	30	29
1859	31	35	30	28	18	19	22	21	36	23	38	31
1860	32	30	31	26	25	29	26	30	29	26	27	25
1861	35	32	34	28	17	16	17	19	22	21	25	28
1862	19	19	21	22	22	23	26	31	24	23	19	21
1863	19	19	26	31	32	27	21	22	16	25	17	20
1864	18	17	22	22	25	23	20	23	17	22	18	18
1865	16	15	18	12	19	19	19	23	19	19	19	16
1866	15	15	16	17	15	15	17	17	17	21	18	15
1867	16	15	16	16	19	18	14	17	17	18	17	17
1868	14	15	19	20	16	19	18	20	17	19	18	17
1869	19	20	22	21	24	32	30	25	26	20	22	19
1870	20	28	30	30	34	38	32	36	35	29	29	23
1871	22	24	29	39	34	31	32	30	24	24	26	23
1872	22	22	23	26	28	31	26	26	24	27	26	16
1873	20	21	25	26	22	22	27	21	19	19	18	16
1874	16	21	19	18	21	20	18	18	19	19	16	14
1875	13	12	13	14	14	15	16	14	14	13	16	15
1876	14	14	12	11	13	13	15	16	14	12	18	13
1877	12	12	12	11	14	13	13	13	14	14	15	14

It may be remarked that if the numbers of Table V. be treated in the way in which the numbers of Table I. were treated to obtain those of Table II., the smoothed values of Table II. would be similarly obtained.

The numbers of the preceding table were now employed to form the two lower curves of Plate 23, taking one minute of arc of declination as corresponding to ·0003 of horizontal force as before. But the coincidence of the zeros of the scales could not, as in Plate 22, be here maintained, because the curves would overlap in such a way as to cause great confusion. The variations of magnitude have, of course, the same relation as before.

The upper curve of Plate 23, that of sun-spot frequency, is laid down, not from the numbers previously used (for Plate 22), but from the monthly values deduced by Dr. Wolf directly from observation, and given in his 'Astronomische Mittheilungen,' Nos. 38, 39, 42, and 46. As these values are probably not so generally available as those used for Plate 22, it has been thought desirable to insert them here. They are the numbers from which, by application of the smoothing process before described, those used in the construction of the sun-spot curve of Plate 22 were obtained.

Table VI.—Numbers expressing the relative sun-spot frequency in each month, as deduced by Dr. Wolf directly from observation.

Year.	Jan.	Feb.	March.	April.	May.	June.	July.	August.	Sept.	October.	Nov.	Dec.
1841	24·0	29·9	29·7	42·6	67·4	55·7	30·8	39·3	35·1	28·5	19·8	38·8
1842	20·4	22·1	21·7	26·9	24·9	20·5	12·6	26·5	18·5	38·1	40·5	17·6
1843	13·3	3·5	8·3	8·3	21·1	10·5	9·5	11·8	4·2	5·3	19·1	12·7
1844	9·4	14·7	13·6	20·8	12·0	3·7	21·2	23 9	6·9	21·5	10·7	21·6
1845	25·7	43·6	43·3	56·9	47·8	31·1	30·6	32·3	29 6	40·7	39·4	59·7
1846	38·7	51·0	63·9	69·2	59·9	65·1	46·5	54·8	107·1	55 9	60·4	65·5
1847	62·6	44·9	85·7	44·7	75 4	85·3	52·2	140·6	161·2	180·4	138·9	109·6
1848	159·1	111·8	108·9	107·1	102·2	123·8	139·2	132·5	100·3	132·4	114·6	159·5
1849	156·7	131·7	96·5	102·5	80·6	81·2	78·0	61·3	93·7	71·5	99·7	97·0
1850	78·0	89·4	82·6	44·1	61·6	70·0	39·1	61·6	86·2	71·0	54·8	60·0
1851	75·5	105·4	64·6	56·5	62·6	63·2	86·1	57·4	67·9	62·5	50·9	71 4
1852	68·4	67·5	61·2	65·4	54·9	46·9	42·0	39·7	37·5	67·3	54·3	45·4
1853	41·1	42·9	37·7	47·6	34·7	40·0	45·9	50·4	33·5	42·3	28·8	23·4
1854	15·4	20·0	20·7	26·4	24·0	21·1	18·7	15·8	22 4	12·7	28·2	21·4
1855	12·3	11·4	17·4	4·4	9·1	5·3	0·4	3·1	0·0	9·7	4·2	3·1
1856	0·5	4·9	0·4	6·5	0·0	5·0	4·6	5·9	4·4	4·5	7·7	7·2
1857	18·7	7·4	5·2	11·1	29·2	16·0	22·2	16·9	42·4	40·6	31·4	37·2
1858	39·0	34·9	57·5	38·3	41·4	44·5	56·7	55·3	80·1	91·2	51·9	66·9
1859	83·7	87·6	90·3	85·7	91·0	87·1	95 2	106·8	105·8	114·6	97·2	81·0
1860	81·5	88·0	98·9	71·4	107·1	108·6	116·7	100·3	92·2	90·1	97·9	95·6
1861	62·3	77·8	101·0	98·5	56·8	87·8	78·0	82·5	79·9	67·2	53·7	80·5
1862	63·1	64·5	43·6	58·7	64·4	84·0	73·4	62·5	66·6	42·0	50·6	40·9
1863	48·3	56·7	66·4	40·6	53·8	40·8	32·7	48·1	22·0	39·9	37·7	41·2
1864	57·7	47·1	66·3	35·8	40·6	57·8	54·7	54·8	28·5	33·9	57·6	28·6
1865	48·7	39·3	39·5	29·4	34·5	33·6	26·8	37·8	21·6	17·1	24·6	12·8
1866	31·6	38·4	24·6	17·6	12·9	16·5	9·3	12·7	7·3	14·1	9·0	1·5
1867	0·0	0·7	9·2	5·1	2·9	1·5	5·0	4·9	9·8	13·5	9·3	25·2
1868	15·6	15·8	26·5	36·6	26·7	31·1	28·6	34·4	43·8	61·7	59·1	67·6
1869	60·9	59·3	52·7	41·0	104·0	108·4	59·2	79·6	80·6	59·4	77·4	104·3
1870	77·3	114·9	159·4	160·0	176·0	135·6	132·4	153·8	136·0	146·4	147·5	130·0
1871	88·3	125·3	143·2	162·4	145·5	91·7	103·0	110·0	80·3	89·0	105·4	90·3
1872	79·5	120·1	88·4	102·1	107·6	109·9	105·2	92·9	114·6	103·5	112·0	83·9
1873	86·7	107·0	98·8	76·2	47·9	44·8	66·9	68·2	47·5	47·4	55·4	49·2
1874	60·8	64·2	46·4	32·0	44·6	38·2	67·8	61·3	28·0	34·3	28·9	29·3
1875	14·6	22·2	33·8	29·1	11·5	23·9	12·5	14·6	2·4	12·7	17·7	9·9
1876	14·3	15·0	31·2	2·3	5·1	1·6	15·2	8·8	9·9	14·3	9·9	8·2
1877	24·4	8·7	11·7	15·8	21·2	13·4	5·9	6·8	16·4	6·7	14·5	2·8
Means.	50·1	53·8	55·4	50·7	53·1	51·5	48·0	53·2	52·0	53·6	51·9	51·4

In laying down on Plate 23 the numbers of the foregoing table, the same relative scale has been employed as on Plate 22, that is to say, one minute of arc of declination is taken to correspond to 20·0 in sun-spot number.

The appearance of Plate 23 is very different from that of Plate 22, the special peculiarity of each month being now fully displayed; at the same time the general eleven year relation is also distinctly apparent. In regard to what may be called minor variations, the correspondence between the curves is not always of a marked character; but in some of the greater and more sudden manifestations of energy, the agreement is very striking. Thus the sudden increase of sun-spot activity in the middle of the year 1847 is accompanied by a no less sudden rise in the declination curve, and, in both cases also, the increased activity is for some time maintained. But no corresponding motion of similar extent is to be seen in the horizontal force trace, although there is a sudden increase, previously, in 1846, and another, afterwards, in 1848, the former of which nearly agrees with a lesser upward movement in the sun-spot curve, and the latter with renewal of activity in the same curve. Again, various correspondences near the epoch of the 1870 maximum are very remarkable; in each of the years 1869, 1870, 1871, and 1872 there are upward motions, the counterparts of which even to some of the smaller bends are to be seen in all three curves. A sudden fall in the year 1873, without upward return, is also shown both in the sun-spot and declination curves, although hardly with equal distinctness in the horizontal force curve. In 1869 the highest point in the sun-spot curve is reached in June, the highest points in the two magnetic curves being reached in July and June respectively; in 1870 the highest point of the sun-spot curve is reached in May, and the corresponding points in the two magnetic curves in July and June respectively; in 1871 the highest point in the sun-spot curve is reached in April, the highest points in the two magnetic curves both occurring also in April. Generally, the variations about the period of the 1870 maximum occur so nearly simultaneously in the three curves that it does not definitely appear that there is any real difference of epoch. The results deduced in Tables III. and IV., from consideration of the epochs of minimum and maximum only, have previously shown the difference to be small. The presumption, in regard to epoch, is that if the various phases of sun-spot and magnetic effect are not entirely coincident, the latter follow the former by comparatively short intervals of time.

It seems worth pointing out that at each of the three epochs of maximum the sun-spot curve exhibits a double maximum; the similarity of the manifestation in the first and third cases, 1848 and 1870, being remarkable. The three curves show in general a much closer agreement during the later years, which seems to confirm the impression produced by the consideration of Plate 22, that a greater accuracy in the magnetic indications has been attained since the instruments have been located in the magnetic basement, that is since the beginning of the year 1865.

A further examination of Plate 23 shows that although the average annual inequality in the magnetic curves has been removed, there yet remains, in some years, a very

sensible inequality (see the declination curve in 1857 and in other years, and the horizontal force curve about the year 1856, and also in other years), not to be explained by direct reference to the sun-spot curve. This suggests, as a matter for inquiry, the possible existence of variation in the annual inequalities. But the general examination of annual inequality is complicated by reason of the existence of the eleven year period. Thus in Table I. for declination the values for the successive months of January, in the years 1849 to 1851 (closely following a maximum epoch), are 9′·0, 8′·0, and 6′·5 respectively, whilst in the years 1867 to 1870 (approaching a maximum epoch), the successive January values are 4′·9, 5′·7, 6′·6, and 7′·4. The annual inequalities have therefore been investigated near to the epochs of minimum and maximum only. Adopting the years 1843, 1856, and 1867 as epochs of minimum, and the years 1848, 1860, and 1870 as epochs of maximum, the means of the numbers in Table I. have been taken at each of these six epochs for periods of three years, both for declination and horizontal force, the middle year in each period being one of those just mentioned. Deducting from the numbers so found for each month, the monthly means for the whole period, 1841 to 1877 (the means of the numbers in Table I.), the annual inequality of each element at the different epochs, as referred to the mean annual inequality, is exhibited. The results for declination are contained in the following table, in which it is to be understood that the number under January 1843 (6′·2) is the mean of the numbers for January (Table I.) in the three years 1842, 1843, and 1844, and similarly throughout the table, the "mean" minimum and "mean" maximum being in each case the mean of the numbers standing in the three columns immediately preceding.

TABLE VII.—Monthly mean diurnal range of declination, and annual inequality of diurnal range as referred to the mean inequality, at epochs of sun-spot minimum and maximum.

Month.	Monthly mean diurnal range at epochs of sun-spot.								Monthly mean for whole period, 1841–1877.	Annual inequality of diurnal range as referred to the mean inequality, at epochs of sun-spot.							
	Minimum.				Maximum.					Minimum.				Maximum.			
	1843.	1856.	1867.	Mean.	1848.	1860.	1870.	Mean.		1843.	1856.	1867.	Mean.	1848.	1860.	1870.	Mean.
January	6′·2	4′·9	5′·7	5′·6	9′·4	6′·5	7′·3	7′·7	6′·8	−0′·6	−1′·9	−1′·1	−1′·2	+2′·6	−0′·3	+0′·5	+0′·9
February	7·7	6·1	7·6	7·1	9·6	8·9	9·1	9·2	8·0	−0·3	−1·9	−0·4	−0·9	+1·6	+0·9	+1·1	+1·2
March	9·4	7·5	8·7	8·5	12·6	13·2	12·3	12·7	10·2	−0·8	−2·7	−1·5	−1·7	+2·4	+3·0	+2·1	+2·5
April	10·7	9·1	11·2	10·3	13·0	14·7	15·3	14·3	12·0	−1·3	−2·9	−0·8	−1·7	+1·0	+2·7	+3·3	+2·3
May	10·6	8·2	9·5	9·4	12·9	12·9	14·0	13·3	11·0	−0·4	−2·8	−1·5	−1·6	+1·9	+1·9	+3·0	+2·3
June	10·6	7·4	9·8	9·3	13·3	13·4	14·5	13·7	11·1	−0·5	−3·7	−1·3	−1·8	+2·2	+2·3	+3·4	+2·6
July	10·4	6·9	9·9	9·1	13·5	11·9	14·5	13·3	10·8	−0·4	−3·9	−0·9	−1·7	+2·7	+1·1	+3·7	+2·5
August	10·7	8·7	10·2	9·9	13·7	14·0	14·6	14·1	11·3	−0·6	−2·6	−1·1	−1·4	+2·4	+2·7	+3·3	+2·8
September	10·1	8·4	8·6	9·0	12·6	11·5	12·6	12·2	10·3	−0·2	−1·9	−1·7	−1·3	+2·3	+1·2	+2·3	+1·9
October	9·0	7·2	8·1	8·1	11·3	10·3	10·9	10·8	9·0	0·0	−1·8	−0·9	−0·9	+2·3	+1·3	+1·9	+1·8
November	6·2	5·9	6·9	6·3	9·4	7·5	9·0	8·6	7·1	−0·9	−1·2	−0·2	−0·8	+2·3	+0·4	+1·9	+1·5
December	5·4	4·9	4·9	5·1	7·4	6·9	6·6	7·0	5·8	−0·4	−0·9	−0·9	−0·7	+1·6	+1·1	+0·8	+1·2
Number of column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

From the numbers contained in columns 1 to 3 have been constructed the lower curves in figs. 1, 2, and 3 respectively on Plate 24; columns 5 to 7 supplying the numbers for the upper curves. Columns 4, 9, and 8 give the numbers from which the lower, middle, and upper curves of fig. 4 are constructed. Columns 10, 11, 12, and 14, 15, 16 contain the numbers used to form the three lower and three upper curves of figs. 5, 6, and 7; and columns 13 and 17 those from which the lower and upper curves of fig. 8 are formed.

The corresponding results for horizontal force, found similarly from the results for horizontal force contained in Table I., are as follows:—

TABLE VIII.—Monthly mean diurnal range of horizontal force, and annual inequality of diurnal range as referred to the mean inequality, at epochs of sun-spot minimum and maximum. (The unit is ·00001 of the whole horizontal force.)

Month.	Monthly mean diurnal range at epochs of sun-spot.								Monthly mean for whole period, 1841–1877.	Annual inequality of diurnal range as referred to the mean inequality, at epochs of sun-spot.							
	Minimum.				Maximum.					Minimum.				Maximum.			
	1843.	1856.	1867.	Mean.	1848.	1860.	1870.	Mean.		1843.	1856.	1867.	Mean.	1848.	1860.	1870.	Mean.
January	97	147	80	108	147	257	133	179	135	−38	+12	−55	−27	+12	+122	− 2	+44
February	93	137	90	107	177	263	180	207	148	−55	−11	−58	−41	+29	+115	+ 32	+59
March	160	157	160	159	227	307	260	265	201	−41	−44	−41	−42	+26	+106	+ 59	+64
April	250	187	247	228	330	343	370	348	274	−24	−87	−27	−46	+56	+ 69	+ 96	+74
May	277	173	227	226	317	260	367	315	269	+ 8	−96	−42	−43	+48	− 9	+ 98	+46
June	260	207	233	233	293	273	397	321	273	−13	−66	−40	−40	+20	0	+124	+48
July	263	180	223	222	360	277	373	337	272	− 9	−92	−49	−50	+88	+ 5	+101	+65
August	283	197	220	233	273	273	343	296	252	+31	−55	−32	−19	+21	+ 21	+ 91	+44
September	237	197	190	208	257	310	303	290	232	+ 5	−35	−42	−24	+25	+ 78	+ 71	+58
October	200	160	183	181	213	240	233	229	198	+ 2	−38	−15	−17	+15	+ 42	+ 35	+31
November	93	150	107	117	157	213	187	186	143	−50	+ 7	−36	−26	+14	+ 70	+ 44	+43
December	87	153	73	104	127	190	127	148	116	−29	+37	−43	−12	+11	+ 74	+ 11	+32
Number of column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

From the numbers contained in this table, figs. 9 to 16, of Plate 24, have been constructed for horizontal force exactly in the same way as those for declination were formed from the numbers of Table VII.

It has been pointed out that, although there is no annual inequality in the sun-spot numbers, in the sense of definite periodical inequality, there is still irregularity, producing, when short periods are considered, sensible inequality, which has the form of annual inequality, the magnitude of which it is necessary to estimate in this more delicate part of our inquiry. We have therefore applied the same treatment to the sun-spot as to the magnetic numbers. The results obtained by such treatment of the numbers of Table VI. are to be found in the following table.

TABLE IX.—Monthly mean sun-spot number, and annual inequality of sun-spot number as referred to the mean inequality, at epochs of sun-spot minimum and maximum.

Month.	Monthly mean sun-spot number at epochs of sun-spot.								Monthly mean for whole period 1841–1877.	Annual inequality of sun-spot number as referred to the mean inequality, at epochs of sun-spot.							
	Minimum.				Maximum.					Minimum.				Maximum.			
	1843.	1856.	1867.	Mean.	1848.	1860.	1870.	Mean.		1843.	1856.	1867.	Mean.	1848.	1860.	1870.	Mean.
January	14·4	8·8	15·7	13·0	126·1	75·8	75·5	92·5	50·1	−35·7	−41·3	−34·4	−37·1	+76·0	+25·7	+25·4	+42·4
February	13·4	7·9	18·3	13·2	95·1	84·5	99·8	93·5	53·8	−40·4	−45·9	−35·5	−40·6	+42·3	+30·7	+46·0	+39·7
March	14·5	7·7	20·1	14·1	97·0	96·7	118·4	104·0	55·4	−40·9	−47·7	−35·3	−41·3	+41·6	+41·3	+63·0	+48·6
April	18·7	7·3	19·8	15·3	84·8	85·2	121·1	97·0	50·7	−32·0	−43·4	−30·9	−35·4	+34·1	+34·5	+70·4	+46·3
May	19·3	12·8	14·2	15·4	86·1	85·0	141·8	104·3	53·1	−33·8	−40·3	−38·9	−37·7	+83·0	+31·9	+88·7	+51·2
June	11·6	8·8	16·4	12·3	96·8	94·5	111·9	101·1	51·5	−39·9	−42·7	−35·1	−39·2	+45·3	+43·0	+60·4	+49·6
July	14·4	9·1	14·3	12·6	89·8	96·6	98·2	94·9	48·0	−33·6	−38·9	−33·7	−35·4	+41·8	+48·6	+50·2	+46·9
August	20·7	8·6	17·3	15·5	111·5	96·5	114·5	107·5	53·2	−32·5	−44·6	−35·9	−37·7	+53·3	+43·3	+61·3	+54·3
September	9·9	15·6	20·3	15·3	118·4	92·6	99·0	103·3	52·0	−42·1	−36·4	−31·7	−36·7	+66·4	+40·6	+47·0	+51·3
October	21·6	18·3	29·8	23·2	128·1	90·6	98·3	105·7	53·6	−32·0	−35·3	−23·8	−30·4	+74·5	+37·0	+44·7	+52·1
November	23·4	14·4	25·8	21·2	117·7	82·9	110·1	103·6	51·9	−28·5	−37·5	−26·1	−30·7	+65·8	+31·0	+58·2	+51·7
December	17·3	15·8	31·4	21·5	122·0	85·7	108·2	105·3	51·4	−34·1	−35·6	−20·0	−29·9	+70·6	+34·3	+56·8	+53·9
Number of column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

The numbers in this table are used to construct figs. 17 to 24, of Plate 24, exactly in the same way in which the declination and horizontal force curves were constructed from the numbers of the two preceding tables.

The scales employed in Plate 24 are relatively the same as before used, that is to say, one minute of arc of declination is taken to correspond to ·0003 of horizontal force, and to 20·0 of sun-spot number.

Considering the diagrams of Plate 24 firstly in regard to the forms of the magnetic curves (figs. 1 to 4 and 9 to 12), it is to be observed (figs. 4 and 12), that a tendency to greater activity is shown in spring and autumn than in summer, the tendency to decline of activity in summer being apparent in many of the separate curves (figs. 1 to 3 and 9 to 11). It appears further that the curves show a tendency to separate more in summer than in winter, indicating variation in the annual inequalities, of periodic character; but in order to determine whether or no they possess this quality, we must consider and estimate the possible influence* of the corresponding sun-spot irregularities (figs. 17 to 20). Comparing these with the magnetic curves, it may be remarked that the unusual rise in the upper curves of figs. 3 and 11 seems (according to what was seen in Plate 23) to be in great part due to the corresponding sun-spot activity shown in the upper curve of fig. 19. It is therefore conceivable that, conversely, the upper curves of figs. 1 and 9 would have ranged higher but for the sun-spot influence indicated by the corresponding upper curve of fig. 17. These points are, however, somewhat better indicated in the supplementary curves (figs. 5 to 7, 13 to 15, and 21 to 23), which show the deviation from the mean annual curve (the middle curves of figs. 4, 12, and 20) in each period for each element.† Thus the influence indicated by the upper curve of fig. 23 is seen in the corresponding upper curves of figs. 7 and 15; the converse influence indicated by the upper curve of fig. 21 having probably operated to lower the middle portions of the corresponding upper curves of figs. 5 and 13. If on making allowance in this way for the accidental sun-spot influences, the upper curves (figs. 5 to 7, and 13 to 15) appear to bend upwards at their middle points, and the lower curves downwards; we have indication that the variation in the annual inequalities of the magnetic elements is really periodic. There is a general accordance in this respect as regards declination, but the agreement for horizontal force is not so good, the lower curve of fig. 13 and the upper curve of fig. 14 being both contradictory. The mean effect is exhibited in figs. 8, 16, and 24. The upper curves of figs. 8 and 16 clearly bend upwards, and the lower curves downwards, and the question now is how far these indications are likely to be modified by consideration of the corresponding sun-spot indications of fig. 24. It is to be remarked that the sun-spot scale was so arranged with regard to the magnetic scales that corresponding motions (see Plates 22 and 23) occupy vertical spaces on the paper of about equal magnitude, and the same relative scale is employed in Plate 24. If anything, the sun-spot scale is somewhat too large. Consideration

* Although, for brevity, it is convenient here and in following sentences to speak of sun-spot influence, the sun-spot phenomena are probably only incomplete manifestations of solar or cosmical action as yet only imperfectly understood.

† The middle curve of fig. 20 represents the average annual inequality or irregularity of sun-spot frequency; and, unlike the corresponding magnetic inequalities (see middle curve figs. 4 and 12), it is nearly a straight line, which explains how reference to it produces so little change in the form of the curves, figs. 21 to 24, as compared with those of figs. 17 to 20.

of the small sun-spot irregularities of fig. 24 would therefore influence, in an insignificant degree only, the forms of the corresponding upper and lower curves of figs. 8 and 16; that is to say, the upper curves would still incline upwards at their middle points, and the lower curves downwards, indicating that after allowance is made for the accidental sun-spot irregularity (in the aggregate small), the magnitude of the diurnal range of declination and horizontal force at the time of a sun-spot maximum, as compared with the value at the time of a sun-spot minimum, is increased more in the summer than in the winter months; or the annual inequality of magnetic diurnal range is increased at the time of a sun-spot maximum, and decreased at the time of a sun-spot minimum, as compared with the average annual inequality. In other words, the annual inequality appears to be increased when the mean diurnal range is increased, and diminished when the mean diurnal range is diminished. If it be desired to examine the question numerically, the materials for so doing may be found in Tables VII., VIII., and IX.; but, having exhibited the results in graphical form, it seems scarcely necessary here to pursue the subject further.

The general conclusions which may be considered to be derived from the whole inquiry are—

1. That the diurnal ranges of the magnetic elements of declination and horizontal force are subject to a periodical variation, the duration of which is equal to that of the known eleven year sun-spot period.

2. That the epochs of minimum and maximum of magnetic and sun-spot effect are nearly coincident, the magnetic epochs on the whole occurring somewhat later than the corresponding sun-spot epochs. The variations of duration in different periods appear to be similar for both phenomena.

3. That the occasional more sudden outbursts of magnetic and sun-spot energy, extending sometimes over periods of several months, appear to occur nearly simultaneously, and progress collaterally.

4. That it seems probable that the annual inequalities of magnetic diurnal range are subject also to periodical variation, being increased at the time of a sun-spot maximum, when the mean diurnal range is increased, and diminished at the time of a sun-spot minimum, when the mean diurnal range is diminished.

Conclusions Nos. 1, 2, and 3 appear to be sufficiently certain, but the evidence in favour of No. 4 is not so decisive.

XVI. *On the Sensitive State of Vacuum Discharges.*—Part II.

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[PLATES 25–29.]

CONTENTS.

Introduction.

IN our previous paper we examined the essential conditions for the existence of sensitiveness in vacuum discharges, and found that sensitive discharges are produced by a rapid succession of discharges of free electricity from one or both of the terminals of the tube, each individual discharge being so small in quantity as to be instantaneous.* The effects which are produced by the approach of a conductor to a tube containing a sensitive discharge were shown to be due to the fact that these individual or component discharges are composed of free electricity, either of a positive or of a negative kind, and that this free electricity during its passage through the tube exercises its ordinary static induction through the space around it; and this, combined with the sudden and instantaneous character of the component discharge itself and the consequent suddenness with which the free electricity appears in any part of the tube, produces impulsive electric action on the sides of the tube and in the space outside it, leading to instantaneous rearrangement of the electricity within the conductor and consequently to discharges from the side of the tube in its immediate neighbourhood, to which these effects are immediately due.

While subsequent researches have confirmed us in our opinion as to the correctness of these conclusions, they have also brought home to us more fully the exact bearing which the examination of the sensitive discharge has upon the general theory of

* It will be understood that the term "instantaneous" is not used in its strict sense of occupying absolutely no time, but in the sense of occupying so short a time that it may be neglected in comparison with the whole period between two discharges.

electricity. It is neither more nor less than the examination of the electric-spark. Each of the separate discharges was shown to pass through the tube in a time so short that it was negligible in comparison with the interval between two such discharges, so that each may be taken as a separate and absolutely isolated discharge; the effect of the rapid repetition of these individual discharges being only to render visible the phenomena which each of them would have presented had it been possible to observe it separately. Thus we are in reality observing the phenomena of the passage of an electric spark through rarefied gas, under circumstances, it is true, which cause great uniformity of conditions throughout the whole series of sparks, but which do not subject any one of the series to any peculiarity of attendant circumstances which would unfit it to be taken as a representative of such electric sparks.

The interest of the investigation rises still higher when we consider it in relation to the continuous discharge. The latter bears precisely the same relation to the electric arc that the sensitive discharge bears to the electric spark; and when we see that these two types of discharge have such striking resemblance, if not identity, of phenomena, we see that a study of them will probably give to us an insight into the relationship of the electric arc and the electric spark which would otherwise be unattainable. Indeed, it is doubtful whether the real intimacy of their relationship would have passed beyond the stage of conjecture but for the light thrown upon them by observing the modifications which each undergoes when the medium through which it passes becomes rarefied. So closely do these modifications resemble one another that there is no doubt that the careful and exhaustive examination of the electric arc in its process of development into the continuous vacuum discharge which we owe to Mr. WARREN DE LA RUE might without much error be taken as giving the development of the electric spark into the intermittent or sensitive vacuum discharge.

Such being the case, we have in the following paper continued our researches into the phenomena of sensitive discharges with the aim of establishing conclusions as to the structure and mechanism of the electric spark. But before so doing it was necessary to extend the conclusions of our former paper to the case of high vacua, the experiments upon which they were based having been exclusively performed with tubes containing gaseous media at a pressure of from about 1 millim. of mercury upwards. The special peculiarities of discharges in high vacua have also required separate examination in order that the phenomena of vacuum discharges might be viewed as a continuous whole. These matters have necessarily occupied much of our attention, but they have not wholly prevented our arriving at certain general conclusions as to electric discharge which we hope may in the future turn out to be of importance.

As we have seen no reason to alter any of the views expressed in our last paper we shall take them as our starting point. So far as the immediate purposes of this paper are concerned, they may be taken to be as follows: that the intermittent discharges which give the sensitive vacuum discharge pass into the tube under the influence of

local tension at the terminal and pass through the tube in the shape of free electricity until they meet with and are neutralised by discharges of an opposite kind, which usually happens at the opposite terminal of the tube. This passage through the tube is not instantaneous, though it is extremely rapid; and the portion of the tube in advance of the discharge is in a state of electric emptiness (or approximately so) while that behind the discharge is filled with the free electricity of the discharge—the discharge spreading out like a pair of lazy tongs and not passing through the tubes in a compact form like a bullet.

The question of the nature and formation of striæ will not here be again discussed; for, in addition to the fact that the explanations given in the previous paper exhaust the subject so far as the authors are at present able to deal with it, the high vacua with which we shall chiefly have to do are not suited to the production of striated discharges. The ideas of positive luminosity and negative dark space will, however, be just as important in this investigation as in the former one, for they are the fundamental elements of all vacuum discharges.

It will be observed that we adhere throughout to the language of the two-fluid theory of electricity. This is done because it not only suffices for our purposes, but is the one that is most naturally suggested by the phenomena. But it must be borne in mind that this is by choice, and for the purposes of clearness and convenience only, and not as expressing any scientific conclusion as to the theory which ought to be preferred. We are not yet sufficiently advanced to be able with any profit to consider the merits of the rival theories.

XIV.—*On the effect of intermittent inductive action of an impulsive type upon continuous vacuum discharges.*

The discharges treated of in our former paper were discontinuous; and the influences whereby the "special" and "relief" effects were produced were more or less directly the outcome of the intermittence of the discharges themselves. In the present section we shall deal with phenomena produced in a different manner. The discharges operated on will be in themselves continuous (using that term in the same sense as in our former paper, *i.e.*, as equivalent to non-sensitive), and the effects will be produced by electrical influences being brought to bear upon them which are due to the intermittence of a wholly distinct electrical system.

The arrangement ordinarily used by the authors for the purpose of examining the phenomena described in this section consists of two large HOLTZ machines, in one of which there are twelve revolving plates of ebonite, and twelve fixed plates of glass, and in the other there are six such plates (Plate 25, fig. 1). The discharge from the larger machine is made to pass through a tube of moderate vacuum, say about 2 millims. pressure, care being taken that no air spark is interposed at any place in the circuit, so that the discharge is neither intermittent nor sensitive. A narrow slip

of tinfoil is then placed round the tube nearly midway between the ends, and is connected by a wire to one of the terminals of the smaller HOLTZ machine. The arrangement is then complete. Its electrical effect is easy to comprehend. When the terminals of the first machine are separated (those of the second machine being closed) the discharge from that machine passes through the tube in the form of an ordinary non-sensitive discharge wholly unaffected by the presence of the tinfoil or its connexion with the terminals of the second machine, inasmuch as these terminals—although serving as the channel through which passes the current generated by the second machine—are electrically inert for all purposes of interference. But when the terminals of the second machine are slightly parted a wholly different state of things is set up. Sparks pass from the one terminal to the other, and each such spark represents an impulsive alteration of electric tension in opposite directions at the two terminals. At the positive terminal it causes a sudden downfall of positive potential; at the negative it causes a sudden downfall of negative potential, or as we might better describe it, a sudden rise of potential. And those changes communicate themselves to all bodies in metallic connexion with these terminals in the form of sudden impulses of negative and positive electricity respectively.

We will assume first of all that the tinfoil on the tube is connected with the negative terminal of the smaller HOLTZ machine, *i.e.*, the machine that is used for producing the interfering system. Then in the interval between the passage of two consecutive sparks between the terminals of the machine, the wire to the tinfoil and the tinfoil itself will be charging up with negative electricity. But when the spark comes this will disappear; in other words, there will be a change equivalent to a sudden rush of positive electricity to the tinfoil. This will be repeated every time a spark passes, which may amount to many hundred times a second if the distance between the terminals of the machine be small. Thus we shall have a like number of sudden impulsive positive chargings-up in each second, the intervals between two consecutive chargings-up being occupied by a gradual and continuous (though of course very rapid) in-pour of negative electricity, the two actions neutralizing each other in each complete period. In other words, we shall have reproduced in the tinfoil ring precisely the action that goes on when we place a similar ring round a tube through which is passing an intermittent discharge with the air-spark in the positive, and connect it with the positive terminal.

Since the electrical action within the tinfoil is identical in the two cases, it follows that the immediate consequences of that action will be alike, however much its ultimate effects may be modified by the difference in the electrical conditions in the interior of the tubes. And this is so. Through the influence of induction each one of these positive impulses causes a similar discharge of positive electricity from the inner surface of the glass beneath it, leaving on that surface a like quantity of negative electricity. This becomes gradually freed during the interval which elapses before the arrival of the next positive impulse—an interval which, though actually occupying

only a very minute period of time, is yet, in all probability, many times as long as the time occupied by the impulse itself. This is precisely the action which takes place within a tube with positive intermittence when the positive special is being produced; the sole difference being that in the present case there are no equivalent discharges of positive electricity synchronously advancing along the tube towards the tinfoil which get satisfied by the negative electricity left behind at the tinfoil, while the positive discharges that are produced there pass out at the negative terminal of the tube.

What, then, is experimentally found to take place in the present case? It is found that if the distance between the terminals, or as it might be termed the *air-spark* of the interfering system, be properly adjusted, we have exactly the same visible appearances as we have in the case of the most perfect form of the positive relief. There is the same complete separation of the positive column, the same sharp bright termination of the truncated portion of it, the same hollow cone of positive discharge separated from the truncated portion of the positive column by the familiar dark space. Nor is it only when the adjustment is very precise that the resemblance exists. With other lengths of air-spark we have other forms of these positive effects, all equally characteristic, although not so readily recognisable as the typical form above described. Just as in producing the positive special a proper adjustment of the length of the air-spark was necessary to produce the typical form, and any excess or defect from this exact length of spark caused the appearances to deviate to a greater or less degree from the typical form, so it is in the present case. And there is so striking a resemblance between the other forms in the two cases that we may express it by saying that they are substantially the same, differing only in the prominence which they respectively give to certain kinds of variance in the details. We thus see that the presence of the *synchronous* discharges proceeding from the positive terminal is not essential to the formation of these typical forms which we have hitherto associated only with the positive special, and its correlative phenomenon the negative relief.

The explanation of this result is very simple. The electricity of the interfering system rushing in accumulated charges to the tinfoil, drives off equal or at all events comparable charges into the tube. These fly to the negative terminal as the goal most suitable for them, thus, as it were, anticipating a portion of the continuous current that would have passed along that portion of the tube in the succeeding interval. This portion of that current is arrested and satisfied by the negative that is left behind on the inner surface of the tube under the tinfoil; and, inasmuch as this negative proceeds from a fixed region (which therefore acts for the moment as a negative terminal or the negative end of a stria), we have the usual phenomenon of a fixed head of luminosity (or, as we may now term it, stria-head), sharp and bright in outline, indicating its reception, such head being separated from the region from which the negative electricity comes by the accustomed dark space.*

* See Phil. Trans., 1879, Plate 16, fig. 10.

But it may be objected that this explanation seems to assume that the sum total of the discharges due to the interfering system is equal to the total of the continuous discharge, an assumption manifestly improbable, seeing that the interfering system is produced by a smaller machine, and, in addition, has an air-spark interposed in its circuit, the effect of which is to decrease the quantity, although it greatly increases the tension of the resulting discharge. The answer to this is, that although considerable change in the length of the air-spark is possible without destroying the typical form of the positive effect, yet in extreme cases we do find indications of just such an imperfect action of the interfering system as would be expected from considerations such as these. The truncated positive column and the hollow cone separated from it by a dark space are seen faintly amid the ordinary positive luminosity of the continuous discharge in the tube, giving the appearance of a superposition of two discharges, the positive luminosity of the one being continuous, and that of the other interrupted in the way above described. But even in the cases where the action is perfect and gives to us the pure typical form of positive effect, it must not be supposed that theory requires an equality between the sum total of the inductive discharges in the tube, and the quantity in the continuous discharge. If the relieving system is sufficient to create the structure we have described, and to maintain it for a portion of the interval between two discharges, it may well be that during the remainder of the time the discharge, finding this structure ready to hand, might use it during the remainder of the interval, seeing that it is of precisely the type that vacuum discharges find most suitable for their propagation. For it must be remembered that the experience we have of stria spaces, and above all of the dark space round the negative terminal, shows us that the normal state of discharge under fixed conditions is an invisible discharge through a space of definite length (depending primarily on the degree of exhaust) lying longitudinally along the path of the discharge between a sharp bright luminosity on the side from which the positive comes, and a hazy luminosity on the side from which the negative comes. All these conditions are present in the structure we have described,* and it may well be that in spite of its being due to foreign influences the continuous discharge makes use of it in the same way as it would make use of a stria of its own creation.

Be this as it may, there can hardly be any doubt of the interpretation of the observed appearances so far as the discharge is concerned. But, in corroboration of it, we may add the following facts gathered from an examination of the discharge in other parts of the tube. If the interference be not too violent, it will be found that the discharge in the portion of the tube lying between the tinfoil and the negative terminal of the tube is sensitive, and gives all the signs of positive intermittence, while that between the tinfoil and the positive terminal is either wholly or almost non-sensitive. This shows that the discharge in the former portion of the tube is

* We shall presently find that this view derives strong corroboration from the behaviour of the continuous discharge when subjected to interference by negative impulses.

carried by bursts of positive electricity, while that in the latter portion is left to go on its even course (Plate 25, fig. 2).

We are brought to the same conclusion if we observe the phenomena produced by placing in circuit with the affected tube two other tubes, one at each end. We then find that the one at the negative end contains a sensitive and the one at the positive end a non-sensitive discharge.* To render the proof quite complete, two pieces of tinfoil were placed on the tube, and were connected, the one to the positive terminal, the other to the negative terminal of the interfering system. When both were so connected it was found that, although they each produced the accustomed effect in the tube, the discharge was not sensitive in the two auxiliary tubes in circuit, but that, if one of the pieces of tinfoil was disconnected from its terminal, sensitiveness appeared in the corresponding one of the tubes in circuit. Thus, when the negative terminal of the interfering system was alone connected with tinfoil on the affected tube, so as to produce positive discharges within the affected tube, it was found that the discharge in the tube at the negative end was sensitive, but this sensitiveness disappeared when the positive terminal of the relieving system was also put in connexion with a piece of tinfoil on the affected tube. The sudden in-rushes of positive electricity from under the one piece of tinfoil were neutralised (so far as the remaining portion of the circuit was concerned) by the synchronous in-rushes of negative electricity from under the other piece of tinfoil, and thus the discharge in the two tubes in circuit with the affected tube remained sensibly uninterrupted.

It is, however, a necessary condition for these latter phenomena that the interference should not be too violent. If the air-spark in the relieving system is increased beyond a certain point, the discharges induced in the affected tube (which we will still assume to be of positive electricity) become so large and violent that they are more than sufficient to satisfy the negative electricity coming from the negative terminal; they consequently no longer go only towards the negative terminal but spread out both ways, and can even be made to pass out at both terminals of the tube. The discharge is then sensitive throughout the whole length of the tube; and the sensitiveness in

* This experiment is very useful in bringing into prominence the essential difference that exists between the passage of electricity through conductors and its passage through gas. It shows us that a current which in some portion of its circuit has to pass through gas can be rendered intermittent in one part of its course while it remains unaffected and continuous, or approximately so, in the other. This is radically different from OHM's law and the general theory of currents through conductors, for these require that at any instant of time the quantity flowing across each section of the conducting circuit should be the same. It is of course true that, whether a portion of the circuit be gaseous or not, the *average* quantity flowing across each section must be the same, but the equality is only in the *average* when taken over a finite and appreciable period of time, and no longer exists at each moment. The tube acts precisely as the air-vessel of a fire-engine. All the electricity that comes into it passes out again, but no longer with the same pulsations. The tube sometimes contains more and sometimes less free electricity, and acts as an elastic or expansible vessel would act if it formed part of the path of a stream of incompressible fluid.

both directions, having been produced by the same interfering impulses, will be of the same character, and will exhibit the same features; thus in the case described above, viz.: positive impulses, we have a double hollow cone at the tinfoil. But the effects are no longer pure, and would not be specially worth notice were it not that some of them resemble very strikingly some of the positive-special effects obtained in the way described in our last paper, with a very long air-spark. It is probable that this same explanation is applicable in both cases. In the present case the power that we possess of altering independently the interfering system without otherwise altering the current affected, enables us thoroughly to test the truth of our explanation. By slightly increasing the air-spark, the limit of the effect of the induction discharges may be made to advance as slowly as we please towards the positive terminal; and in a well striated discharge the authors of this paper have succeeded in putting out one by one the striæ lying between the tinfoil and the positive terminal. The portion of the discharge that remained striated was still continuous and non-sensitive, while that between the striated portion and the tinfoil (as well as that between the tinfoil and the negative terminal) became highly sensitive, showing positive intermittence.

If the wire from the tinfoil be connected to the positive terminal of the machine in the interfering system, we shall of course have negative discharges within the tube, and all the intermittence will be of a negative type. With this arrangement we are able to produce the well known phenomenon of repulsion of the positive luminosity; and, with proper adjustments, to get excellent examples of what we have called the ring-terminal form of negative effect, and of all the forms with which we are familiar in positive-relief or negative-special effects.* Again, as in the case of the positive air-spark, if the air-spark in the interfering system be long, so that the intervals between the impulses are very considerable, or if in any other way we deviate far from the conditions which give us the best effects, we are apt to find mixed results, belonging partly to the original discharge and partly to the interfering system. This fact makes us incline more strongly to accept the explanation given above of the ease and completeness with which the typical positive effect can be obtained, and the strong resemblance which the results of positive interference bear to those of the positive-special through a great range of air-spark. The negative is very inferior to the positive in this respect, and naturally must be so, since the structure set up by the negative impulses, though suited to the needs of their own circumstances, is wholly unlike the structure that the discharge would shape for itself in order to facilitate its passage through the tube; and hence there is a greater tendency for it to reassert its former shape and appear superimposed upon the visible results of the interference.†

* See Phil. Trans., 1879, Plate 17, figs. 12, 13, 14.

† This distinction between the positive and the negative effect of an interfering system is so marked that when the interfering system is of small quantity it becomes very difficult to recognise the effects produced at the tinfoil by negative impulses. The best method of showing the effect of these negative impulses in causing intermittence in the main discharge is to place the finger on the tube between the

Before leaving this part of the subject we must mention that this arrangement enables us to obtain striking examples of artificial striæ. The mode of doing so, viz.: by connecting two rings of tinfoil upon the affected tube to the negative terminal of the machine in the interfering system, will be obvious, after the description given in our former paper of the mode of obtaining artificial striæ by means of the positive-special effect. But in some instances it has not been found necessary to employ two rings. So perfectly does the inner surface of the tube beneath the tinfoil and the interior of the hollow luminous cone perform the functions of a negative terminal, that the portion of the discharge between the positive terminal of the tube and the tinfoil has been found to be organised in every way like an ordinary discharge. When the adjustments were suitably arranged, there might be seen standing at the proper distance from the tinfoil a perfect stria of a flat disc-like shape (Plate 25, fig. 3). This stria was the representative of the negative glow, but it took the ordinary shape of a stria because it was not constrained to assume a distorted form by the presence of any rigid metallic negative terminal, but had in place of it a system almost, if not precisely, what any non-terminal segment of a striated discharge in such a tube would naturally have, viz.: a hollow gaseous structure specially framed to receive the positive discharge which came from the bright head of the stria next to it. Behind this stria came a long dark space, the representative of the ordinary negative dark space; and behind this again the positive luminous column starting from the positive terminal.* It will of course be understood that such perfect effects as these are not common; in order to produce them it is necessary that the magnitudes of the interfering and affected systems should have due relation to one another as well as that the air-spark in the former should be of proper length.†

tinfoil and the positive terminal. The usual indication of negative intermittence will then appear in the form of luminosity proceeding from the inner surface of the tube beneath the finger.

* The significance of this last result as bearing upon the general theory of the striated discharge given in our former paper is very great. In the first place we find the bright termination of the truncated positive column (to which our theory assigned the functions of a stria) actually taking the form of a single perfectly formed stria. In the next place we see that this stria, having just the same advantages that a negative glow has in respect of fixity of conditions, and probably also in respect of the readiness with which negative electricity is supplied to it, is like a negative glow, followed by a long negative dark space. It is in fact in much the same position, electrically and in all other respects, as the negative glow of a tube where the negative terminal is a concave metal plate occupying the whole of the tube. This strongly confirms the view that the negative glow and the negative dark space form a physical unit of discharge (following the language of our former paper) identical in nature and function with the unit composed of any stria and its dark space, but modified by local circumstances; since we find that when we produce a stria under the same circumstances under which a negative glow is formed it is attended by a similar long dark space.

† In connexion with what has been above stated about the proportion between the strength of the main and the interfering discharges, and also with the experiments described in Section IX. of our former paper (p. 201, and Plate 18, fig. 16), the following fact deserves mention. In some cases where two rings of tinfoil were used, and two striæ corresponding to them were formed by connecting them with

It must not be thought that the arrangement described above is the only arrangement by which these effects can be produced. Instead of the Holtz machines, any two sources of high tension electricity giving out continuous currents may be used, providing that it is possible, as must generally be the case, to render the interfering system intermittent, either by the introduction of an air-spark or otherwise. For some purposes it is very convenient to use a current from a large condenser which is filled by a coil for illuminating the tube. Other arrangements might be made to give the rapid impulsive electrical actions which are requisite in the interfering system. But all these modes of arriving at the results would be electrically equivalent, and do not require further mention here.

XV.—*On the standard-tube method of examining intermittent vacuum discharges.*

The results given in the last section have been utilized by the authors of the present paper to furnish them with a new method of testing the intermittence of vacuum discharges, which has been, and promises still to be, of very great utility in the investigation of the mechanism of such discharges. It differs essentially from almost all the modes of testing intermittence described in the former paper in that the luminous effects, by which the nature of the intermittence is recognised, are produced, not in the tube under examination, but in a separate tube which may be chosen on account of its manifesting such effects very readily, and which, being used throughout the whole of the experiments, may be termed the standard-tube. This avoids the great difficulty that otherwise must have been faced in extending the results of our former paper to vacua of a different character to those used in the experiments there described. So long as the nature of the intermittence could only be judged from the appearances in the tube itself, each separate tube presenting phenomena in any way differing from those which had been previously observed and classified, had to be subjected to a separate examination until the observer became familiar with its peculiarities. But with the new method this labour is avoided. Only one tube has to be known thoroughly, and all others, however much they may differ among themselves, are made to express the nature of the intermittence of the discharge that is passing through them in terms of the appearances in the standard-tube.

the negative terminal of the interfering machine, it was found that the formation was rendered more perfect by leading a wire from the positive terminal of the interfering machine to a point on the tube between the rings near the stria due to the first ring (*i.e.*, that nearest the negative end of the tube). This, in fact, supplied synchronous impulses of negative electricity in the manner figured in fig. 16A of the plate above mentioned. The circumstances under which these auxiliary impulses were found useful were doubtless those of an excess of positive thrown in from the surface of the tube below the tinfoil, or as it may also be described a deficiency of negative to satisfy the positive coming up from the second ring.

On the one hand these facts supply an interesting corroboration of what has been said above; and on the other they illustrate the additional power which the present method (viz.: that of an independent source of electricity) furnishes for experiments on interference in general, and especially with discharges not in themselves sensitive.

The way in which this is attained is as follows:—We have seen in the preceding section that rapid electrical impulses of a positive or negative type upon the outside of a tube carrying a continuous discharge, produce within the tube the familiar positive and negative effects respectively; that is to say, the phenomena of positive-special and relief effects. Now it is clearly a matter of indifference in what way these rapid electrical impulses are produced, provided that they have the sharpness and rapidity requisite for producing the luminous effects which are associated with sensitive discharges. Hence, instead of producing these impulses in the manner described in the previous section, it suffices to bring the wire from the tinfoil into contact with the surface of a tube through which a sensitive discharge is passing. If that sensitive discharge be one of positive intermittence, the positive charges that burst through the tube will drive off like discharges of positive electricity along the wire that is in contact with the tube, and we shall have positive effects (that is to say, phenomena of the same class as positive-special effects) in the standard-tube, *i.e.*, the one through which the continuous discharge passes. If, on the other hand, the sensitive discharge be one of negative intermittence we shall have negative effects in the standard-tube, *i.e.*, phenomena of the type of negative-special or positive-relief effects. Thus, the nature of the intermittence in the sensitive discharge will express itself in the appearances in the standard-tube. Nay, it is not only the type of the intermittence in the sensitive discharge that will thus be indicated by the standard-tube. It is clear that the violence of that intermittence will affect the result, and this may be ascertained by examination of the standard-tube; and there is no doubt that all other qualities which a sensitive discharge can possess will in some way or other express themselves in the phenomena which they cause in the standard-tube.

It will be seen from what we have already stated that if the appearances in the standard-tube were thoroughly known it would be possible to read with accuracy the exact nature of the electrical disturbances that are going on in any tube. It is probable that at some stage of the investigation into the mechanism of vacuum discharges it may be necessary to do this, and it will then be desirable to ascertain what form of tube, what kind of gas, and what state of exhaust is best for a standard-tube. But at present this test has only been used by the authors of this paper to establish certain broad fundamental principles; and as this could be done under circumstances very favourable to the action of the test, they have not yet pursued these questions further. They have, however, tried various sources for the continuous discharge that is used in the standard-tube, and on the whole they find that the most sensitive is obtained from a large condenser maintained at a very low state of charge so that the discharge through the standard-tube is not great in quantity. The use of a HOLTZ machine for the source of the discharge in the standard-tube has, however, so great an advantage in the matter of convenience that most of their experiments have been tried with discharges so produced.

The arrangement, therefore, is as follows (Plate 25, fig. 4):—The standard-tube,

which is generally a hydrogen or nitrogen tube of very low resistance, but of considerable diameter and length, is placed in circuit between the terminals of a HOLTZ machine, so that the discharge from the machine passes in a continuous manner through it. A narrow strip of tinfoil is placed round the standard-tube and connected by a wire with a flat piece of metal fixed to the extremity of a glass rod which is held in the hand. To test the intermittence of the discharge in any tube one has only to bring this piece of metal into contact with it.* The intermittent discharges in the tube, of whatever nature they may be, drive off from the flat piece of metal and through the wire in connexion with it electricity of a like sign to that of the pulses of free electricity that pass through the tube. This electricity, rushing to the tinfoil upon the standard-tube, produces by induction discharges within that tube, which are of course recognisable. Thus, as we have already said, positive intermittence in the tube under examination produces in the standard-tube what we have termed positive effects, *i.e.*, positive discharges from the interior surface of the tube either in the form of simple positive luminosity, or in the more perfect form of the hollow cone accompanied by the truncated positive column and the intermediate dark space. Negative intermittence, on the other hand, produces constriction of the positive column, and in cases in which the action is more intense it gives the ring-terminal effect.

Used as above described, this test leaves nothing to be desired so long as the intermittence in the tube under examination is positive in type. It would be difficult to exaggerate the sharpness with which all the details of the positive effects that have so often been referred to in this and our former paper come out in the standard-tube so soon as any positive intermittence appears in the other. But with regard to negative intermittence the case is different. It is, for the reasons given in the last section, very difficult to get good negative manifestations at the tinfoil upon the standard-tube. Even when they do appear they are often confused by the superposition of luminosity due to the discharge in the standard-tube. The best way to deal with the difficulty is to observe the nature of the discharge between the tinfoil and the positive terminal by placing the finger upon the tube as mentioned on page 571. We are then examining a discharge composed of a continuous and a negatively-intermittent portion superimposed one on the other. The latter will, of course, cause positive luminosity to appear within the tube beneath the finger, just as much as though the former was not present, and thus the presence of negative intermittence in the original tube will be demonstrated. With this mode of procedure the standard-tube method is probably as delicate a test for the existence of negative as it is of positive intermittence, though it still labours under great disadvantages when it is desired to learn something

* If it is desired to augment the effect on the standard-tube a piece of tinfoil of tolerably large size can be placed upon the tube that is under examination, and contact can be made with it instead of with the naked surface of the tube. If it is desired to diminish the effect it can be done by allowing the piece of metal affixed to the glass rod to remain at a suitable distance from the tube under examination without coming into actual contact with it.

more about the nature of the intermittence, for we have no longer the power of detecting slight changes in the effects produced in the standard-tube as is the case when the sharply defined positive effects are under examination.

It is obvious that if the standard-tube be at a considerable distance from the tube that is to be tested, the relieving system, composed of the tinfoil and the wire, will have considerable capacity, and thus the impulse on the tinfoil will lose proportionally in intensity. It is necessary, therefore, that they should not be too far apart. But there is a certain difficulty in placing them near together, for the tube containing the intermittent discharge acts by induction on the discharge in the standard-tube and makes it behave as though it were intermittent, even when there is no metallic connexion between the surfaces of the two tubes. But this is not of very serious importance, for the effect is of the same nature as that produced by the tinfoil, and differs from it only in being feebler and more diffused.

It is often useful to place a second ring of tinfoil round the standard-tube unconnected with the tube that is being examined (Plate 26, fig. 5). By touching this we shall discover the nature of the intermittence produced in the standard-tube by the inductive discharges which are caused (either through the medium of the tinfoil, or directly as mentioned in the last paragraph) by the influence of the tube under examination. If the intermittence in the standard-tube be of a positive type, then it is clear that such also was the original intermittence, and *vice versâ*. In cases where the electrical disturbances are very violent this is a useful addition to the tests already described, as the results it gives are of a milder type than those produced at the ring of tinfoil which is in direct connexion with the other tube.

This method is not confined to cases in which the electric pulsations in the system under examination are actually in the form of discharges. It enables us to contrast any systems in which there are electric variations of a suitable type. Thus, for instance, we can by its aid compare the relative intensity of the disturbances at the terminals of the tube, at a point on its surface and at an intermediate terminal not in connexion with any portion of the external circuit. The effect at the air-spark terminal is always by far the greatest. But no general rule holds as to the others, save that there is but little difference between the intensity of the disturbance at an intermediate terminal (wherever it be situated) and that upon the surface of the tube. Nor is there, as a rule, much difference between the disturbance at the non-air-spark terminal and an intermediate terminal. On the contrary, it would seem that tube and terminals are all on much of an equality so soon as the electricity has once launched itself into the tube from the air-spark terminal.

XVI.—*On the Leyden-jar effect of vacuum tubes.*

We are about to touch on a subject which merits a much more complete investigation than we have as yet given it, not only because the influence of this property

of vacuum tube ramifies in all directions throughout the whole of the phenomena which they present, but also because there is no doubt that an attentive study of it would throw great light on the nature and capabilities of a vacuum tube considered as a system capable of being affected by electrical influences.

It is well known that a vacuum tube after being used to convey a discharge is often strongly charged; the gas within the tube, therefore, or the interior surface of the glass can retain a considerable quantity of free electricity. And further than this, a vacuum tube is capable of giving considerable relief to a sensitive discharge in another tube, just as a conductor would, showing that the inductive discharges which take place within the tube have much the same effect that the displacement of electricity in a conductor would have. This is rendered evident by the inductive discharges that become visible in the neighbouring tube, which must represent relief given to the sensitive discharge. The exact amount of this effect is difficult to measure, but it is plain that the result is to make the tube act for the moment much as a Leyden jar would act in which the inner tinfoil was in connexion with earth, for the superfluous electricity on the inside of the tube, though it cannot be driven out of the tube, is driven off from the inner surface of the jar, and remains for the moment as a charge in the rarefied gas within the tube.

The capacity of a vacuum tube to act as a relieving system is immensely increased by passing a continuous discharge through it. This is seen in experiments with the standard-tube. If we mark the effect produced on the tube under examination by connecting a piece of tinfoil upon it with a piece of tinfoil upon the standard-tube (which is of course a relief-effect) and then suddenly stop the discharge in the standard-tube, we shall see an immediate diminution of the effect. In positive-relief, for instance, we have found the discharge-effect pass into mere repulsion, a change which as we have shown in our previous paper indicates a diminution in the capacity of the relieving system. And in some cases this difference is very marked, showing that the passage of the continuous discharge greatly facilitates the redistribution within the tube of the free electricity which is developed by the action of induction from the discharges that are passing through the sensitive tube. That this should be the case is not surprising when one has regard to the phenomena which have been described in Section XIV.

The attention of the authors of the present paper was drawn specially to the subject of the capacity of vacuum tubes as receivers of electricity by a very peculiar phenomenon. They noticed that on touching tubes containing a sensitive discharge, severe shocks were sometimes experienced, while at other times no such shocks were felt. It was found that this was not due to any greater length of air-spark in the former case, for the two effects would occur with the same length of air-spark. On examination it was found that the severe shocks were felt when that terminal of the machine was connected to earth which was separated from the tube by the air-spark interval, but not otherwise (Plate 26, fig. 6). In other words, we may say that

it occurred when the earth connexion was *behind the air-spark*, reckoning from the tube.

A little consideration suffices to explain how this is caused. Let us suppose that the air-spark is in the negative, so that the positive terminal of the machine is in metallic connexion with the positive terminal of the tube, while the air-spark is situated between the negative terminal of the tube and that of the machine. Then, as we have said, the strong shock will occur when the negative terminal of the machine is put to earth, and not otherwise.

Let us consider how the discharge takes place. The negative terminal of the machine and all bodies metallically connected with it are maintained at potential zero by means of the earth connexion. Hence the negative side of the air-spark interval is at potential zero, and would remain destitute of free electricity if free positive electricity did not collect upon the surface of the metallic body forming the other side of the interval. But, as such is the case, free electricities of opposite signs accumulate on the two surfaces until, through their tension and their inductive action upon one another, the discharge is brought about. Now, how has this positive electricity accumulated upon the positive side of the interval? It is not due to the influence of the negative side, for that, as we have seen, would have been inert. It must, therefore, have become charged solely through the accumulation of electricity at the positive terminal of the machine which has flowed from thence into the tube and, passing through it, has emerged at the negative terminal and charged the positive side of the air-spark interval. In this process the tube has become fully charged. When a discharge comes, the tube, like a Leyden jar, will empty itself, and thus the amount of electricity passing will be augmented by the whole of the charge which the tube can hold. The shock, therefore, which is the inductive effect of this sudden discharge of the large stock of accumulated electricity, will be proportionately severe.

Compare this with the case in which (with the same arrangement of air-spark) the positive terminal is to-earth. The tube is now kept at potential zero,—or, more correctly, its positive terminal is. The active cause of the discharge is now the electricity accumulating on the negative terminal of the machine, and, although it no doubt produces a displacement of the electricity in the system on the other side of the air-spark interval, it is clear that it cannot cause the tube to become charged, connected as it is with earth at the positive terminal. Hence, when the discharge takes place, it is only the interchange of the free electricity on the two sides of the air-spark interval, and there is no accumulated store in the tube to take part in the discharge. The shock is accordingly very much smaller.

There are two very interesting experiments which serve to confirm the justness of these conclusions. The first is an application of an experiment described in Section XIII. of our previous paper (Phil. Trans., 1879, p. 220, and Plate 20, fig. 26). In order to ascertain the condition of a tube in advance of the discharge, a piece of tinfoil was fastened to a glass rod which was then laid along the tube, the piece of tinfoil

resting upon it at a spot near to the non-air-spark terminal of the tube. A piece of tinfoil was then laid on the tube near the air-spark terminal and connected with the former piece by a fine wire. The consequence of this is, as we know, that the piece near the air-spark terminal derives relief from the other piece ; and, in the experiment above referred to, it was shown that the efficacy of the system to give relief was greater as thus arranged than it would be if the glass rod with the tinfoil were moved into a position at right angles to the tube, so that the tinfoil would be as far as possible from the tube. The conclusion deduced from this was that the further end of the tube was in the act of charging up in the contrary sense to the air-spark pulse at the moment that the discharge occurred. Now if this be tried with the Leyden-jar arrangement (*i.e.*, with the earth connexion behind the air-spark) it will be found the effect is intensified, and that the tinfoil gives very much greater relief when upon the tube than in any other position ; showing that the further end of the tube is rapidly charging up at the time of the air-spark discharge. Thus if the air-spark is in the positive the relieving tinfoil will supply more negative when on the tube than otherwise ; in other words, the negative end of the tube must be receiving a rapidly growing negative charge. And this is exactly what must be the case if the explanations given above are correct.

The other experiment is a very remarkable one. We take a tube of moderately high exhaust through which is passing a discharge of considerable quantity, with the air-spark in the negative, and place, as before, the earth connexion behind the air-spark by connecting the negative terminal of the machine to earth (Plate 26, fig. 7). The consequence of this arrangement is that we get a violent negative intermittence. If we place upon the tube a piece of tinfoil and connect it with earth we shall get the usual negative-relief effects, *i.e.*, positive luminosity on the inner side of the tube beneath the tinfoil. But if we withdraw the earth connexion from the tinfoil, and hold it at long sparking-distance from it, bright phosphorescence will appear opposite the tinfoil. The explanation of this is that during the interval between two air-sparks the tube is rapidly charging-up with positive electricity. Rapid, however, as this charging-up is, it is not of a sufficiently impulsive type to give rise to a relief-discharge from the tinfoil sufficient to produce luminous effects. If, however, we hold a wire connected to earth at such a distance from the tinfoil that the accumulated effects of this charging-up are able at intervals (either once, twice, or even more frequently in each interval of the main air-spark) to draw a spark from the earth wire, we have corresponding to each such spark an impulsive relief-discharge of negative electricity from under the tinfoil which produces the phosphorescence observed.

This Leyden-jar effect is found with all lengths of air-spark and in all kinds of tubes. When we come, however, to tubes of very high vacuum the whole becomes so complicated by other considerations that, though there are traces of the effect, it is so masked as to require special examination to detect it. In fact, the tube represents in many cases a much greater resistance than the air-spark interval, so that the above

reasoning would not apply without great modifications. It is true that there is a similar contrast between discharges which give strong shocks and those which do not do so, but that may arise from wholly different considerations, which will be noticed in due time (see Section XXVII.). It is partly on account of the great importance of distinguishing between this latter phenomenon, which is peculiar to high-vacuum tubes, and the phenomenon to which this section has chiefly been dedicated, which is common to all tubes, that has made us examine the latter.

XVII.—*On the phenomenon of phosphorescence in vacuum tubes.*

Before entering into the investigation of vacuum discharges through tubes of high exhaust, it will be advisable to consider from all points of view a phenomenon which is a very marked accompaniment of discharges in high vacua, though, as we shall see, by no means confined to them. We allude to the well-known phosphorescence which appears on the inside of the tube, especially in the neighbourhood of the negative terminal. This phenomenon required only incidental notice in our former paper, not because it does not belong to the sensitive state (for we shall find that it is quite as marked a feature in the case of the intermittent as in that of the continuous discharge), but because it is of comparatively rare occurrence in connexion with the low exhausts to which our researches were there confined. But the remainder of the present paper will be chiefly devoted to the consideration of discharges in high vacua, and it is therefore necessary that we should start with a clear understanding of the nature and laws of this phenomenon.

The immediate cause of phosphorescence in vacuum tubes is ascertained beyond controversy. GOLDSTEIN and CROOKES have shown that it arises from streams of molecules or small particles of some kind which pour off from the negative terminal during the continuance of the discharge. These while moving at high velocities strike against the glass and by their impact impart sufficient energy to the glass to render it luminous, and also to raise its temperature very considerably. The peculiar colour of the light thus generated has been conclusively shown to depend solely on the composition of the glass; it is indifferent to the substance of which the terminal is composed; in fact, as will be seen later, the very glass itself may serve as such a terminal. Its configuration undoubtedly depends upon that of the terminal as well as upon that of the surface of the glass upon which it is actually formed; but as the phosphorescent light has no direct connexion with the luminosity of the gas itself, it gives us no direct information as to what is going on within the tube, save so far as it testifies to the existence of these streams of material particles coming from the negative terminal.

The relation of this to other electrical phenomena seems not to have been clearly understood, and it has been supposed by some to indicate that the gas within the tube is in some special and peculiar state differing widely from the ordinary gaseous

state in its physical qualities, and especially in the length of the free path of its molecules and the frequency of the collisions between them. As we are of opinion that there are no sufficient grounds for such a supposition, but that, on the contrary, the phenomena are compatible with the ordinary molecular theory of gases, we shall proceed to state our views upon the subject, and the experimental facts upon which these views are based.

It is by no means an unusual phenomenon to find streams of particles driven off from the surface of bodies highly charged with electricity. The very familiar phenomenon known as the "electric wind" is an instance of this kind. In this case particles of air are driven off from the pointed terminal of an electric machine or a highly charged Leyden jar with such force that they produce a perceptible wind; and their reaction can be made to turn a vane much in the same way as CROOKES' electric radiometer is turned by the electric discharge in a vacuum. This phenomenon is admitted to be due to the repulsion between the highly charged conductor and the neighbouring particles of air which have become charged in a like sense by coming in contact with it.

There are, it is true, many peculiarities of this "electric wind" which prevent our accepting it as an exact analogue of the molecular streams which produce phosphorescence in high vacua. In the first place it is common to both the positive and the negative poles, and is indeed more easily produced at the former than at the latter. Then, again, the velocities of the particles seem to be much less than in the case of phosphorescence, although the whole pressure produced by them is relatively considerable owing to the greater density of the medium affected. But these are not differences which weigh very heavily in the consideration of the matter as a whole. The typical peculiarities of the negative terminal only begin to manifest themselves as the pressure of the surrounding gaseous medium is lessened, and it is only in very high vacua that they attain their full proportions. Moreover, the lower velocity of the particles is exactly what we should expect to find in a medium so much denser than the high vacua in which phosphorescence is usually observed.

These are not the only cases in which, at the ordinary atmospheric pressure, we have phenomena of this nature. It is well known that in the electric arc there is a constant stream of particles of carbon from the positive to the negative pole. And of late a method of coating glass with platinum has, we believe, been invented and carried out, as a commercial process, which depends on a like principle. The platinum is deposited from an electrode of that metal held near the glass and connected with some source of high tension electricity. And doubtless if experiments were made upon the streamers which are seen between the two poles of an electric machine when they are beyond striking distance some very closely analogous phenomena might be observed.

But if these phenomena fail in being strictly analogous to the molecular streams that produce phosphorescence in that they show no special preference for the negative pole, or are even characteristic of the positive pole rather than of the negative at

ordinary atmospheric pressures, the analogy becomes much more strict when we come to discharges in rarefied gas. It has long been known that if small metallic particles are lying loose on the negative terminal of an exhausted tube, a strong electric current will drive them along the tube towards the positive terminal. A convenient form of the experiment is obtained by enclosing some platinum black in the tube. If this be shaken down to the negative end of the tube, so as to lie upon the terminal, a shock from a coil of fairly large size will drive it along the tube in spite of the great specific gravity of the particles of which it is composed. No such phenomenon will be seen if the platinum black be placed on the positive terminal.* And all who have used vacuum tubes with platinum terminals will remember how commonly it is the case that the portion of the tube round the negative terminal becomes coated with a thin film of platinum due to the small particles of the metal that are driven off from the terminal by the electric discharge.

In the opinion of the authors of this paper, there are no sufficient grounds for looking upon the molecular streams which produce phosphorescence in vacuum tubes as anything other than or different from the phenomena above referred to.†

The hypothesis that the existence of the molecular streams that produce phosphorescence depends upon some special modification of the gaseous structure of the medium through which the discharge passes, apparently owes its origin primarily to the belief that this phosphorescence is peculiar to tubes of high exhaust, and secondarily to the belief that in the tubes in which it is found to occur the particles which cause it (and which are presumably molecules of the gas within the tube) are exempted from the usual interference which gas molecules exercise upon one another in their motions through the space which contains them. Both these beliefs we consider to be unsupported. So far as our observation goes, phosphorescence can be produced in almost any vacuum tube. The sole condition is that the violence of the discharge from the negative terminal should be sufficiently great, taking into consideration the form and size of the tube, and of its negative terminal and its degree of exhaust. This degree of violence is attained in the case of tubes of very high exhaust without any special arrangements, and a continuous current sufficiently

* Plumbago, lampblack, and finely divided steel have been used with success in this experiment. Lycopodium and sand, and apparently non-conductors in general, are not similarly affected. These experiments were suggested some years ago by Mr. WARD.

† A very remarkable confirmation of the theory that these molecular streams are identical in their nature with the phenomena above described is obtained from the fact that, as PLÜCKER has shown, the metallic deposit in the neighbourhood of the negative terminal will follow the magnetic curves if the deposit be allowed to take place in a magnetic field, thus showing that the particles of platinum are affected by a magnet in precisely the same way as the particles in these molecular streams. Other experimental facts which in the opinion of the authors of this paper conclusively demonstrate the substantial identity of the two phenomena will be given in a subsequent portion of this paper (see page 648), but it is not convenient to insert them here as it would require us to anticipate in some measure the results of several of the sections that follow.

strong to pass through such tubes appears always to excite phosphorescence to a greater or less distance from the negative terminal. In tubes of a less degree of exhaust this is not the case, but the difficulty can be got over by intensifying the violence of the negative discharge by means of the introduction of an air-spark. This divides the current into isolated individual discharges of great violence; and if the air-spark be taken long enough this process generally results in the production of phosphorescence in the neighbourhood of the negative terminal. If this does not succeed, yet more violent methods must be resorted to; and if even all means fail to produce phosphorescence we trust to be able to show that it is solely on account of the interference of the surrounding gas; and that the absence of phosphorescence is not due to the non-existence of the requisite molecular streams, but to their not travelling with sufficient velocity to enable them to impinge on the glass with the requisite violence. In other words, we hope to establish experimentally that these molecular streams are present in all vacuum discharges, and that their behaviour under various conditions of vacuum and discharge, so far from pointing to any unusual state of the gaseous medium in which they occur, shows a perfect continuity of variation throughout the whole of the wide range of circumstances under which they appear.

It will be convenient in demonstrating these propositions to show first of all that phosphorescence can be produced in vacuum tubes, in which it would not otherwise occur, by increasing the violence of the discharge. By far the best example of this is obtained in the way to which we have just referred, viz.: by introducing an air-spark. If a tube of very moderate exhaust be placed in circuit with a large HOLTZ machine, and an air-spark of considerable size be introduced into any part of the circuit, it will generally be found that phosphorescence appears in the neighbourhood of the negative terminal, even though there was not the slightest appearance of it while the discharge was passing continuously. This is clearly due to the fact that intermittent discharges are necessarily much more violent during the very short period of time which they occupy than are continuous discharges; and hence the velocity imparted to the molecular streams is sufficient to make them impinge on the glass with the velocity requisite to produce phosphorescence. It will be noticed that we have here no change in the degree of exhaustion, but only in the violence of the discharge; and it is further to be remarked that this increased violence can be obtained, either directly as in the case of the negative air-spark, or by way of response to a violent positive discharge in the tube, as in the case of the positive air-spark.*

But it is not only thus that the introduction of an air-spark can be made to produce phosphorescence in a vacuum tube. It is possible to obtain similar phosphorescence in other portions of the tube than those immediately surrounding the negative terminal, and it is by these methods that we can show most clearly that the pheno-

* This fact alone is a sufficient warning against viewing the emission of these molecular streams as indicating in any way a special direction of the electric discharge. It is, in our opinion, fatal to the idea that these molecular streams prove that an electric discharge in a vacuum tube is a "negative flow."

menon does not depend on the existence of any specially high degree of exhaust. If a strong current be made to pass through any tube of not too poor a vacuum to be capable of giving a positive luminous column, and a considerable air-spark be introduced in the positive portion of the circuit, so as to cause the positive electricity to pass through the whole length of the tube in strong charges, the contact of the finger with the tube will, in almost all cases, cause a bright patch of phosphorescence to appear on the opposite side of the tube (Plate 26, fig. 8). The reason is obvious. The interior of the tube beneath the finger acts as a negative terminal, *pro tem.*, to the advancing positive electricity, and in the act of thus sending off negative electricity it sends off also the streams of molecules that accompany negative discharge, and thus produces phosphorescence in the tube. This will be the case even in tubes which are full of bright luminosity, and the molecular streams will drive through this luminous mist without necessarily dispersing it: a phenomenon which is in itself a sufficient proof that they do not require a specially high degree of exhaust, since we shall find that this is incompatible with the existence of bright luminosity.*

There is yet another arrangement which enables us still further to increase the violence of the negative discharge so as to obtain phosphorescence in tubes in which it does not ordinarily appear. This is the arrangement referred to in our former paper, on page 170 and Plate 15, fig. 2, and is due to Mr. WARD, our assistant. It consists in bringing a wire from the positive terminal of a HOLTZ machine to a small tinfoil patch on the outside of a tube, one of the terminals of which is in metallic connexion with the negative terminal of the same machine. On separating the terminals of the machine to a distance of, say, half-an-inch, a stream of violent sparks will of course pass from the one to the other. Each of these will cause, at the positive terminal, and therefore at the tinfoil of the tube, a sudden downfall of positive tension or rise of negative tension, and thus will be equivalent to an impulsive negative charge there. As we have seen, this will make the inside of the tube act as a negative terminal for the instant, and with this there will be the accompanying molecular streams, and phosphorescence will appear on the opposite side of the tube. Inasmuch as in this way we are able to attain to a much greater degree of violence in the individual discharges, we are by it enabled to demonstrate the existence of phosphorescence due to negative discharge in tubes in which all other methods fail to show it. We shall now give a few experiments to show the very wide range of exhaust through which, by some or all of the above methods, we have been able to obtain it.

We first of all tried the tubes which were most frequently used by us in our former investigation, viz.: tubes of a moderate exhaust representing some 1 or 2 millims.

* This experiment is interesting also for another reason. It shows that the action to which the emission of these streams is due must take place on the bounding surface of the solid and gaseous matter, for in this case the electricity can only come from the surface of the glass or the gaseous film in immediate contact with it, and not from the interior of the solid body, as is the case with the negative terminals of tubes.

pressure, and giving an amorphous positive column, the exhaust not being sufficiently high to give stratification. Every one of these readily gave phosphorescence under any one of the foregoing methods. As it was of course unnecessary to examine tubes of higher exhaust, we then set to work to examine tubes of lower exhaust. Of these we had but few instances; for if the exhaust be much less than that described above, the special phenomena of vacuum discharges are very imperfectly manifested, and hence such tubes would be useless as vacuum tubes. One, however, was found about an inch and three-quarters in diameter, containing vapour of bromine, in which the exhaust was so moderate that the luminous discharge consisted of a thin red line extending from the positive terminal up to within a very short distance of the negative. This, on being subjected to the last of the tests above described, gave splendid phosphorescence.

By way of a crucial experiment a tube of about 1 inch external diameter was taken which had a cavity at one end filled with potash. In its ordinary state this tube is one of very great resistance, and displays splendid phosphorescence throughout almost its whole length (Plate 26, fig. 9). By heating the potash with a spirit lamp gas is driven out from it into the tube, and thus the degree of the exhaust can be lowered to any desired extent. This was subjected to the method last described, and the potash was heated until it melted, when we were compelled to desist from fears for the safety of the tube. Very bright green phosphorescence was manifested throughout the whole of the time, and it was clear that we had not yet reached the limits of pressure at which it could have been obtained. It was of course difficult to estimate exactly the pressure of the gas in the tube at the termination of the experiment, but from the resistance of the tube, and the appearance it gave when a current was sent through it, we judged it to be equal to at least half-an-inch of mercury.

Feeling the importance of demonstrating conclusively that these molecular streams are not dependent on the existence of any specially high state of exhaust, we next took a tube of about 2 inches diameter which contained nitrogen at a pressure of about 2 millims., and permitted air to enter slowly through the stopcock which closed it. The arrangement for producing the phosphorescence was that last described. We found that it gave marked phosphorescence until air had been entering for a considerable time. When at length the pressure became so great that no phosphorescence appeared, we examined the tube by passing a current through it, and found that it gave no luminous phenomena save in the immediate neighbourhood of the two terminals: an appearance which is well known to signify a very moderate exhaust. The phosphorescence faded gradually as the air entered, and when at last we decided that the phosphorescence had disappeared, it was merely because it had faded to such an extent that we could no longer certainly recognise its presence by the eye. But there was no sudden or discontinuous change marking the exact epoch of its disappearance, nor was there anything to lead us to believe that there had been any sudden cessation of its existence at the moment when it ceased to be visible.

These experiments, which could be multiplied to any extent, show that phosphorescence can be produced in tubes of all degrees of exhaust by sufficiently increasing the violence of the negative discharge. But there is an experimental fact which has been repeatedly observed and should be mentioned here, which shows that it is not essential that there should be any increased violence of the whole discharge, but that a sufficient intensification of the local action at the negative terminal is all that is necessary. When a tube is being exhausted to a high vacuum the phosphorescence always appears first in the neigbourhood of the negative terminal. But it often happens that one of the terminals of the tube is much smaller than the other. In such cases there is invariably a stage in the exhaust in which phosphorescence is visible when the smaller terminal is the negative terminal, but in which no phosphorescence appears when the current is reversed. And similarly when there is this inequality of size in the terminals, a smaller air-spark will suffice to produce phosphorescence when the smaller terminal is negative than when it is positive. Now it is well known that negative discharge is greatly facilitated by increasing the size of the terminal, so that we have a case in which, when all the other circumstances remain the same, we can produce phosphorescence merely by restricting the size of the negative terminal so as to render more violent the local action there.

The above experiments show that the phenomenon of molecular streams can be produced at pressures so considerable as to deserve to be called ordinary gaseous pressures. We shall now endeavour to show, in the second place, that there are no sufficient reasons for supposing that the gaseous molecules which form the discharge are in any way exempt from the ordinary laws that govern gaseous media. It is true that their original projection is an exceptional phenomenon, and that their consequent motion has no analogue in ordinary gaseous media, but there are many phenomena which show clearly that these molecular streams are interfered with in their course by the circumjacent gas, much as other currents (whether of gaseous or solid matter) would be under like circumstances. This is, we think, made evident by the following observations and experiments.

In the first place, when a tube is being exhausted and a discharge is maintained through it, phosphorescence appears first in the immediate vicinity of the negative terminal. This is so well known that it seems to be a matter of course that such should be the case, and yet it is difficult to understand why it should be so, except on the hypothesis that the gas in the tube obstructs the path of the molecular streams, and lessens their velocity. The only other explanation, viz.: that it is due to the greater obliquity of impact on the sides of the tube farther removed from the negative terminal, though of course it has a very decided effect, would not in our opinion be sufficient to account for it.

But we are not left to conjectural explanations to determine that the molecular streams are obstructed by the gaseous media through which they pass. We shall proceed to describe a series of experiments which put this beyond the reach of doubt,

and at the same time show how this clogging effect increases with the density of the gas in the tube, just as would be the case if streams of any kind of small particles were trying to force their way through it.

In order to observe the effect of the resistance of the gaseous medium upon the molecular streams to which phosphorescence is due, it was necessary to have an arrangement by which these streams could be examined at various distances from their source without any alteration being made in the other circumstances of the discharge. To effect this the following experiment was devised. A tube was constructed (Plate 27, fig. 10), having loose inside it a second piece of tube, whose external diameter was about $1\frac{1}{2}$ inches, while the internal diameter of the main tube was about 2 inches. When the tube was placed horizontally there was a distance between the two tubes of about half-an-inch on the upper side, decreasing down to zero on the under side, where the two tubes lay in contact. The arrangement adopted for producing the phosphorescence was that due to our assistant, Mr. WARD, referred to above, and with this arrangement we could, by moving the small patch of tinfoil upon the tube to a suitable spot, give to the streams of molecules thrown off from the interior surface of the outer tube any range we pleased from half-an-inch to zero. We then attached the tube to an ALVERGNIAT air-pump and ascertained the maximum range at which relief-phosphorescence could be obtained at different pressures of the gas by moving the tinfoil about until we got to a position where phosphorescence just became visible on the outer surface of the inner tube. This gave us the distance through which the molecular streams occasioned by the impulsive inductive action on the exterior of the tube were able to force their way through the gas without having their velocity reduced below the limits necessary to produce phosphorescence.

A series of precise numerical results would have involved an accurate determination not only of the pressure of the gas and the range of the molecular streams, but also of the quantity of electricity given off by the HOLTZ machine, as well as the length of air-spark used. But as our principal object was to show the existence of a maximum range dependent on the pressure, it will be sufficient to subjoin a few approximate results, which are, however, derived from a considerable number of actual observations.

The pressures of gas (atmospheric air) and the corresponding maximum ranges at which phosphorescence could be obtained with the 12-plate HOLTZ running at 300 revolutions per minute, and an air-spark of about half-an-inch in length, were as follows:—

Pressure.	Range.
5 millim.	12 millim.
22 "	5 "
24 "	2 "
26 "	almost contact.

This shows that although these molecular streams exist at a pressure of an inch of mercury, they are unable to force their way through the gas except to a very short distance, and that if the pressure be reduced the distance to which they can penetrate rapidly increases. So far, then, from being exempt from the action of the surrounding gas, they are very highly susceptible to its influence.

Another experiment which shows how completely these molecular streams are subject to the ordinary laws of gaseous resistance was made by us while working with the tube described on page 24, into which air was allowed to enter in order to discover the superior limits of the pressure within the tube at which we could obtain phosphorescence. When the phosphorescence had entirely disappeared it occurred to us to examine the effect of a magnet placed beneath the tube with its axis pointing in the direction of the tinfoil, so that the molecular streams, if any existed, would be moving towards it in directions nearly parallel to its axis. We knew that the effect of a magnet in such a case is to constrict the molecular streams and cause them to move in a more compact body, so that if the disappearance of the phosphorescence was merely the effect of the loss of velocity of the particles through their having to pass through gas of such considerable density, the magnet might have the effect of enabling them to penetrate to the other side of the tube so as to produce phosphorescence. Accordingly when phosphorescence had completely ceased to be visible a strong electromagnet was placed with its pole near the tube, diametrically opposite to the place where the wire from the positive terminal of the machine rested upon it. The experiment proved the justness of the conjecture, for while the magnet was in action a small bright and well-defined green patch was observed in the place where the phosphorescence would naturally appear, and this disappeared as soon as the current within the magnet was stopped. We then connected the tube with an ALVERGNIAT's air-pump fitted with a siphon gauge to measure the pressure within the tube, and repeated the experiments while the tube was in connexion with the pump. We found that without the assistance of the magnet we could produce phosphorescence at a pressure of a quarter of an inch of mercury, and with the assistance of the magnet at a pressure of at least three-eighths of an inch. Considering the very large diameter of the tube (something more than two inches) and the moderate magnetic power which we were using, these measurements, as well as those previously given, strongly confirm the estimate of the pressure in the case of the tube with potash mentioned above.

There is another interesting experiment of a different kind which shows clearly how readily the moving particles lose their velocity on passing through the gas in the tube. A tube containing a number of loose films of glass of extreme tenuity was exhausted till it gave very fine striæ, soft in outline, and also gave, with an air-spark, good phosphorescence. A discharge with a long positive air-spark was made to pass through it. On touching the tube with the finger (which, as we have already mentioned, has the effect of causing these molecular streams to pour off from

the interior surface of the tube at the spot on which the finger rests) phosphorescence appeared on the films opposite to the finger. These films were moved by the impact of the molecules as in the case of CROOKES' mill; but they were only moved very slightly. If, however, the finger was placed close under one of the films it was moved readily, showing that though the momentum of the molecules after they had crossed the tube was not sufficient to move the films, yet their initial momentum was amply sufficient to do so. The irregular shapes of the films gave opportunities of testing in a variety of ways the truth of this conclusion, and in all cases it was confirmed.

The importance of these results is twofold. They not only demonstrate that phosphorescence can be obtained at pressures so comparable with ordinary gaseous pressures that it is unnecessary, and indeed inadmissible, to have recourse to the supposition of an alteration of the ordinary laws of gases; but they also show that these streams of molecules are strongly under the influence of gaseous resistance, and that they rapidly lose their velocity from its action, so that, even in cases where phosphorescence is not visible, the same molecular streams exist, and may be made to produce it if proper means are taken to prevent their velocities being checked too much by the density of the vapour through which they have to force their way. Thus we may fairly conclude that the above-mentioned pressures by no means necessarily represent the limit at which these molecular streams exist. If it were desired to obtain phosphorescence at still higher pressures, all that would be necessary would seem to be to bring the glass intended to be affected into close proximity to the place of discharge, and still further to augment the violence of the electric impulses. No doubt in this way it would be possible to trace the presence of these molecular streams at much higher pressures; and if the thermal instead of the luminous effects of their impact on the glass were taken, it is probable that the range of pressures might still farther be increased. But it is sufficient for our purpose to show that no special condition of gas is necessary for the genesis of these molecular streams, and that they enjoy no special exemption from ordinary gaseous action in their subsequent path, since our object is not to determine the exact condition under which they occur, but to establish the close analogy between the molecular streams that produce phosphorescence and the other instances to which we have above referred, in which streams of particles are driven off from the negative terminal, and thereby to divest these streams of molecules of the character of an unprecedented phenomenon which would justify the hypothesis of any considerable change of conditions to account for its presence.

It may perhaps be said that it is unnecessary to give experimental proof that these molecular streams are obstructed by the medium through which they pass, so that they may actually exist even when no phosphorescence is manifested. This is in one sense common to all theories respecting them. The experiments of CROOKES with the electric radiometer show that the molecular streams seldom penetrate beyond what is

known as the negative glow, with sufficient force to affect the radiometer.* Thus it would appear that all are agreed upon the point. It seems, however, to be thought that the phenomena in tubes of extremely high vacua show a freedom from this retardation. But it must be remembered that these vacua are estimated to be equivalent, or at all events comparable, to one millionth of an atmosphere. Now we have shown that at a pressure of a quarter of an inch or thereabouts we can get bright phosphorescence at a distance of two inches from the origin of the molecular streams. Is it a matter of wonder, then, that at a pressure of ten thousand times less than this we should find that these streams move through a distance of a few inches without appreciable retardation, especially when we consider that we have no certain means of detecting whether they are retarded or not? The conclusion to be drawn from the above is, we think, that whether or not the ordinary gaseous laws suffer any modification in high vacua there is nothing in the phenomenon of phosphorescence in such vacua which entitles us to suppose that they do so.†

* It must be borne in mind that neither the negative glow nor positive luminosity necessarily bar the passage of these molecular streams. They often (as has been mentioned in connexion with some of the previous experiments) pass through bright positive luminosity for a considerable distance, and very frequently penetrate through a clearly marked negative glow and render phosphorescent the glass behind it. On the other hand, the experiment with the tube containing the glass films shows that it is not necessary that there should be luminous matter in the tube in order to stop the molecules. When a sufficient air-spark was used there was no positive luminosity at many of the places in the tube where the experiments were made, and there was only a very faint haze in the remainder, and yet the retardation of which we have spoken was clearly manifested.

† In a letter published since the reading of this paper Mr. CROOKES has made a further statement of his views on the existence of a fourth or ultra-gaseous state of matter.

We have never expressed any opinion as to the possibility of such a state, and have only dealt with the question whether the phenomenon of "molecular streams" furnishes evidence of its actual existence.

It may readily be conceded that if we could "by some extraneous force infuse order into the apparently disorderly jostling of the molecules in every direction by coercing them into a methodical rectilinear movement," we should fundamentally alter the physical properties of a gas. But our experiments furnish no evidence that any such action as this takes place in the formation of molecular streams. Before the discharge the particles of the gas are moving about in a perfectly irregular manner, and the effect of the discharge is to impress on them a very rapid proper motion in a definite direction. But we see no ground for supposing that the lateral motions, and the collisions consequent thereon, are in any way affected. Every wind furnishes us with an instance of gas the particles of which have an average proper motion, but no one would contend that such proper motion lessened the number of collisions in the gas or interfered with its gaseity. And we can see no reason for regarding a molecular stream as anything else than an exaggerated form of the well-known electric wind, or a mass of gas with an extremely rapid proper motion the magnitude of which is evidenced by the heat imparted to the body on which the gas impinges.

It is shown in the text that molecular streams can be produced with an intermittent discharge in tubes at comparatively high pressures where the gas is certainly in its ordinary state, and it may be added that in an intermittent discharge the periods of action are in all probability very small in comparison with the periods that separate them. Thus in all probability the greater part of the molecular stream would be composed of gas which had not been subjected to the direct action of the electrode, and which, therefore,

But there is another point of view from which the results given above are important. They show that no conclusions can be drawn from the length of the path of these molecular streams as to the average free path of the molecules of the gas or the frequency of collisions between them. We know enough of gases to be certain that at a pressure of a quarter of an inch of mercury the ordinary laws of gases are in full force; that the average free path of the gaseous molecules is infinitesimal; and the number of collisions between them in any finite time inconceivably great. And yet at that pressure we can get phosphorescence at a distance of at least two inches.

Although, therefore, we can no longer regard phosphorescence as so exceptional a phenomenon as has been generally supposed, we are far from intending to underrate its importance as a characteristic phenomenon of electric discharge. But this importance is due to the fact that it becomes more and more prominent as the degree of exhaust increases, and not to its specially appertaining to any type of exhaust. And this increase of importance is greatly enhanced by the consideration that the other characteristics of the discharge, such as positive luminosity and the like, become gradually less and less marked as the degree of exhaust increases, till at length almost the sole visible phenomenon of the discharge is the phosphorescence* in the tube caused by the streams of molecules which its passage excites. And in one respect the indications given to us by phosphorescence are more definite than those of any other of the luminous phenomena, because it always speaks to the existence of a negative discharge; and if it is possible, by the method of shadows or otherwise, to determine the direction of the streams of molecules, it tells us with considerable accuracy the position of the source of that discharge. And this renders it, as we shall presently see, of the greatest value in researches which have for their object the discovery of the mechanism of the discharge, and indeed constitutes it the main source from which we derive information in the matter.

must retain its normal state of intermolecular motion. This gas must be inextricably mixed up with that which has undergone the direct influence of the electrode, so that it is well nigh inconceivable on any hypothesis that there can be anything like order or directed motion in the molecular stream. And yet it is found to produce all the effects of a molecular stream produced by a more continuous discharge in a higher vacuum.

From these considerations and from the entire absence of anything which points to the suppression of the lateral motions, we conclude that the molecular streams furnish no evidence that the gas of which they are composed is in any other than its ordinary state. [July, 1880.]

* It is needless to repeat that the colour of this phosphorescence depends upon the substances used in the manufacture of the glass. The most convenient and easily distinguishable kind of phosphorescence is the green phosphorescence of German glass, and all the experiments for this paper have been performed with tubes of this glass. We shall therefore speak of phosphorescence as being green, although, as we have said, it is not necessarily so.

XVIII.—*The sensitive state* exists in discharges through tubes of high exhaustion when the current has the sharp intermittence which is the essential condition of its existence in tubes of lower exhaustion.*

When we examine a discharge in a tube of high vacuum which gives phosphorescence, we usually find that there is present an ill-defined column of haze of a greyish or purple colour, extending from the positive end of the tube. This must be taken as the representative of the positive column of the ordinary discharge, and it can be shown experimentally that such is the case by exhausting a tube while a discharge is passing through it, when it will be found that the positive luminosity passes continuously into the haze of which we are speaking.

So long as there is no interruption in the circuit external to the tube, these luminous appearances may† be non-sensitive, *i.e.*, may be indifferent to the approach of a conductor to the tube.

But if an air-spark be inroduced, the green phosphorescence becomes decidedly more brilliant, and the haze is found to be highly sensitive. With a positive air-spark the haze behaves on the approach of a conductor in all respects like the positive column in an ordinary sensitive discharge with a positive air-spark, excepting that its sensitiveness is usually more intense. When the air-spark is in the negative it is more difficult to establish the identity of behaviour of the haze and the ordinary sensitive luminous column under similar circumstances, but this does not affect the question of whether it is sensitive or not. As to this there can be no doubt, for the faint luminosity changes its conformation in a very marked way on the approach of a conductor to the tube.

If the vacuum be very high the tube appears almost wholly destitute of the haze of which we have spoken, and of course it then becomes difficult to demonstrate the sensitiveness of the discharge in the manner which we have just described. The very term itself seems to require an extended meaning, inasmuch as the true luminous discharge to which it was originally applied no longer exists. But it is not difficult to decide on the meaning which must now be given to it, for it will be found that the only luminous phenomenon that still remains, viz.: phosphorescence, undergoes changes when a conductor is brought near to or in contact with the tube so that we may fairly apply to it the same term "sensitive" that we have used with regard to luminous discharges in tubes in which the vacuum is not so perfect. It is true that in one class of cases (viz.: those in which the air-spark is in the negative) the sensitiveness of the

* It will be remembered that the definition given in our former paper of the *sensitive state* is "*the state in which the discharge is affected by the presence or approach of a conductor.*" This definition will be adhered to throughout.

† We say that the discharge *may* be non-sensitive when there is no interruption in the external circuit because we shall see that it is not necessarily so, just as in the case of tubes of lower exhaust a tube of high vacuum may itself cause the discharge passing through it to become intermittent and sensitive.

phosphorescence* when a conductor is brought into contact with the tube is not very strongly marked, and is in fact often difficult to detect; but in these cases it will be found that the phosphorescence is highly sensitive to the approach of a conductor which is in metallic connexion with the negative terminal of the tube, a property which is quite as distinctive of the luminous phenomena of the sensitive discharges of which we have treated in our former paper as is their sensitiveness to the approach of a conductor which is not in connexion with any portion of the circuit. We can thus apply the term "sensitive state" to discharges through tubes of high vacua, even though the phosphorescence should constitute the main or even the only visible portion of the phenomena.

Taking, then, this extended conception of sensitiveness, we find that it appears in tubes of high vacua under precisely the same conditions as in the cases with which we have dealt in our previous paper. A machine giving a continuous current produces a sensitive discharge when an air-spark is introduced; and so does a coil which does not give too much quantity. Indeed, it is in some respects easier to obtain a sensitive discharge in the case of tubes of high vacua than in that of tubes of low vacua on account of the great resistance they present, and the consequent need of considerable violence in the discharge if it is to pass through them. In the case of a coil, for instance, it is only the first and more violent part of the discharge that has force enough to penetrate into the tube, and consequently the discharge within the tube has often the sharp impulsive character necessary for sensitiveness, when in a tube of less exhaustion the discharge from a similar coil would be more prolonged and probably non-sensitive. This property has, however, its disadvantages as well as its advantages. Many of the methods by which we succeeded in obtaining sensitive discharges in tubes of moderate vacua and therefore small resistance, are inapplicable to the case of tubes of high vacua where the resistance is necessarily very much greater. Such a method as the use of the wheel-break with the HOLTZ machine† would seldom if ever succeed in causing a current of any kind to pass through a tube in which the vacuum was very high. Before the machine had charged up sufficiently to give a current capable of passing through the tube the next division of the wheel-break would have come into contact with the platinum spring, or would have approached it sufficiently to induce the charge to adopt that path in preference to passing through the tube.

There is therefore no need of an elaborate investigation to show that the sensitive

* A full account of the phenomena due to the sensitiveness of the phosphorescence in the intermittent discharge and of the effects produced on it by a conductor in metallic connexion with one or other of the terminals of the tube will be given in the subsequent sections. It is not necessary here to do more than refer to the fact that changes can be produced in the phosphorescence by the approach of conductors which are either uninsulated or in metallic connexion with some part of the tube. The nature of those changes does not concern us at this stage.

† See Phil. Trans. 1879, Part I., p. 170.

state of discharge in high vacua is dependent on intermittence. The arguments in favour of this are precisely identical with those that have been previously adduced in the case of discharges in low vacua, and the evidence is just as conclusive. As in the former case, sensitiveness is never found except in the presence of circumstances which render it extremely probable if not certain that the discharge is intermittent; and on the other hand, whenever the circumstances are such as to cause an intermittence of the proper type, the resulting discharge is found to possess sensitiveness. The telephone gives exactly the same indications of intermittence when placed in circuit between the earth and a piece of tinfoil laid upon a tube containing a sensitive discharge, and the revolving mirror gives exactly the same direct evidence of the intermittence of such discharges. In short, so far as has been observed, the whole of the evidence in favour of the connexion between intermittence and sensitiveness that can be adduced in the case of tubes of low exhaust is equally applicable to the case of tubes of high exhaust, excepting so far as instrumental difficulties or special peculiarities of the discharge (as, for instance, the extremely faint luminosity of the positive haze) make it impossible to apply the same tests. But although the amount of evidence is somewhat diminished by the limitation of our methods of producing intermittence, yet the nature of the evidence remains the same, and it is sufficient to show conclusively that in discharges through high vacua sensitiveness is just as much the invariable accompaniment of sharp intermittence and just as inseparable from it as is the case in the discharges of which we treated in our former paper. And, further, all the considerations which render the examination of the intermittent discharge of importance in the analysis of ordinary vacuum discharges exist and if possible possess yet greater force in the case of discharges in high vacua. There is the same identity of phenomena in the continuous and the discontinuous discharges, and there is the same ground for seeking in the discontinuous discharge the explanation of the various phenomena of the continuous discharge. No excuse will therefore be necessary for subjecting the sensitive state of discharges in high vacua to an investigation of the same type as that which is contained in our former paper.

We shall, for the sake of simplicity of language, assume during the remainder of this paper that the intermittence and consequent sensitiveness is produced in all cases in the simplest and most convenient way, *i.e.*, by an air-spark situated either in the positive or negative portion of the external circuit.

XIX.—*When the air-spark is in the positive the discharge passes through the tube in the shape of positive electricity, and* vice versâ.

It will be seen that this amounts to saying that the general results of our former paper hold good for tubes of high vacua.

The importance of demonstrating this is very great. For it signifies that there is

no radical difference between the nature of the discharge in high and low vacua, and it does away with the idea (apparently suggested by the phenomena of phosphorescence) that in high vacua the discharge is only derived from the negative terminal. In addition to this, the establishment of the fact that even in the absence of positive luminosity the positive terminal may be the prime source of the electric discharge, and the negative discharge may be only a response to the positive, sheds important light upon the functions of the molecular streams which accompany the negative discharge, and affords a strong argument in favour of the view that they do not represent in any sense the discharge itself, nor have any necessary connexion therewith, save as being accompanying phenomena of the passage of the negative electricity from the negative terminal, however such passage may be brought about.

There is a good deal of difficulty in applying directly the results of our former investigations to tubes of high exhaustion. The positive luminosity, which in tubes of lower exhaust constitutes the main feature of the discharge, and from the behaviour of which we obtained the indications of the nature of the discharge, fades away, as we have seen, into a thin haze with outlines so vague and shadowy as to be with difficulty discerned; indeed, when the exhaust is very high, the positive luminosity is either so faint as not to be discernible in the presence of the more brilliant luminous effects of phosphorescence, or else is actually absent. But so long as it is present it enables us to obtain evidence as to the character of the discharge which, if not equally convincing with that obtained under the more favourable circumstances of lower exhaust, would at all events suffice to give great probability to the hypothesis that there is no radical difference in the laws of the discharge so far as its capability of possessing either sign is concerned. With a positive air-spark the haze (whenever it is present) is repelled by the finger, and is constricted by a ring of tinfoil which is touched by the finger, just as the positive luminosity would be in a tube of lower exhaust; indeed, the sensitiveness seems rather to increase with the degree of exhaust than otherwise. And if the finger be passed along the tube there is the same continuity in the characteristics of the phenomena, showing that whatever be the nature of the disturbance it is the same from end to end of the tube. If the air-spark be in the negative the appearances are markedly different from those with a positive air-spark, and there is the same continuity of characteristics, but it is more difficult to identify the actual appearances with those which we have been accustomed to see in the case of other tubes. This is not to be wondered at, because we have already had to notice the want of sharpness of the effects with negative air-sparks, and even in our previous investigations we were frequently compelled to work with positive air-sparks in order to ensure good definition. This difficulty can, however, be overcome to a certain extent by having recourse to special rather than to relief effects; and if it were necessary we have no doubt that a great amount of evidence in favour of the proposition at the head of this section might be obtained in this manner.

It is possible, however, even by the use of the methods of our previous paper, to

obtain direct evidence of the truth of the proposition in question.* If we place a somewhat broad piece of tinfoil round the tube, and connect two wires from it to the ends of a suitable tube of moderate exhaust, we shall get clear signs of the appropriate double unipolar discharge (Plate 27, fig. 11). The sole drawback of this test is that it is only applicable in cases where the action is of considerable violence, so that it gives no result where the air-spark is very small, or where the tinfoil is very near to the terminal remote from the air-spark. This proof can, however, be extended to cases to which it is not directly applicable, by passing to them in a continuous manner from cases in which it can be used. Thus, where the action is sufficiently violent near the air-spark terminal to enable us to use the unipolar test, but not sufficiently strong at points further removed from that terminal, we can show that the nature of the electrical disturbances at the latter is the same as at the former, by passing the finger along the tube and observing that the appearances at the different points are substantially identical.

There is another method by which we can raise a strong presumption as to the applicability to high vacua of the principles we established in our previous paper. This is by observing continuously the phenomena presented by the discharge during the process of passing from a low vacuum to a high vacuum while the tube is being exhausted. If an intermittent current of either type be allowed to pass through the tube during the whole of the operation, the phenomena observable in low exhausts will pass in such a gradual and continuous way into those which we are accustomed to meet with in high exhausts, that it becomes well nigh impossible to doubt that the *modus operandi* of the discharge is the same throughout. And in the same way we can extend the test from the case of a large air-spark to that of a small one. And if it were not that the direct methods of which we are about to speak render these less direct evidences unnecessary for the establishment of the truth of the proposition in question, such considerations as these would be of the highest value as raising a strong presumption in favour of the radical identity of the modes of discharge in the two cases. As it is, however, we need not dwell on them further, and we have only referred to them in order to show that the methods of our former paper would have enabled us to solve the difficulties of the new subject-matter with which we are dealing had it been necessary that we should have recourse to them.

All the foregoing evidence, though valuable as confirmation of the theory, and interesting in connexion with our previous results, is insignificant in importance compared with the direct evidence afforded by the standard-tube method described in Section XV. A tube of high exhaust is taken, and its terminals are connected with those of a HOLTZ machine. A patch or ring of tinfoil is placed anywhere upon the tube, except in immediate contact with either of the terminals, and a wire is taken from it to a ring of tinfoil upon the standard-tube. No effect will be produced on the standard-tube unless the high vacuum tube is of a nature to cause by its own action

* See Phil. Trans., 1879, p. 216.

an intermittence in the current—a peculiarity which we have seen may also occur in tubes of lower degrees of exhaustion. But if a positive air-spark be introduced into the circuit which passes through the tube of high exhaust, the standard-tube will at once show positive effects, *i.e.*, the positive luminosity will be severed, and the well-known hollow cone and highly striated termination of the truncated column will at once be visible. This demonstrates conclusively that the action within the tube of high exhaust is such as to cause charges of positive electricity to be driven from the tinfoil upon it in the sudden intermittent and impulsive way that is needed to produce the ordinary sensitive effects in a continuous current; or, in other words, it shows us that charges of positive electricity are rushing through the tube of high exhaust and affecting the tinfoil upon it. And wherever the tinfoil be placed upon the tube of high exhaust (unless, perhaps, in the immediate neighbourhood of the negative terminal, where the results may be somewhat affected by the special circumstances of the case) the same effects will be found to be produced. Thus, in the case of the positive air-spark the discharges pass through the tube in the shape of positive electricity.

The above phenomena present themselves when the air-spark is in the positive, whenever care has been taken that neither the peculiarities of the tube nor those of the discharge introduce a second type of intermittence; and to ensure their appearance it will generally be found sufficient to connect the negative terminal to earth, and to introduce an air-spark into the positive. It happens, however, not unfrequently, that either from the shape or size of the terminals, or their relation to the degree of exhaust, or from some other similar cause, a tube possesses an inherent power of causing an intermittence of a negative type in a discharge even though a positive air-spark has been introduced into it, and in such cases we, of course, get mixed results. And it is of great importance that this should be borne in mind, for when the resistance is very great the danger of results becoming mixed in their character is very much increased, and further complexities are doubtless produced by the escape of large quantities of electricity from different portions of the circuit outside the tube into the air. We shall have to speak of certain types of these exceptional results; but in the meanwhile it is important to remember that such peculiarities as these do not furnish any argument against the conclusions above referred to; they should be looked upon as a kind of instrumental error, and they are only exaggerated forms of difficulties with which we have already become familiar in dealing with ordinary vacuum tubes.*

Conversely, if a negative air-spark be introduced into the principal circuit the effects in the standard-tube will be negative in type, *i.e.*, produced by negative impulses on the outer surface of the glass. If the air-spark be of a moderate size the well-known constriction or "ring-terminal" appearance will often be very clearly manifested; and

* The only cases of such peculiarities that we have hitherto met with are those of tubes of extremely high vacuum, in which the negative terminal is of small size. These sometimes impart to the discharge a negative intermittence of the most violent type.

if the air-spark be considerable we often shall find phosphorescence produced in the test tube in the neighbourhood of the tinfoil ring that is upon it. In many cases, however, the effects will be less easily recognized, and recourse must be had to the various methods of examining the intermittence in the standard-tube described in Section XV. But these difficulties are common to tubes of all kinds of exhaust when negative effects are being examined. The effects are less sharp than those of positive intermittence, and we must be content with less perfect definition. The result is, however, sufficient to enable us to detect with certainty the existence of negative impulses in the tinfoil upon the standard-tube; and reasoning as in the other case, this shows that the negative charges which burst into the tube pass through it in the shape of negative electricity.

It must not, however, be supposed that in all cases the one discharge passes quite to the other terminal of the tube without exciting a response. If this were the case, high-tension tubes would present a uniformity of behaviour which even low-tension tubes do not possess. It is often very difficult to trace the evidences of the discharges in the immediate vicinity of the opposite terminal, and in some tubes it would seem that the response can come from the opposite terminal a little before the original discharge has reached it. But these last cases are of an exceptional character, and do not at all affect the conclusion that the discharge in general passes through the tube up to the immediate neighbourhood of the opposite terminal before exciting a response. In most tubes evidences of positive relief can be obtained (with a positive air-spark of considerable length) close up to the negative terminal.

The evidence obtained by this method is so direct and so unmistakable in its signification, that it leaves no room for doubt. And just as in tubes of low exhaust we found that these properties of the intermittent discharge rendered possible a variety of other phenomena, all of which were explicable by this theory of the discharge, so also in the case of tubes of high exhaustion all the other phenomena which follow from these properties of the discharge are also manifested, and serve in their turn to demonstrate the identity between discharges in high-vacuum and in low-vacuum tubes. Thus very good unipolar and double unipolar effects can be obtained, manifesting, with certain modifications, the same peculiar phenomena which we are accustomed to see in connexion with them in low-vacuum tubes. So easily are these effects attainable that (as will be seen later on) they afford to us the readiest way of obtaining luminous discharges suitable for an important part of our investigation. And if we recollect how intimately the existence of unipolar and double unipolar discharges in ordinary vacuum tubes is connected with the fact that the discharge at each terminal is independent of the action elsewhere than in its own neighbourhood, we shall see that the existence of similar phenomena in high-vacuum tubes is strong evidence of the substantial identity of the *modus operandi* of the discharge in the two cases.

The various types of evidence which we have already given represent, after all, only

a portion of the evidence in favour of the proposition of which we are treating. There are phenomena of the relief and special effects in tubes of high exhaust, which when we come to consider them will afford, if possible, still more conclusive evidence of the truth of the hypothesis. But it is for many reasons desirable that we should not anticipate the examination of the relief and special phenomena of high exhaust tubes, because we propose to deal with them separately as they merit a much more minute examination than we could conveniently give them in this section. Moreover, in our opinion, the tests and other experimental proofs that we have already given, and the complete absence of any phenomena which would lead us to believe in any breach of continuity in passing from the vacua used in our previous experiments to these high vacua, sufficiently establish the proposition that the discharge is in general carried along the tube by electricity of the same sign as that of the air-spark.

XX.—*Phosphorescence exists in sensitive as well as in continuous discharges; and when occurring in sensitive discharges it is intermittent in a like manner with the other luminous effects.*

If a tube, which is exhausted to a degree sufficient to produce phosphorescence when a continuous discharge is made to pass through it, be placed in circuit with the terminals of a HOLTZ machine (Plate 26, fig. 8), and an air-spark be introduced into the circuit, the only effect that is produced upon the phosphorescence is that it grows brighter, and is sometimes slightly altered in its position and distribution. Substantially, the phenomenon remains the same as before the introduction of the air-spark, thus showing that the intermittence of the discharge does not prevent the negative pole from sending off the streams of molecules to which this phosphorescence is due, but, on the contrary, favours its so doing. This is in accordance with what has already been stated in Section XVII.

To the eye the phosphorescence of the intermittent or sensitive discharge is just as continuous as that of the continuous discharge. But this is easily proved to be a mere optical effect, and that the phosphorescence is, in fact, like the discharge itself, intermittent. To demonstrate this, it is only necessary to take a revolving mirror and examine the tube through a narrow slit in the ordinary way. Whether there be a positive or a negative air-spark it will be found that the green luminosity is intermittent. If the air-spark be large a very small velocity of rotation will show the green lines which are the images of the slit clearly separated by dark bands; and with an increased velocity of rotation the intermittence can clearly be shown even when the air-spark is very small.

It might be thought that since phosphorescence is supposed to last for a short time after the excitement which causes it has ceased, the green light would be continuous even though its cause were intermittent. And there are traces of this when the very bright portions of the glass are examined with the revolving mirror, when the air-

spark is small and therefore the intermittence very rapid. It is in such cases difficult to decide satisfactorily whether the bright portions are absolutely continuous or not. But this difficulty is in reality of very slight importance, inasmuch as there is no doubt that the other portions of the phosphorescence are intermittent, and these brighter spots do not differ from the rest of the phosphorescence in the origin of their luminosity, but only in their being more favourably situated for the concentration of molecular streams upon them. Hence there can be no reasonable doubt that this apparent continuousness is due to the persistence of the phosphorescence. In the case of large air-sparks the mirror when revolving at a high velocity exhibits a slight haziness on one side of the image of the slit, showing that the dying out of the phosphorescence is not quite instanstaneous, though very nearly so.

It will thus be seen that the phosphorescence produced by the separate discharges of an intermittent current must be very intense, seeing that the periods during which the glass is brightly illuminated by it are extremely short compared with the periods that intervene, and yet the intensity of the apparent illumination of the glass is greater than with the continuous current. This shows that the particles must be driven off at a greater velocity or in greater numbers during the short period occupied by an individual pulse of the intermittent discharge than during the continuous discharge—a difference that would naturally follow from the fact that the electricity in each of these individual discharges represents the total accumulation of the period between two discharges. But it is remarkable that the vanes of a radiometer, when used as the negative terminal of a vacuum tube, as in CROOKES' experiments, revolve very much more quickly under the influence of a continuous discharge than an intermittent one. If an air-spark of increasing length be introduced into the current of a HOLTZ machine that is driving a radiometer electrically, the driving power will gradually diminish, and ultimately cease. This is, perhaps, equivalent to saying that the action upon the radiometer is dependent on the quantity of the discharge, and not upon its tension, since, of course, the introduction of a long air-spark into the circuit must necessarily have the effect of enormously decreasing the quantity of electricity passing, although it similarly increases the tension. But then we are met with the difficulty that the intensity of the phosphorescent illumination in similar cases is increased by the introduction of an air-spark in spite of the diminution of the quantity of the discharge. A possible explanation of this law is obtained by supposing that the effect on the radiometer is proportional to the *momentum* with which the particles leave the terminal while the phosphorescence depends on their *energy*. This would mean that a diminution of quantity diminishes the number of particles leaving the terminal while the increase of tension increases the velocity; such increase being insufficient to prevent the total momentum of the particles from being decreased by the introduction of an air-spark, but sufficing to cause a marked increase in their total energy.

It is a very significant fact that the intermittence of the phosphorescence exists

with a positive as well as with a negative air-spark. In the former case the discharge at the negative terminal is of the nature of a response, and is presumably of a much less sharp and impulsive type than it is when that terminal is the air-spark terminal. Yet so brief is the whole time occupied by a single discharge that even the time required for the less instantaneous response is not sufficient to give any perceptible broadening out of the bands in the revolving mirror. It may, however, be said with much justice that it is probable that it is only the first burst of the response that is intense enough to give phosphorescence, and that the absence of continuous phosphorescence in this case is no proof that there is not a gentle continuous discharge from what we have previously called the connected terminal. But even if we allow for this it is clear that the main part of the discharge from the negative terminal in the case of a positive air-spark must take place in an extremely short space of time.

Although phosphorescence is intermittent, like all the other luminous phenomena of the sensitive discharge, there are many reasons which make it desirable to treat it separately from the luminous discharge. It is not, like striæ or the negative glow, a part of the phenomena of gaseous discharge properly so called, but it is the effect of a mechanical radiation which accompanies the discharge. It is a diagram, a projected image of that radiation upon the surface of the glass; while itself, it is the quasi-accidental effect of the fact that glass is the most convenient substance for transparent tubes, and that glass is usually made of substances which will give such phosphorescence when exposed to streams of rapidly moving particles such as those in question. Hence it is only a secondary phenomenon of gaseous discharge, and for clearness it will be well to treat it quite separately, and as in no way a part of the luminosity due to the discharge. But just as the relief and special effects in the luminous discharge are due to interference with the actual main discharge by means of artificially produced discharges of like period, so it is found that the phosphorescence which is due to the main discharge is capable of being affected in a similar way, though, of course, the results are wholly different to those special and relief-effects of which we treated in our former paper, and which relate to the luminous column itself, *i.e.*, to phenomena existing in the gaseous medium through which the discharge passes. And as phosphorescence is the most marked of all the phenomena that accompany the discharge in high-vacuum tubes, we shall, in studying the sensitive state in high vacua, and the nature and circumstances of discharges therein, pay special attention to the phenomena that depend on phosphorescence, and shall in the subsequent sections examine these first, and subsequently proceed to consider those phenomena which are the true analogues of the luminous appearances in tubes of lower exhaustion.

XXI.—*On the relief-effects in tubes of high exhaustion with a positive air-spark.*
I. *Relief-phosphorescence.*

If a tube of moderate exhaust be taken, and a continuous current be passed through it, no phosphorescence will, it is well-known, be visible. But if an air-spark of considerable length be interposed between the positive terminal of the machine or other source of electricity and the tube, the usual green phosphorescent light will be seen near the negative terminal. When this is the case if the finger be placed upon the tube, a bright green patch will ordinarily appear on the further side of the tube immediately opposite the finger. In order to produce this phenomenon, the finger may be placed at any part of the tube, except in the immediate neighbourhood of the negative terminal; but as a rule the green patch will be brighter the nearer the finger is to the positive terminal of the tube. If the same experiment be tried with a tube of high exhaustion it will be found still easier to produce this phosphorescent patch; for the air-spark necessary to produce it in a tube of high exhaust is much shorter than that which would be necessary in a tube of lower exhaustion. There is, however, the correlative disadvantage that it is not so easy to distinguish it, inasmuch as the whole of the inside of the tube of high exhaust may itself be phosphorescent from the action of the negative terminal.

It is not difficult to interpret this phenomenon after the theory of the intermittent discharge is once comprehended. The inner surface of the glass beneath the finger acts as a negative terminal under the influence of the advancing positive electricity that has come from the positive terminal, and the relief-discharge from it is sufficiently violent to send off streams of molecules capable of causing phosphorescence on the opposite side of the tube.

The importance of the phenomenon is much greater than at first sight would appear. In the first place, it affords direct evidence that there is negative discharge from the inside of the tube close to the finger, for these molecular streams only occur as an accompaniment of negative discharge. This alone is a result of great value. The phenomena of positive relief which in the previous paper were attributed to negative discharge could only be identified with the phenomena characteristic of negative terminals by a long and intricate process, and even after this identification had been satisfactorily made it was often difficult to trace the resemblance between the two sets of phenomena so as fully to realise their identity. But here we have no such difficulty. The indication cannot be mistaken; and we are enabled to affirm just as certainly that there is discharge from the side of the tube, and that such discharge is of negative electricity, as if we could test the electricity actually coming from it. And again, the fact that the same indication of negative discharge is obtainable from all parts of the tube (except perhaps in the immediate vicinity of the negative terminal) is direct evidence that the discharge passes throughout the tube in the form of positive electricity, since the response is throughout in the form of negative discharge. And

the importance of these considerations is greatly enhanced by the fact that this relief-phosphorescence occurs in tubes of every degree of high exhaust, showing conclusively that when the air-spark is in the positive the discharge is carried in these tubes by bursts of positive electricity, which pass throughout the whole length of the tube just as in tubes of lower exhaust. No stronger confirmation of the results of Section XIX. could be desired so far as regards positive intermittence.

Nor are these the only important conclusions that can be drawn from the appearance of this relief-phosphorescence. It shows that it is not necessary that there should be a discharge actually passing from out of a solid body to cause these streams of molecules. The discharge in question comes from the inner surface of the glass, not from the interior of its mass. This would go to show that the action takes place at the bounding surface of the terminal, or perhaps in the layer of gas that lies immediately upon it, and forms a kind of border-land between the solid and the gaseous, and that it is really an action between the gas and the solid terminal. It would seem clear that it does not lie in the free gas itself, or consist of an action between the particles of the gas merely, as we should then expect to find that there was phosphorescence on the surface of the glass from which the discharge proceeds, caused by the backward recoil of the particles of gas in that layer derived from their violent disruption from those of their fellows that go to form the molecular streams. And there is certainly no trace of this, for as we shall presently see, the spot from which the discharge comes is denuded even of the phosphorescence that it would receive from other sources.

Before we pass on to examine this relief phosphorescence in other particulars we may remark that it is well shown, even in tubes of comparatively low vacua, by the use of the positive unipolar or double unipolar arrangement. By this method we are able to obtain intermittence of a much sharper and more violent type than with an effective current, so that the violence of the relief-effects is proportionally increased. A finger placed on such a tube will give well-marked phosphorescence. This serves, as in the former case, to show the correctness of the conclusions as to the cause of the unipolar phenomena, since it demonstrates that there are sharp periodical bursts of positive electricity into the tube. The relief-phosphorescence with these unipolar discharges is extremely bright, and the whole of the other relief-effects seem greatly intensified. By the use of this double unipolar arrangement we are able to produce phosphorescence in tubes of much lower exhaust than with an effective current, but both of these methods are inferior in this respect to the method described in Section XVII., which is a combination of a unipolar with an inductive discharge.

Another very convenient arrangement for showing the phenomenon of relief-phosphoresence consists in the use of a tube in which the terminals are near together at one end. The introduction of an air-spark will fill the remaining portion of the tube with a luminous discharge or haze (according to the degree of exhaustion), which is in fact a unipolar discharge (Plate 27, fig. 12). If the air-spark be in the positive circuit,

this will be in a positive unipolar discharge and it will give relief-phosphorescence and all the other phenomena of positive intermittence.

The phenomenon of relief-phosphorescence supplies us with a very convenient method of determining experimentally the laws that govern the production of these molecular streams and the direction in which they travel. By varying the shape and size of the patch of tinfoil on the tube we vary the negative terminal from which the streams proceed, and by varying the length of the air-spark we vary the violence of the discharge. The opposite side of the tube acts as a very conveniently placed screen on which the effects are shown, so that we are able with a minimum of labour to examine the various phenomena to which these molecular streams give rise. We shall now give the results of this examination so far as we have had time to carry it.

A very cursory examination suffices to show that it is impossible to accept the hypothesis that presents itself to the mind most naturally, viz.: that these molecular streams move in straight lines, or nearly so, starting in a direction normal to the surface of the terminal. Were this the case, then the shape of the phosphorescent patch due to a piece of tinfoil on a tube when employed to produce relief-phosphorescence would be of a shape similar to that of the tinfoil. Experiment, however, shows it to be quite otherwise, and indeed it is at first sight extremely difficult to arrive at precise conclusions as to the law of the distortion observable in this patch, which we may term the phosphorescent image. There are certainly two causes at work in producing this distortion which are of great importance, but it is probable that at least one other cause is present, the existence of which, however, we cannot demonstrate at this stage.

The first cause which is undoubtedly at work to prevent the molecular streams forming an exact image of the tinfoil patch is that they do not all leave normally. It has been assumed rather than proved that such is their natural tendency, but we very much doubt whether even with the continuous discharge this is the case except to the extent that it is approximately true for those parts of the terminal that are not very near its edge. And it is certain that when we come to the intermittent discharge the direction in which the molecular streams which cause relief-phosphorescence can be made to go is inclined at such an angle to the normal that they must possess considerable initial obliquity. A ring of tinfoil connected to earth was placed round a tube in which a current with positive air-spark was passing, and which gave the usual phosphorescent phenomena. A small piece of glass lying loose within the tube was then shaken to such a position in the tube that the line from the piece of glass to the edge of the tinfoil opposite to the place where the glass lay made an angle of about half a right angle with the axis of the tube. A fine sharp shadow of the piece of glass was seen, and its direction was (so far as could be judged by the eye) just what it would have been had it been made by streams proceeding directly to the piece of glass from the tinfoil on the opposite side of the tube. Now had the streams started in a direction more nearly normal to the surface of the tinfoil, and subsequently become deviated to

so large an extent as to reach the piece of glass, the shadow must have taken a direction much more oblique than was actually the case; and furthermore, the extreme obliquity of the streams of molecules that caused the shadow renders it very improbable that they should all have started normally.

In this particular instance there was a peculiarity which rendered the experiment very interesting, as showing how completely the relief phosphorescence is independent of that which comes from the negative terminal. The piece of glass was between the tinfoil and the negative terminal, so that the shadow pointed towards the negative terminal instead of away from it.

If a very small patch of tinfoil, connected to earth, be placed upon a tube of considerable diameter and not too high exhaust to permit the details of the relief phosphorescence to be readily distinguished, it will be found that it produces a bright central patch of phosphorescence at the point of the tube exactly opposite to it (Plate 27, fig. 13), and that this bright central patch is surrounded by an annulus of feebler intensity but considerable breadth ending in a fine bright line of phosphorescence serving as its outer edge. The breadth of the whole of this phosphorescent area is such that it subtends a very considerable finite angle at the patch of tinfoil, the semi-vertical angle of the cone of rays being from 20° to 30°. Now in this case the patch of tinfoil is so small that it may in considering the direction of the resulting molecular streams be taken to be a point, and thus we see that the molecular streams from a small elemental area would, if unaffected by any other circumstances than those necessarily present in a tube, pass off in all directions comprised within a solid angle of finite size (depending probably upon the degree of the exhaust and the violence of the discharge), surrounding the normal in an approximately symmetrical way, *i.e.*, forming a right cone of which it is the axis.* We are not able to speak definitely as to the intensity of the streams in the different directions. Those that proceed strictly normally are probably the most intense either from the greater density of the streams or the greater velocity of the particles, for we find that there is a very bright patch in the centre. But this may be partly due, as we shall see, to the fact that the patch of tinfoil has a finite though small area. A more difficult matter to account for is the apparently sharp limit which bounds the phosphorescent area on its outer side. It is difficult to imagine that there can be an abrupt limit to the angular extent of these molecular discharges. The most probable hypothesis is that it is due to a wholly

* It will probably be objected (and with perfect justice) that we are reasoning as though all the molecular streams that leave the gas beneath the tinfoil arrive at the opposite side of the tube and make themselves visible there. This is of course not the case, and if we were to take a screen situated very much nearer to the tinfoil than is the opposite side of the tube we should no doubt get a wider limit to the directions of the molecular streams. But the general reasoning is not affected by this, although it is most important to bear in mind that our tests do not exclude the possibility of feebler streams of molecules issuing at still greater inclinations to the normal, and that it is only the streams that have a certain intensity that are rendered visible by phosphorescence.

different cause and that it is a result of certain phenomena which are discussed in Section XXIII.

Next take a narrow strip of tinfoil about 2 inches in length and place it longitudinally along the tube. If it be connected to earth we shall find that the relief-phosphorescence produced is in the shape of a broad patch occupying, say, half the circumference of the tube (Plate 27, fig. 14). It is, however, far from being uniform in brightness. Exactly opposite to the strip of tinfoil there is an ill-defined longitudinal band that is brighter than the surrounding parts, and which evidently corresponds to the bright central patch that was observed in the former experiment. From this band there branch out narrow bright streamers all perpendicular to the general direction of the band and giving to the glass a striped or striated appearance. These extend throughout the whole of the phosphorescent area, the spaces between them being less bright, though doubtless they are also phosphorescent, but to a less degree than the striations.

It will be at once seen that no mere superposition of the phosphorescence of the elemental areas composing the strip could produce such a distribution of luminosity in the phosphorescence due to the whole strip. The streams that proceed from the different elements of the strip must therefore interfere with each other. And it is easy to see how this interference leads to the configuration described above. The strong repulsion between an element of the surface of the glass beneath the tinfoil and the particles of gas in contact with it can no longer drive them off at all azimuths, for in the direction of the length of the strip there are equally active elements exercising an equally strong repulsion upon these particles. The only directions in which the streams can spread out are therefore those which are comprised in a plane normal to the direction of the strip through the element under consideration. Any accidental cause creating a difference in the intensity of the local action at any point of the strip (as, for instance, a slightly better contact with the glass) will cause the streams from one element to be rather brighter than those from another, and hence the phosphorescent image of the tinfoil will appear to be striped or striated in the manner above described.*

If the strip of tinfoil be placed at right angles to the direction of the axis of the tube and partially surrounding it, the same striated appearance will be visible in the relief phosphorescence, but the stripes or striations will now stretch along the tube in

* It is impossible to see this effect without being reminded of the appearances produced by the volatilisation by the electric spark of a fine metallic wire placed upon a sheet of white paper or cardboard. From the general line of the wire there branch out in a direction normal to its length fine lines or striations precisely similar in shape and arrangement to those that appear in the phosphorescent image of the strip of tinfoil, only on a smaller scale. The cause of this striated structure is probably the same. The violent disruptive action that dissipates the metal in the wire is forced to drive it off in normal planes because the particles would be turned back into the plane by the action at the neighbouring elements of the wire were they to commence to move in an oblique direction.

the direction of its axis and not perpendicular to that direction as in the former case, and, similarly, the irregular bright central band will have its general direction along a normal section of the tube. So marked is the extension in the breadth of the image due to these stripes of phosphorescence (Plate 27, fig. 15) that if the strip of tinfoil be a short one its phosphorescent image seems to be long and narrow and to have the direction of its length at right angles to the direction of the tinfoil. The same is the case when a piece of tinfoil is placed longitudinally upon the tube; the greatest elongation of the image seems to be at right angles to that of the tinfoil. It will be understood that there is no actual rotation of the image through a right angle; the effect is solely due to the fact that the spreading out is at right angles to the direction of the length of the tinfoil, and that it is so great as to cause the breadth of the phosphorescent patch which it forms to be greater than the length of the strip of tinfoil.

That the above conclusions as to the law that governs the spreading out of the molecular streams are correct was shown by placing strips of tinfoil in oblique positions, when it was invariably found that the stripes or striations ran perpendicularly to the direction of the tangent to the strip of tinfoil at the point from which they proceeded. In order to test it rigorously, it was determined to place a strip in such a curve that the normal planes to the curve would pass through the tangent at the corresponding point of the *image* of the curve, *i.e.*, the curve on the opposite side of the tube, each point of which is exactly opposite to its corresponding point on the tinfoil. In such a case all the striations must lie along the curve formed by the locus of the central patches of phosphorescence, and the result should be a single bright curved line of phosphorescence without any spreading out or striated margin (Plate 28, fig. 16). The curve in question is evidently a helix, whose pitch is half a right angle. On trying the experiment these anticipations were found to be exactly fulfilled. It also occurred to us that as the striations are in the normal plane to the strip of tinfoil the locus of their consecutive intersections would give the evolute* of the curve formed by the strip of tinfoil, and that as such locus it would probably be represented by an especially bright line. We tried the experiment with a fine copper wire in the form of an ellipse, which was bent round so as to lie on the tube, and it was found to answer perfectly. The four-cusped shape of the evolute was distinctly marked by a bright line of phosphorescence (Plate 28, fig. 17).

It is easy to advance from the case of a strip of tinfoil to that of a patch whose length and breadth are alike considerable. The molecular streams from beneath the elements in the interior of the tinfoil will be hindered from spreading out in any direction by the action of the circumjacent elements, and they will therefore be concentrated and forced to pass off in a normal direction. The elements at the edge of the tinfoil are, however, in a different position, they can still spread out in directions normal to the edge of the patch. They are in fact much in the same condition, so far

* This term is of course not used in its strict sense, for the wire and the phosphorescence are upon curved surfaces, and not upon planes.

as spreading out in directions outwards from the patch, as they would be if they were separated from the rest of the patch and formed a narrow slip of the same form as its edge. There is, however, the further complication (when the patch is of finite size) that all the normals at the various points of its surface pass through the axis of the tube, and would, if uninterfered with, form a *reversed* image on the opposite side of the tube. In so doing, they doubtless interfere with one another to a greater or less degree. It is not necessary, however, to examine carefully the results of this fresh element of complexity; speaking generally, the consequence is that the phosphorescence assumes the shape of a central bright patch surrounded by a border of smaller luminosity.

The second cause that operates to prevent the streams of molecules from making a perfect image of the patch of tinfoil which excites them is the property that such streams possess of interfering with one another during their passage through the gas. It is no doubt difficult to separate this cause from the last, for the streams only make themselves manifest at their extremities where they strike the glass, so that it is well-nigh impossible directly to distinguish between an initial obliquity and an obliquity that has been acquired during flight in consequence of the interference of other molecular streams. But there is abundant evidence of the independent existence of this latter cause of the obliquity of molecular streams. If we take a piece of tinfoil on a tube of not too large diameter and connect it to earth (the air-spark being, of course, in the positive) we shall, as we have said, produce a patch of green light on the opposite side of the tube. Let the outlines of this patch be carefully observed, and then let another patch be placed on the same section of the tube, but distant, say, a quadrant from the other. If this be also connected to earth, the former patch will be found to have altered in shape, and of course a fresh patch will have appeared corresponding to the second piece of tinfoil. If, now, the first piece of tinfoil be disconnected from earth, it will be found that the second patch has altered in shape. Thus the streams from the two loci of discharge must have interfered with each other, and as there was no community of origin this interference must have taken place during their passage through the gas.*

There is one form of this interference which is so marked, and which is so unmistakably a matter of interference, that it deserves special mention. We refer to the case in which the two pieces of tinfoil referred to in the last experiment are diametrically opposite, so that each is placed where the phosphorescent image of the other would naturally fall. In such a case the streams proceeding from each appear to beat back

* The existence of this interference has also been demonstrated in the following way. A helix of tinfoil upon the tube connected to earth in the usual way was taken, the pitch being half a right angle. It gave the sharp phosphorescent helix of which we have already spoken. The tube was then touched at a point midway between two threads of the helix of tinfoil, so that the molecular streams from the finger cut normally through the surface formed by the molecular streams from the tinfoil. This was found to cause a distinct shifting of the corresponding part of the phosphorescent helix.

or turn aside those proceeding from the other, for just over each piece of tinfoil there is a patch of glass, of about the same shape as the tinfoil though rather larger than it (inasmuch as it overlaps it a little on all sides), which is wholly devoid of phosphorescence. The best example of this is got by putting a ring of tinfoil on the tube and connecting it with earth. No phosphorescence appears beneath the tinfoil, showing that no streams proceeding normally from the surface of the glass have reached the opposite side. All the phosphorescence is arranged in two rings, one on each side of the ring of tinfoil and parallel to it but separated from it by a space whose breadth is pretty uniform and varies in different cases from one-eighth to half an inch. Here it would seem that the streams that started normally, or nearly so, have interfered with one another with the result either of neutralising one another or deviating one another from the normal course and causing incidence on the glass at some distance on one side or the other of the tinfoil ring.

This capability of interfering with one another possessed by these molecular streams is one of great importance, both as giving us an insight into their nature, and also as an assistance in examining the mechanism of vacuum discharges. But we shall not dwell further upon it now as it belongs more properly to the next section, and is only mentioned here incidentally as one of the causes at work in determining the shape of the relief-phosphorescence.

The third cause which is probably at work to distort the phosphorescent image of the tinfoil is the influence of the position of the exciting positive electricity upon the negative discharge that responds to it. It has been thought that the direction of these molecular streams depends solely on the shape of the negative terminal, and is wholly independent of the position of the positive terminal, that is to say, of the direction from which the demand for negative electricity comes. This may be approximately so in the comparatively mild action that accompanies the continuous current (which may be compared to a case of steady motion in dynamics), but it certainly is not so in the more violent actions which accompany the impulsive discharges of the intermittent current. But, just as in the former case of the interference of molecular streams during their flight, the examination of this point belongs more properly to another branch of the investigation, so that we shall not notice it here at any length.

Leaving the question of the causes which determine the direction of these molecular streams, there is no doubt of their identity of nature with those that accompany the continuous current. They cause shadows in the same manner as do those proceeding from the negative terminal. In this way we can obtain shadows of any loose object in the tube and even of the positive terminal itself. The negative terminal, however, does not ordinarily cast a shadow properly so called, inasmuch as it is itself giving off like streams, and thus the streams that proceed from it turn aside any other streams that would otherwise impinge on it. These shadows due to relief are wholly independent of those that are due to the discharge from the negative terminal. If the object casting the shadow be sufficiently near the negative terminal to cast a shadow in the

ordinary way, a second shadow of it can be produced by casting relief-phosphorescence upon it, and these two shadows will in general co-exist without to any great extent interfering with one another. In a similar way two shadows of the same object may be produced by placing two pieces of tinfoil connected to earth on the opposite side of the tube, one a little nearer the positive terminal and the other a little nearer the negative terminal than the object, so that the latter is within the relief-phosphorescence produced by each of the pieces of tinfoil. Two oblique shadows in opposite directions will then be seen, though their definition is not so good as in the former case, since the two systems of molecular streams have under such circumstances a strong tendency to interfere with each other.

XXII.—*On the relief-effects in tubes of high exhaustion with a positive air-spark.* II. *Virtual shadows.*

If, while a discharge is passing through a tube of high exhaust with positive air-spark, we place the finger on the tube, the green light is seen to fade away from that part of the nearer side of the tube (*i.e.*, the side on which the finger rests) which lies in the direction of the positive terminal, giving the effect of a shadow falling upon that part of the surface of the tube. As the shadows are produced, not by any object actually intervening in the path of the gaseous particles, but by a body affecting them from outside, we have termed them *virtual shadows*. If the air-spark be small the region over which the shadow extends is bounded by a plane almost parallel to the tangent plane at the point where the finger rests and at a little distance from it (Plate 28, fig. 18), but if the air-spark be large the bounding plane is inclined at a considerable angle to the tangent, and cuts the tube obliquely. But it is only in very rare cases that the virtual shadow extinguishes the green phosphorescence from the opposite side of the tube, or from the end of the tube round the positive terminal, although it will often diminish the phosphorescence in that portion of the positive end which lies towards the side upon which the finger is placed.

A virtual shadow has a well-defined outline, the edge of which is generally brighter than the rest of the tube. It starts from the side of the finger nearest the negative terminal, but broadens considerably in the direction of the positive. Sometimes, when the air-spark is very large, it even seems to start from a point a little to the negative side of the finger, leaving that side of the finger surrounded by the outline of the shadow. The area of the shadow appears to be nearly, but not completely, deprived of the phosphorescent light. It is, however, difficult to judge how far the appearance of residual light within the shadow is due to reflexion, and how far to direct illumination. It is probable on theoretical grounds that some phosphorescence remains within the shadow; and this is confirmed by observation, for when the discharge has recently commenced it is often difficult to trace the outline distinctly. It would seem as though the glass needed to lose a little of its sensitiveness to show the full influence of positive relief in producing this phosphorescent shadow.

Now we have already established the fact that the immediate consequence of affording relief to any portion of the surface of a tube through which a discharge with positive air-spark is passing is to cause rapid impulsive discharges of negative electricity from the inside of the tube, and we have also seen that these discharges are accompanied in high vacua by the usual streams of molecules. Hence it is natural, in seeking to account for the phenomenon we have described above, to look to these negative discharges and their accompaniments for the solution. And that this is the proper source is shown by the fact that a similar phenomenon appears with negative special where there are also similar impulsive negative discharges, while in positive special and negative relief which give rise to positive discharges there is either no such phenomenon, or it is manifested on so much more insignificant a scale as to point to its being only a secondary effect.

Considering then that the positive relief and the negative special give effects which are as identical in the case of phosphorescent discharges as in that of ordinary discharges, we may fairly consider that we are on safe ground in applying our previously obtained results to them; and we therefore conclude that a negative discharge from the inside of the tube transversely to its length is the necessary condition for the existence of this phenomenon of virtual shadows. We shall now show that it is due to a beating down of the streams of molecules coming from the negative terminal (which would otherwise impinge on the side of the tube and there cause green light), this beating down being caused by the transverse streams of similar molecules coming from the inside of the tube.

In the first place, it is certain that such streams of molecules do interfere with each other when their paths cross. The experiments referred to in the last section suffice to show this. If two patches of tinfoil giving positive relief be so arranged on the tube that their green lines cross, it will be found that they displace each other, and that neither of the green patches produced by the pieces of tinfoil is in the position that it would be were the other not present. It is true that it is difficult to draw conclusions as to the exact nature of this interference from the mode in which they are displaced, for both the patches themselves and their displacements are very irregular, but the experiment is decisive to show the existence of interference between such streams of molecules when they are synchronous and when their paths cross.

In the instance just given the two sets of molecular streams are both due to the relief discharges that come from the side of the tube. We shall now give some instances in which one of the interfering streams is due to the discharge at the negative terminal of the tube.

We have already described relief-phosphorescence, and have shown that it is usually situated exactly opposite the place where the relief is given. But if the finger is placed in the immediate neighbourhood of the negative terminal, a little in front of it, we shall find that the patch of phosphorescence formed by it is no longer immediately opposite it, but some distance farther down the tube (Plate 28, fig. 19). In other

words, the streams of molecules that were crossing the tube to form the relief-phosphorescence have been swept down the tube by the streams that were proceeding from the negative terminal.*

A much more striking form of what is substantially the same phenomenon was observed in a tube the negative terminal of which consisted of a straight wire fixed at right angles to the axis of the tube and passing through it to a point about half-way between the axis of the tube and the opposite side (Plate 28, fig. 20). The discharge passing through the tube had a positive air-spark of considerable size. On placing the finger upon the tube the usual relief-phosphorescence appeared. But when the finger was placed upon the tube at the spot where the wire forming the negative terminal of the tube would have, if produced, cut the surface of the tube, it was found that the relief-phosphorescence took the form of an annulus round the root of the negative terminal. The inner boundary of this annulus was well defined and formed approximately a circle round the root of the negative terminal as centre, but the external boundary was of course irregular. This showed beyond a doubt that the streams of molecules from the sides of the negative terminal had caused the streams from the interior of the glass beneath the finger to deviate from their course, and, instead of passing along parallel to the negative terminal, to be inclined at an angle to it, and thus to form the annular patch already described. And the truth of this conclusion was made still more evident when the finger was placed on the side of the tube so as to be at the point on the normal section through the negative terminal at the greatest distance from that terminal. The relief-phosphorescence then appeared to be cut in two by a broad and comparatively black space with roughly parallel sides, showing that the molecular streams from the sides of the negative terminal had diverted the streams that were going to form the relief-phosphorescence.†

We will now describe certain experiments which although they closely resemble the case which we have just mentioned have got an individual value from the remarkable way in which they support the whole theory of the intermittent discharge as put

* A splendid example of the interference of molecular streams is obtained by the same means when the negative terminal is in the very usual and convenient form of a hollow cone. The molecular streams that proceed from it first strike the sides of the tube at a little distance from the negative terminal, thus leaving a zone quite destitute of phosphorescence. If the finger be placed upon this zone (the air-spark being of suitable length) the whole phosphorescence on the tube is affected. The molecular streams from the finger coming across the cone of molecular streams from the negative terminal cause them to deviate *en masse* from their previous course, and thus throw the phosphorescence upon the other side of the tube. The effect is generally very striking, and this property of the dark zone near the negative terminal in such tubes (and also in a lesser degree in tubes that have their negative terminal in the form of a disc) has led us to give to it the name of the *sensitive zone*.

† It is instructive to compare this with the behaviour of the positive pole under similar circumstances. If relief-phosphorescence be thrown across the positive terminal its shadow is as fine and sharp as though it were a non-conductor. The reason is obvious. If it has at the moment any electrical function at all it is that of receiving and not of giving forth negative electricity. Hence there are no molecular streams proceeding from its surface which could cause those that pass near it to deviate from their course.

forward by the authors of these papers. A loose skeleton-tetrahedron of copper wire was enclosed in a tube of high exhaust. A discharge with a positive air-spark of considerable length was made to pass through the tube and the finger was placed on the tube just opposite to the place where the tetrahedron lay. The shadow of the wires forming the tetrahedron was cast upon the relief-phosphorescence in precisely the same clear sharp way that would have been the case had the molecular streams proceeded from the negative terminal (Plate 28, figs. 21 and 22). A conductor was brought into contact with the outside of the tube exactly at the point where one of the angles of the tetrahedron was in contact with the inside of the tube. Instantly all the shadows of the wires bulged out to a breadth depending on the distance from the glass of the part casting the shadow. This was so precisely the counterpart of the experiment above referred to that it was impossible to mistake its meaning. The conductor permitted negative electricity to pass off from the inside of the tube (and therefore from the angle of the tetrahedron, which was situated there) in obedience to the demand for negative electricity that was created by the positive discharges through the tube. But the whole of the tetrahedron being metallic, a supply of negative electricity to one portion of it enabled the whole to act as a negative terminal, and hence negative electricity streamed from all portions of it in obedience to the general demand for it. With this negative electricity there streamed of necessity molecules, and these streams of molecules diverted from their course the similar streams that were passing by on their way to produce relief-phosphorescence, and hence came the bulging out of the shadows. Those parts that were most distant from the glass diverted the molecular streams at an earlier period of their course than the other parts; and it was in the shadows of the former that the bulging was most conspicuous, showing that the effect was a true diversion or alteration of the direction of motion of the molecules.*

It scarcely needs further discussion to show that the above is the true explanation of the phenomenon. But there are one or two cognate experiments which serve to establish this yet more clearly. A similar experiment having been tried with other pieces of metallic wire with like results, it was then tried with a wire-shaped piece of glass. No such effect followed. The glass being a non-conductor, the electricity could not pass from one part of it to another. The air-spark was then changed to the nega-

* A slight peculiarity in the phenomenon ought to be here mentioned as indicative of the way in which intermittent discharge is obedient to local circumstances. One of the edges of the tetrahedron chanced to be made of two very fine parallel wires. The sharp shadow of the tetrahedron represented this perfectly, having a narrow green line between the two black lines which formed the shadows of the two fine wires. When the conductor was brought into contact with the tube, as before described, the shadows of these two fine wires bulged on the outside but not on the sides where they were nearest to one another, and thus the narrow green line was left intact. The inductive repulsion of the negative electricity in the two wires prevented any discharge taking place from them towards one another, and hence there were no molecular streams from those sides of the two wires that lay closest to one another, and there was nothing to impede the transverse molecular streams that sought to pass between these wires.

tive and a piece of tinfoil immediately over the tetrahedron was connected with the negative terminal, thus forming the negative special which, as we shall see (and as is otherwise obvious) produces precisely the same phenomenon as the relief-positive. Just as before, the shadow of the wires of the tetrahedron was cast, sharply and clearly, upon the bright patch of phosphorescence which was formed thereby. A conductor was then brought, as before, in contact with the tube at one of the corners of the tetrahedron. No bulging out followed. The relief afforded by the conductor only permitted positive electricity to stream from the wires of the tetrahedron, and as this was unaccompanied by any molecular streams it was powerless to divert those that were passing across the tube.*

Secondly. It is experimentally evident that the shadow is produced by a depression or deviation of the streams of molecules in question which come from the negative terminal. An examination of the appearance is well nigh enough to establish this, for it is easy to see how the rays are bent down, and how the parts upon which they are constrained to fall grow brighter thereby.† But there is an experiment which puts this beyond a doubt. In a tube kindly lent by Mr. CROOKES there is an intermediate terminal, nearer to one end of the tube than to the other, partly composed of a flat piece of aluminium, the flat sides of which are turned towards the terminals of the tube, and in this flat piece of aluminium there chance to be some little holes (Plate 28, fig. 23). When the more *distant* terminal is made *negative*, a bright image of these small holes appears on the side of the tube in the midst of the general shadow of the intermediate terminal. When the tube is touched on the side on which this image appears, but at a point on the negative side of the image, it is found that the image is splayed out, part being thrown down or farther along the side of the tube. If the finger be moved a little to one side the splaying out moves towards the other side.‡ This seems to show distinctly that the effect of the finger is to push away from it the rays going to form the image.

But it is possible to prove in an even more direct way that the virtual shadow is

* An interesting variation of these experiments is obtained by placing the finger beneath the tetrahedron and casting the shadow of the apex on the opposite side of the tube. If the tetrahedron be a non-conductor a somewhat magnified shadow will be seen; but if it be a conductor this will be splayed out to an enormous extent, for the finger which supplies the negative electricity which permits the production of the relief phosphorescence also enables the edges of the tetrahedron to become sources of negative discharge, and thus to splay out the shadows which they cast upon the relief-phosphorescence.

† This is clearly shown when we form a virtual shadow at a spot that lies within an already existing virtual shadow. If the former would, if it alone existed, extend to any part of the surface beyond the limits of the latter it will still do so, and the bright line that marks the edge of the virtual show will have an angle at the spot where their boundaries would cross and will thereafter follow the outline of the outer shadow. This shows that each relieving system produces its effect independently of the other.

‡ Similar effects are produced upon the edges of the shadow of the intermediate terminal itself, but the phenomenon is best observed in the way described in the text as the displacement or distortion of the small isolated bright spots which form the image of the little holes in the aluminium is more capable of being accurately observed.

due to a turning aside of the molecular streams that come from the negative terminal. A tube was taken in which there were a number of small objects, such as pieces of glass. The tube being of a very high exhaust there was phosphorescence over the whole of the interior of the tube, and all these small objects cast shadows, the pieces of glass also giving out phosphorescence. If the finger was brought in contact with the tube a little on the negative side of any one of these small objects (Plate 29, fig. 24), it was found that the molecular streams that were proceeding down the tube from the negative terminal were so far turned aside by the cross currents of molecules that proceeded from the interior surface of the tube beneath the finger that they no longer struck the object but passed completely over it, so that it gave no shadow at all. And the fact that the molecular streams did not reach it was not only shown by the absence of a shadow but it was also directly demonstrated in the case of the small pieces of glass by the fact that they ceased to manifest phosphorescence ; so that it was established beyond contradiction that the molecular streams that formerly reached them no longer did so.

On these grounds, then, we may accept the hypothesis that the virtual shadow is due to the beating down of the streams of molecules that pass along from the negative terminal. It will be remembered that they are moving at a very slight angle to the sides of the tube, so that a very small deflection would suffice to account for this phenomenon. This last point seems to be an essential condition for the formation of the virtual shadow, for in the case of short tubes, in which the negative terminal is a wire placed axially and projecting a considerable distance into the tube so that the molecular streams would no longer proceed at a slight angle to the sides of the tube, it has been found difficult to get good virtual shadows even with a fairly long positive air-spark.

An experiment ought to be mentioned here which is serviceable in rendering more complete the chain of proof, viz.: the production of a double virtual shadow when the negative is bifurcated. This was done in the tube of Mr. CROOKES, to which we have already referred, which was furnished with an intermediate terminal which was made positive, while the two terminals at the extremities of the tube were made negative (Plate 29, fig. 25). When the finger was placed upon the tube two shadows were clearly seen pointing in opposite directions, showing that these shadows point *from* the negative and not *to* the positive ; a law the truth of which might perhaps have been anticipated, but which, so far as it can, serves to show the correctness of our conclusions.

XXIII.—*On the relief-effect in tubes of high exhaustion with a positive air-spark.* III. *The positive luminosity and its attendant phosphorescence.*

If a tube be watched during the process of exhaustion, and the phenomena be carefully noted which it presents when a current with a positive air-spark is passing through it, the positive luminosity will be found to go through very marked changes

of form. As it approaches the high state of exhaustion which is usually associated with phosphorescence, the positive column will be found to diminish in diameter until it is reduced to a small pencil-like column very sensitive to the approach of a conductor, which, on account of the difficulty of keeping the tube free from all disturbing influences, usually stretches along the glass on one side of the tube instead of taking the normal centrical position. Its extreme sensitiveness often causes it to be crooked, and it may sometimes be seen crossing the tube from side to side in a zigzag manner.

This luminosity can be driven against the opposite side of the tube by the approach of the hand or any other conductor to the tube. Contact with the tube is so far from being necessary, that the violent action which it causes entirely masks the phenomena of which we are about to speak. The best effects are got when the hand is from three to six inches from the tube. When this pencil-like column is thus in contact with the tube it will be found that a long streak of the well-known phosphorescence marks its line of contact, and if the hand be brought a little nearer, so as to force the column, as it were, with more violence against the side, this green streak becomes very bright and vivid (Plate 29, fig. 26). It is extremely mobile, for it moves with the positive column, and is beyond all doubt attendant upon it. We shall hereafter see that they can be made to separate slightly by special precautions being taken, but this only serves to prove more clearly how closely they are attendant the one on the other, for it is with difficulty that the separation takes place, and as soon as the disturbing influence is removed they come together again.

But it is not only in this way that the phosphorescence attendant upon the positive luminosity becomes visible. If from any cause the positive column, taking a zigzag course, impinges, so to speak, on the side of the tube, a green patch of phosphorescence appears. And when the finger is placed upon the tube so as to form a patch of what we have termed relief-phosphorescence, although this seems to blot out the feebly luminous positive column, yet its presence is shown by its attendant phosphorescence. If some other conductor be brought near the tube, not on the same side as the finger, it will be found that the true relief-phosphorescence remains unmoved, but that there is an extremely mobile portion of the phosphorescent image of the finger which by its sensitiveness shows that it has a different origin and nature to the immovable relief-phosphorescence. If this portion be traced out, as can be easily done since its mobility enables it to be readily distinguished, it will be found to be in the line of a prolongation of the positive column, and thus it is evident that it is the attendant phosphorescence of which we have been speaking.

Before we proceed to examine in detail this most peculiar phenomenon it will be well to mention that one of the most convenient modes of obtaining it is by means of the positive unipolar or double unipolar discharge. Using this arrangement, the thin pencil-like form of the positive column appears in tubes whose exhaust is not sufficiently perfect, or in which other circumstances are not sufficiently favourable to give it when a current passes through the tube. And just as in less perfectly

exhausted tubes each of the two positive columns of the double unipolar discharge assumes a thick tongue-shaped form, the tips pointing towards each other, so in higher exhausts we find two long pencil-like columns each tapering away to a point, and each accompanied by its attendant phosphorescence. Either of these can be used for the purpose of examining the properties of this phosphorescence, and the results will hold good in all cases, however the pencil-like column be obtained. There is no such pencil-like column when a negative unipolar is observed.

The first thing to do in the analysis of this attendant phosphorescence is of course to ascertain the direction and, if possible, the source of the streams of molecules which produce it. This was done by means of a tube into which had been introduced a number of crooked pieces of wire and small bits of glass. When the attendant phosphorescence was brought up to these it was found that (subject to the qualifications to be given later on) *the shadow of each object lay in the same normal section of the tube as the object itself*, and was, roughly speaking, in the position in which it would have been had the seat of the discharge been the opposite side of the tube. This conclusively showed that the streams producing this attendant phosphorescence did not come from the negative terminal, but were due to local action in the tube, and that the *direction of these streams was normal to the axis of the tube.*

Further and more minute investigation has since shown that these results require some modification when the part of the attendant phosphorescence which is being examined is situated at some bend or corner of the zigzag course of the positive luminosity, and that they can only be taken as accurate where the luminosity is not very greatly curved. This result, so far from affording any argument against the theory that this phosphorescence is due to local action, strongly supports it, as showing that peculiarities in the direction of the molecular streams are only found when we should, from the circumstances of the case, expect to find peculiarities of local action. No doubt in such cases the direction of the streams is a resultant of conflicting tendencies. We shall, therefore, neglect such cases for the present, and confine ourselves to the case of a fairly straight positive column where, as we have already said, the streams at each point come to the glass along paths lying in the normal section of the tube at that point.

We see at once that we have here a phenomenon of almost unexampled importance in the analysis of the *modus operandi* of the positive discharge. We find that its passage is accompanied by discharges of negative all along the tube, the molecular streams accompanying which move in directions normal to its axis. Taking as a provisional hypothesis* that these start from the sides of the tube, we see that we have a process of relief continually going on. But the most remarkable point seems to be that these streams either start in the direction of or are subsequently directed to the thin column of positive luminosity. This leaves scarcely any room to doubt that

* We shall presently see that there are experimental grounds for believing this hypothesis to be a correct one.

it represents the *locus of the exciting cause of this local action.* We, therefore, have direct indications of this being a locus where the positive electricity (which alone can be the exciting cause of these negative discharges) obtains relief from the negative given off in the tube.

It will here be the place to mention an observation of a very remarkable kind which was made upon the shadow thrown by a crooked wire within the tube upon this attendant phosphorescence. The portion casting the shadow was nearly straight and had a direction lying in a normal section of the tube somewhat inclined to the radius of that section on which its foot lay. As the narrow green line was driven round the tube towards the wire the shadow of the wire appeared upon it, but not crossing it. When it was driven still nearer to the foot of the wire the shadow, instead of remaining fixed, and thus more and more completely crossing the green line, was found to shrink in and grow shorter, and continued only partially to cross the green line, although the latter had moved so much that it would have been completely divided had the shadow retained its original length. When, however, the green line was driven still closer up to the foot of the wire the shadow ceased to shrink below a certain length, and finally passed quite across the green line. The solution of the peculiarities here exhibited will be found, we believe, in the hypothesis that the streams of molecules come off from a considerable segment* of the surface of the tube situated in the region opposite to the green line, and they by their mutual action, or by the direction in which they originally start, stream towards the thin column of positive luminosity without their paths crossing at the axis of the tube or at any other part of their path.

We have said that this thin luminous column with its attendant phosphorescence is sensitive in the highest degree, and is strongly repelled when the hand is brought anywhere in the neighbourhood of the tube. It might be thought, therefore, that it is nothing but a special form of relief phosphorescence, and that it differs from what we have treated of in a previous section only in that it is the portion first formed and not needing actual contact for its formation. But this is by no means the case. The two are wholly different in genesis and in properties. Nothing is easier than to distinguish them. Relief phosphorescence is quite fixed in position unless another centre of production is formed by touching the tube in its immediate neighbourhood, in which case the two systems of streams of course interfere the one with the other. But even when contact is made with the tube it is quite easy to drive this attendant phosphorescence about by bringing other conducting systems into the neighbourhood of the tube, although they do not produce the slightest effect upon the relief phosphor-

* This is confirmed by the fact that it is impossible to cast a shadow upon this attendant phosphorescence unless the object casting it be very near to the side of the tube where the phosphorescence appears. And even then it has a tendency to be blurred and hazy. This indicates that the molecular streams that cause it do not proceed from any one spot, but that they converge upon it from a considerable region, all their directions being, however, approximately situated in a normal section of the tube.

escence. Then, again, the shadows cast by the two will often be in different directions. Moreover, if an object cast a clear shadow upon the relief-phosphorescence and the mobile portion of the phosphorescence (which we have seen belongs to the attendant phosphorescence) be brought up to it, we shall generally find that it blurs it so completely as to render it difficult to make out its outlines, thus showing that the streams which give rise to it do not take their origin in the same spot as those which occasion the relief-phosphorescence. In short, however we examine the matter we are continually met by fresh proofs of the radically distinct nature and origin of the two kinds of phosphorescence.

Perhaps the most convincing experiment upon this point remains to be described. A tube was taken in which were a quantity of very thin films of glass lying loose in the tube. A spot was selected where one film lay over another with a very shallow interval between them. The relief-phosphorescence was thrown upon the spot of the tube where they were lying. It caused bright phosphorescence on the surface of the upper one, but left wholly unilluminated the one that lay beneath it. The positive luminosity with its attendant phosphorescence was now driven behind the films. The lower as well as the upper was now illuminated by phosphorescence. Now we know that the streams producing this phosphorescence are normal to the axis of the tube. Hence it is clear that the exciting cause of the negative discharges which caused these streams must have been below the upper film (and probably below the lower one also) since it caused streams to impinge on the lower film which could have come from nowhere but the lower surface of the upper film. Thus we see that we have the most cogent evidence for supposing that the thin column of positive luminosity really represents a locus of demand for negative electricity and excites discharge from the sides of the tube, and possibly from the gas, to satisfy it.

In order to establish more firmly the position that this attendant phosphorescence is caused by streams that come towards it in directions normal to the axis of the tube, and principally in directions inclined but slightly to the diameter of the normal section through the point, an electro-magnet was placed with its axis perpendicular to that of the tube and its pole near to the tube, but not near enough to produce relief-phosphorescence. The position of the green line was carefully observed, and then the current through the coils of the magnet was turned on. The instantaneous effect was to twist the green line into the shape of an S. Now if it be remembered that, roughly speaking, the directions of these streams of molecules are diametral, and therefore in the original position were parallel to the axis of the magnet, it will be seen that the effect observed was precisely what should have been expected. Molecular streams parallel to the axis of the magnet would be bent into helices, and accordingly those on one side would appear to be thrown one way and those on the other would appear to be thrown the other way. In order to establish experimentally that this effect would be produced upon a system of molecular streams constituted as described, a broad slip of tinfoil not quite long enough to go round the tube was placed on the

tube. It left, of course, an opening along its length, and when it was connected to earth this became phosphorescent from the relief-phosphorescence. The directions of the streams producing this phosphorescence were substantially the same as those to which we have seen reason to believe the attendant phosphorescence is due. An electro-magnet was tried upon this strip of phosphorescence, and it produced similar results to those above described, thus strongly confirming the truth of the theory we have adopted to account for the genesis of the attendant phosphorescence.

One more peculiarity remains to be noticed with regard to this attendant phosphorescence. It is possible, as we have said, to cause it to separate slightly from the positive luminosity with which, as we have said, it is so closely bound up. One way is by bringing the positive luminosity across some piece of wire or other conductor in the tube; but as this phenomenon is obscure and difficult to observe we shall not dwell on it further. The other way is by approaching a conductor to the tube on one side or the other of the position which surrounding circumstances have constrained the positive luminosity to take up. It is often possible by doing this to drive the green phosphorescence a little distance off on the further side of the positive luminosity to that on which the conductor is. It would seem as though the phosphorescence were more strongly repelled than even the positive luminosity. A straight wire in connexion with earth was placed at an angle to the axis of the tube, and made to approach it. The green line assumed a serpentine form, cutting the positive luminosity at its point of inflexion (Plate 29, fig. 28). The solution doubtless is connected with the fact that the presence of the conductor renders negative discharge easier from the part of the tube nearest to it, and that the streams of molecules, in consequence of their inertia, persist in maintaining a nearly diametral path although the direction from which comes the demand for the negative discharge which is inclined at an angle to this.*

We will now return to the excepted case in which the positive luminosity is made to pass in a zigzag direction through the tube (Plate 29, fig. 27). As we have said, the places where it appears to impinge on the side of the tube are marked by patches of phosphorescence. These patches are as a rule exceptionally brilliant. If an object within the tube be situated so that the column of luminosity comes into contact with it on thus crossing the tube, it will be found that it casts a shadow in the direction in which the column strikes it—that is to say, in the direction of the course of the column as we proceed towards the positive terminal. In fact, in such cases the shadow is much as it would be if the column were threaded throughout its whole length by molecular streams coming from the negative terminal and proceeding towards the positive terminal.

Now before we proceed further we must call attention to the fact that it is only

* If the conductor be brought towards the other side of the tube exactly opposite to where the luminosity and its attendant phosphorescence is situated it often splits the line of phosphorescence into two, one situated on each side of the thin column of positive luminosity.

where the positive luminosity has such a tortuous course as has been described that this phenomenon presents itself. Repeated experiments have demonstrated beyond the possibility of doubt that where the positive luminosity lies along the inner surface of the tube (as is usually the case) the image of a small object lying over it will be cast perpendicularly upon it. Hence the case we are about to discuss must be taken as an exceptional case due to the presence of special circumstances, the nature of which it is one of our aims to discover, but not in any way casting doubt on the conclusions which we have already drawn from observations upon the normal behaviour of the positive luminosity and its attendant phosphorescence.

If the object which is brought into the path of the positive luminosity in its passage across the tube be a non-conductor, there is no special peculiarity in the shadow case. It will be an ordinary phosphorescent shadow, whose direction is defined in the way we have given above. But if it be a conductor, and especially if it be a portion of a conductor of some little magnitude such as the end of a piece of wire, it will be found that the shadow is bulged out to a very considerable extent. And if the finger be placed against the outside of the tube exactly opposite to the other end of the small conductor, this bulging out will become immensely increased.

Now there can be but one possible interpretation of these phenomena. The bulging out must be caused (as we saw in a previous case) by a discharge of negative electricity from the sides of the wire, and such discharge must be in response to a demand for it in the tube. But here we come to a very remarkable peculiarity of the present case. This demand must be intensely local; for while in the case of relief phosphorescence there was no perceptible bulging out of the shadow of a conductor that was partly within the range of the streams that were crossing the tube, such a bulging out not only occurs in the present case but is a most marked phenomenon. Thus we have direct proof that the positive luminosity marks a locus of intense demand for negative electricity.

A very curious variation of this experiment may here be referred to in order to strengthen the conclusions just drawn. The zigzag positive luminosity was made to cut the thin projecting wire that formed the positive terminal. Its shadow gave no sign of bulging out, and behaved as though it was a non-conductor. It is obvious that no negative electricity could be drawn from it as a response, and hence there was no bulging out of the shadow. To test the matter still further, the metallic object that had previously been experimented on was shaken down into contact with the positive terminal, and its shadow was observed. It was found to have no bulging, but to be thin and sharp like the shadow of a non-conductor.

We have thus direct evidence of the intense local demand for negative electricity in the track marked by the positive luminosity.* It seems paradoxical that this can co-exist with streams of molecules proceeding along it. But it must be remembered that we have no evidence that they are (at all events throughout the whole of their course)

* We shall return to this subject in Section XXVIII.

carriers of negative electricity, and the experiments which we have just described seem to show almost conclusively that such is not the case.

The property which, as we have just seen, conductors within the tube possess of becoming negative terminals and thus giving out negative electricity on all sides, seem to account for a peculiarity in the attendant phosphorescence which merits remark. If the hand be passed along the under side of the tube at a little distance from it, the line of attendant phosphorescence will be seen sharply and clearly defined along the top of the tube until the hand comes to a place where a conducting object is lying in the tube. The line of phosphorescence will there be split in two and become irregular in outline, and will join again beyond the spot where the conductor lies, thus enclosing within it a space with no phosphorescence. This experiment is also of use as showing strong grounds for holding that the streams which produce the attendant phosphorescence come from the opposite side of the tube.

It will now be seen how special is the importance of the evidence given us by these molecular streams. Without the definite evidence which they give of the existence of sources of negative discharge, it would have been left to speculation to determine the nature of the action in the tube which accompanies a discharge. And as we have seen that these molecular streams to which phosphorescence is due are not peculiar to tubes of high exhaust, but probably exist as an accompaniment of negative discharge in tubes of all degrees of vacuum, we see that we have here a fresh step in the analysis of the mode of propagation of vacuum discharges in general.

XXIV.—*On the special effect in tubes of high exhaustion with a positive air-spark.*

We have seen in our former paper that the characteristic peculiarity of the special effect in tubes of low vacua with a positive air-spark is that the luminosity is attracted instead of repelled. If a wire from the positive terminal be carried parallel to the tube, and at a little distance from it, a line of luminosity will appear on the side of the tube nearest to the wire throughout its whole length.

If the same experiment be tried in high vacua precisely the same effect is produced. The thin pencil-like column of which we have spoken in the last section will be found pressed close to the side of the tube nearest to the wire. It follows all the movements of the wire, and takes a curved direction when the wire does so. This alone would furnish a strong presumption of the radical identity of the modes of discharge in low and high vacua, were any further proof needed.

There is, however, an apparent peculiarity in the behaviour of this thin pencil-like column, when the special effect is produced, which must be mentioned. When the piece of metal attached to the wire which is in connexion with the positive terminal is brought close to the tube there is no longer an attraction of the positive luminosity, but a strong repulsion (Plate 29, fig. 29). There is nothing in this phenomenon to invalidate the above conclusions. An exactly analogous phenomenon is observed in the case of tubes of

less perfect vacuum. When the body producing the special effect is brought up to the tube, the positive discharge caused by it stretches away towards the negative terminal, and the part of the positive luminosity on the positive side is depressed, owing to the formation of an imperfect negative dark space beneath the wire or tinfoil—a depression which would, if symmetrical all round the tube, cause the positive column to pass into the thin central column which leads up to the termination of the truncated portion. There is no doubt that the phenomenon observed in tubes of high vacua is substantially the same, except that the characteristic feebleness of the positive luminosity in such tubes prevents there being sufficient definition in the positive discharge caused by the special effect to enable the eye readily to recognise the identity of the two phenomena; while, on the other hand, the greater breadth of the negative dark space in tubes of high vacua makes this repulsion of the positive column a more striking phenomenon than its analogue in tubes of lower vacua. That the above is a substantially accurate interpretation of the phenomenon there is no doubt, for if the special effect be produced by a ring of tinfoil we find the well-known truncated positive column; the hollow cone around it being, however, as we should expect from what has been said above, faint and very ill defined.

We see then that positive special causes the positive luminosity to locate itself on that side of the tube along which runs the wire in connexion with the positive terminal. And we have also seen in the previous section that there is a constant discharge of negative electricity towards this positive luminosity from the surrounding parts of the tube. It follows, therefore, that we may expect positive special to be marked by the appearance of phosphorescence at the place where the positive special is being produced, and not, as in positive relief, on the opposite side of the tube; and this has been found to be the case. The thin pencil-like column of positive luminosity that appears along the line of the wire is accompanied by the same attendant phosphorescence that we have discussed in the previous section. Moreover, the inside surface of the tube immediately beneath a piece of tinfoil connected with the positive terminal has been observed to be covered with the well-known green phosphorescence, and even a shadow has been thrown upon it from a film of glass within the tube, the edge of which was over it and very near to it, and also from other small objects lying in similar positions. This completes the proof both that the positive special is in effect the creation of a virtual positive terminal on the inner surface of the tube, and also that where there is such a virtual terminal or centre of instantaneous discharge the portions of the tube near thereto give off negative electricity to satisfy it; and with this negative discharge there are the usual molecular streams. And it further confirms the view that neither the direction of the negative discharge nor that of the molecular streams is independent of the position of the spot from which the demand comes.

The action of the positive special is of course of considerable violence. This enables us in a very convenient way at once to demonstrate the substantial identity of origin

of the two forms of attendant phosphorescence described in the previous section, and also to obtain them at will. If a long slip of tinfoil be laid along a high vacuum tube and connected to the positive terminal, and a considerable positive air-spark be used, the thin positive luminosity will be seen to lie along the slip till we arrive at the end of the strip nearest to the negative terminal. Here it will naturally cease clinging to the side of the tube and slant off to a more central position (Plate 29, fig. 30), thus causing an angle in its direction. The attendant phosphorescence can be distinguished under favourable circumstances all along the line of the slip of tinfoil; and the spot where the angle is formed in the direction of the positive luminosity as above described will be found to be covered with bright phosphorescence, precisely as was the case when the positive luminosity was examined by the methods detailed in the previous section. The best arrangement for securing these effects is by the use of the positive unipolar discharge. So clearly are all the details of the phenomenon shown thereby that the attendant phosphorescence is readily distinguished in the shape of two bright green lines running along the two edges of the slip of tinfoil, these edges being doubtless the operative parts of the tinfoil considered as creating a quasi-positive-terminal in the interior of the tube. By this arrangement it is perfectly easy to obtain these effects in a sufficiently brilliant form to permit of the ready determination of the direction of arrival of the molecular streams which cause the phosphorescence. It will be found that the results in all cases are in conformity with the conclusions already arrived at.

But it is not only in the above manner that positive special produces phosphorescence. If we recall to mind the special effect in low tension tubes with a positive air-spark, we shall remember that the hollow cone of luminosity which marked the positive discharge in the tube directly caused by the positive impulses within the tinfoil, was separated from the truncated positive column by a dark space, across which occurred the action by which the advancing positive discharge became satisfied by the negative electricity left behind in the tube beneath the tinfoil. The discharge of this latter from the glass under the influence of the advancing positive electricity is a case of true negative discharge, and accordingly we find that in tubes of sufficiently high tension with sufficiently long air-sparks it is accompanied by the usual molecular streams, and that it produces the usual phosphorescence. This phosphorescence must be due to a negative discharge commencing almost synchronously with that of positive relief, for they both take place through the action of the advancing positive electricity, but it is of much less violence. If a piece of tinfoil be placed on the spot where the positive-special-phosphorescence falls, and then be connected to earth, it will be found that it clears the glass beneath it and for a short distance round its edge from phosphorescence; while, on the other hand, it is able to throw phosphorescence over the glass immediately below the tinfoil that is connected to the positive terminal. This superior intensity of relief phosphorescence will be found to be of importance later on.

Since positive special is accompanied by a negative discharge which commences almost synchronously with that which gives rise to relief phosphorescence and virtual

shadows, we might expect that it would also give rise to virtual shadows. And such is found to be the case, although the effects are decidedly more feeble than in the case of positive relief. They are, however, generally present, and under favourable circumstances are clear and distinct (Plate 29, fig. 31). They have not formed the subject of any special examination as yet, but so far as can be determined by inspection they do not materially differ from those due to positive relief in any other respects than would naturally flow from the inferior intensity of the action to which they are due.

Thus we see that positive special effects are clearly distinguished from relief effects by the attraction of the positive luminosity and the appearance of phosphorescence *on* the tinfoil. It is true that they both produce virtual shadows and phosphorescence on the glass opposite to the tinfoil, but these are given only in a feebler degree, and as secondary phenomena, by the positive special. It is worthy of remark that in the latter case we have the phosphorescence both on and opposite to the tinfoil, a phenomenon of which no instance will be found in the experiments on interference which have hitherto been described, and which at first sight appears to contradict the laws of interference of molecular streams already established (see page 609). It is probable that the solution of the difficulty is, that the arrival of the molecules which causes the one is not synchronous with the discharge which causes the other, but the authors of the present paper have been unable to come to any definite conclusions as to the way in which this occurs.

XXV.—*On the relief and special effects in tubes of high exhaustion with a negative air-spark.*

We have already noticed that the main peculiarity of discharges in high as compared with low vacua is the prominence of the special characteristic of negative discharge, viz.: molecular streams, and the comparative insignificance of the special characteristic of positive discharge, viz.: positive luminosity. So much is this the case that while in the tubes of low exhaust we found it necessary to rely chiefly on the effects of positive discharges, as they alone rendered themselves plainly visible, we are, in the case of tubes of high exhaust, compelled to rely chiefly on the effects of negative discharges to guide us in our investigations.

It follows naturally from these considerations that negative-relief is a comparatively uninteresting subject of investigation, for all the impulsive discharges produced thereby being discharges of positive luminosity, are extremely feeble in their luminosity, if not practically invisible. It is in fact exactly similar to positive special, except that the unaffected state of the discharge is different, the luminosity in the case of a negative air-spark being in the form of a diffused haze, and not in the form of a thin pencil-like column. And even this difference contributes to make the phenomena of negative-relief still less impressive and striking.

But, as we have already said, it is not only the negative-relief that has its counter-

part in the effects with a positive air-spark. The negative-special presents precisely the same phenomena as the positive-relief. We get perfect virtual shadows and bright patches of phosphorescence on the other side of the tube. Indeed, so perfect is the identity of effects that it would be just as possible to work with one as with the other, were it not that it is so much more convenient to examine relief effects than special effects on account of the simpler character of the manipulation.

There is, however, in all cases a very superior sharpness of action in the positive effects of which we shall have to speak later; but, allowing for this, it would be difficult to overstate the perfect correspondence that there is between the positive-special and the negative-relief, and between the positive-relief and the negative-special. The splendid phosphorescent effect and virtual shadows produced by the two last mentioned are so strikingly alike as to need no further comment; but this resemblance extends to the less striking phenomena. For instance, if we produce the negative-special by the aid of a wire from the negative terminal brought parallel to the tube, and near to it, we shall find that the previously diffused haze is driven into a thin column on the other side of the tube, closely resembling the pencil-like column of the positive air-spark. This can be driven about in like manner, and even attendant phosphorescence has at times been faintly visible. If the negative-relief be produced by a similarly placed wire connected to earth there will be a concentration of the haze on the side of the tube nearest to the wire, as in the case with the positive-special. We have little doubt that further investigation will show that there is here also attendant phosphorescence, though it is difficult to detect it.

The patch of phosphorescence on the glass immediately beneath the tinfoil of which we have spoken in treating of positive-relief has also been seen in negative-relief. The existence of phosphorescence on the glass opposite the tinfoil and that of the correlative virtual shadow are, however, more difficult to determine. We have not obtained the former in a satisfactory manner, and though we have often seen feeble virtual shadows with the negative-relief it has always been a matter of doubt with us as to whether some slight positive intermittence had not crept in and caused it. Our own opinion is that these two secondary phenomena are in the case of the negative-relief too feeble to be generally visible, though the virtual shadows (which require a much less violent action than does the production of phosphorescence on the opposite side of the tube) may occasionally do so.

But although for these reasons we do not find it profitable to examine with special care the phenomena of the discharge with negative air-spark, the importance of the general identity of these effects with the converse effects in the case of the positive air-spark is very great as a proof of the main proposition in the theory which the authors of this paper have put forward with regard to the disruptive discharge in its intermittent form, viz.: that it takes place from either terminal, and that, in general, the electricity passes along the whole of the tube in the form of a discharge of the same name as that of the terminal from which it proceeds, and only meets with a

response from the other terminal when it has arrived in its immediate locality. The argument here is the same as that used in the former paper, and therefore need not be repeated, and it is only referred to in order to show that the evidence and the reasoning that sufficed to demonstrate it in the case of tubes of low exhaust are equally applicable and equally effectual here. And it is important to notice how persistently the characteristics of the positive luminosity in the case of the two air-sparks which we met with in tubes of moderate exhaust remain unchanged when the exhaust is pushed further. We know that with a positive air-spark the tendency is for the luminosity to shrink from the sides of the tube into a bright central column, smaller than the interior diameter of the tube, while the tendency of the positive luminosity with a negative air-spark is to spread out through the whole of the interior of the tube (or, at all events, to fill all the peripheral parts of it) and to become hazy in so doing. The pencil-like column of the positive intermittence and the diffused haze of the negative intermittence in high vacuum tubes represent the extreme forms of these peculiarities.

We shall not, therefore, pursue the question of negative discharges any further at present. The general result that they correspond to the converse effect of the positive intermittence, except that the definitión is less perfect, and that the feebler and secondary effects are difficult to obtain, will render the remarks in the previous sections applicable to the case of the negative intermittence. The special peculiarities of negative discharge will be dealt with in a subsequent section.

XXVI.—*General conclusions as to the electric discharge.*

I. *The comparative magnitudes of the small time-quantities of the discharge.*

The problem of the physical nature of electricity is so closely bound up with the question of the distinction between positive and negative electricity, that the most hopeful way of approaching the greater problem is by solving the lesser. Now there is no class of electrical phenomena where the differences between the two kinds of electricity manifest themselves so strikingly as in the disruptive discharge, and hence this is the best field for studying the contrasts between the two kinds of electricity, with a view of ascertaining the source of this contrast. The subject is naturally a very wide one, and we do not purpose to deal with it generally in the present paper. Our object is at present simply to record a few conclusions to which we have come bearing upon the *modus operandi* of the discharge.

We have frequently had occasion to remark upon the extremely short duration of the phenomena of the electric discharge. This discharge is, however, not equally instantaneous in all its phenomena. In the present section we shall examine the various small time-quantities of the discharge, in order to get a clear idea of the relative shortness of the periods which they occupy, for the purpose of guidance in our future speculations as to their relationships one to another.

These small time-quantities are as follows :—

1. The period occupied by a discharge.
2. The interval between two discharges.
3. The time occupied by the discharge of the positive electricity from its terminal.*
4. The time occupied by the discharge of the negative electricity from its terminal.
5. The time occupied by molecular streams in leaving a negative terminal.
6. The time occupied by positive electricity in passing along the tube.
7. The time occupied by negative electricity in passing along the tube.
8. The time occupied by the particles composing molecular streams in passing along the tube.
9. The time occupied by electricity in passing along a wire of the length of the tube.

To these there might be added the time that electric induction occupies in traversing a finite space. This we have taken to be zero, for we have not been able to discover any symptoms of its being durational ; or perhaps we should rather say that we have considered it either negligible in comparison with any of the above quantities, or included in them so as to be indistinguishable in order of magnitude from the shortest of them.

Taking, then, the small time-quantities which are enumerated above, we know, in the first place, that the interval between two discharges is incomparably greater than any of the other small quantities to which we have referred. This is shown by the revolving mirror. For although this instrument easily separates the intermittent flashes, it never shows any splaying out in the luminous phenomena of the individual discharges ; for the haziness which it sometimes shows as attendant on the phosphorescence does not testify to any durational character in the period of arrival of the molecular streams, but only to the power of the glass to retain phosphorescence for a short time after the exciting cause of it has ceased. We may therefore take the interval between two discharges as by far the largest of all the small quantities of which we have spoken.

This conclusion is what we should naturally have anticipated. For although we are not at present in a position to say whether a discharge through a vacuum tube occupies a greater or smaller period of time than an electric spark in air, yet they are phenomena of a like nature, and they probably occupy periods which are to some degree similar in point of duration. Now it is well known that the time occupied by an electric spark in air is almost inconceivably small. However fast a wheel be rotating, it will appear to be at rest when illuminated by an electric spark. And

* It will of course be understood that under the term "terminal" are included all sources of electrical discharge, whether effective terminals or quasi-terminals.

although the interval between two of the discharges considered in this paper may seldom exceed a thousandth part of a second, it is not surprising that it should be out of all proportion greater than the time actually occupied by the discharge. It is to this fact that we owe the isolation of the individual discharges of the sensitive discharge, which has formed the basis of the present investigation: an isolation which renders the examination of this type of discharge equivalent, as we have already mentioned, to the examination of the electric spark itself.*

We need not examine the magnitude of the period occupied by the whole discharge. It would be obtained by adding together a proper selection of the other small time-quantities with which we are dealing, and which are in reality the component parts of which the whole discharge is made up. And, further, it would possess no scientific value so far as the investigation of the mechanism of the discharge is concerned, for it is the relative duration of the various processes which go to constitute the complete discharge that it is important for us to learn.

We shall next consider the magnitudes of the periods occupied by the positive and negative electricities in passing along the tube, by which we mean the interval that elapses from the instant of their emission from one terminal to the instant of their arrival at a point situated in the neighbourhood of the opposite terminal. And we shall first show that *the time occupied by the passage of either electricity along the tube is of a greater order of magnitude than the time required for it to pass along an equal length of wire.*

The experiments given in Section XIII. of our former paper suffice to demonstrate this proposition. For when a piece of tinfoil near the air-spark terminal was connected by a wire with a piece near the opposite terminal, the former derived at least as much relief from the latter as if it had not been on the tube, while the special effect was manifested at the latter. This showed conclusively that at the time the electric disturbance arrived at the former piece of tinfoil the latter was unaffected by it; and further that there was time for the impulsive electrical action which was exerted upon the former piece of tinfoil to be communicated along the wire to the latter, and to affect it and the tube near it before the electricity itself passed up within the tube to the place where the latter lay. A similar phenomenon was presented when a long strip of tinfoil was placed along the tube. In that case there was a gradual shading from the relief to the special effect. It will be remembered that in this way we got perfect examples of the typical positive effect near the negative terminal by means of impulses from a piece of tinfoil laid on the tube near the positive end (the air-spark being in the positive), showing that the positive impulses had had time to run along the wire, to form the hollow cone of positive luminosity by means of the positive dis-

* In our previous paper an experiment was described in which the luminous phenomena of a single discharge from a coil were observed, such single discharge having the characteristics which an individual discharge of a sensitive vacuum discharge must have. It was found to exhibit all the phenomena of sensitiveness.

charge that was created by their arrival at the second piece of tinfoil, and to leave the corresponding negative electricity free to meet the advancing positive discharge before that discharge had time to come up. And similar results were obtained, *mutatis mutandis*, when the intermittence was of a negative type.

We have no means of comparing directly the velocities of positive and negative electricity in a tube. But in some experiments* which we described in our previous paper, in which the discharge was due to the action of a small coil, it was found that when all the circumstances were alike at both terminals the two electricities met about the middle, and that the neutral zone was situated there. Further, it was found that the neutral zone could be shifted toward one end or the other by the use of a thunder plate or small condenser which was hung on the terminal, the discharge from which it was desired to retard. These phenomena point clearly to the theory that the velocities of the two electricities are the same in such a tube, and we have seen no phenomena which would lead us to imagine that such is not the case generally.† At all events, in the absence of any evidence to the contrary, the most probable view is that their velocities do not differ largely, or in other words, that the times they respectively take to pass along the tube are small quantities of a like order of magnitude.

The next question is the comparison of the times that the discharges take to pass along the tube with the times that are occupied by their emission from the terminals. And here we are met by a difficulty in defining what is meant by the period of emission of a discharge. It may well be that if the discharge contains only a certain quantity its emission may be very rapid, while if it is greatly larger in quantity it may be durational, and the degree of exhaust may similarly affect it.‡ Something of the sort is pointed at by the contrast between the experiment last referred to and those about which we are about to speak. The former seems to point to considerable equality between the two electricities both as to velocity and rapidity of discharge. The latter will be shown to point to great inequality in rapidity of discharge, the positive having by far the advantage. The difference may probably be accounted for by the fact that in the former case the discharges were of a very gentle character and in a tube of low exhaust, while in the experiments to which we are about to refer the discharges are sufficiently violent to produce phosphorescence.

A further question arises here as to the time occupied by molecular streams in

* Phil. Trans., 1879, p. 210.

† It is true that the experiment has only been made in tubes of low exhaust, and that it is not safe to conclude that relations which exist in low exhausts between the properties of positive and negative discharges will hold good also in high exhausts. For example, we shall hereafter see that the degree of exhaust has a marked effect on their relative rates of emission. But in the present case we have direct experimental evidence that the velocity of negative electricity along the tube does not participate in this change, but that, on the contrary, negative electricity in tubes of high exhaust continues to behave as though its velocity were the same as that of positive electricity, or at least of the same order of magnitude as it. Hence we shall take such to be the case.

‡ We shall return to this question in the next section.

leaving a negative terminal. Now it is very difficult to imagine that these molecular streams can commence to leave the terminal before the negative discharge begins, or that they can continue after it has ceased; so that in the absence of anything pointing to an opposite conclusion, we are justified in assuming that the period of emission of the molecular streams is included within that of the negative discharge. There is not the same justification for assuming that these molecular streams continue to be emitted during the whole of the period during which the negative discharge is leaving the terminal, but we think it probable that such is the case, though it may be that they are not of the same intensity throughout so as to be equally capable of causing phosphorescence. Seeing, then, that it is probable that the two periods of emission are identical, and that this supposition will not in any way affect the validity of the argument, inasmuch as the period of the emission of the negative discharge is undoubtedly as long as that of the molecular streams, we shall treat these two periods as the same.

In order to compare the periods of time occupied in the actual emission of the two discharges respectively, we shall commence by showing *that the negative discharge occupies a period greater than that required by the particles composing the molecular streams to go the length of the tube but comparable with it.*

This proposition is placed beyond doubt by the various phenomena depending upon the interference of the molecular streams and especially virtual shadows. It is quite clear that if we give relief to a portion of the tube near the positive terminal during the progress of a discharge with positive intermittence, the streams of molecules from the sides of the tube must start before those from the negative terminal do so. It is true that the difference may be infinitesimal, but at all events it exists. Now we find that the streams of molecules that come from the negative terminal are interfered with and diverted by streams from the side of the tube. Hence these latter streams must have continued to flow at least as long as it has taken the molecules from the negative terminal to arrive at the point of the tube where relief is given. It may be that the main portion of these relief streams has passed across the tube before the arrival of the streams from the negative terminal, but the relief streams must be still continuing, or they could not interfere with the others. This last remark may serve to explain the fact that relief phosphorescence is usually diametrically opposite to the place where the relief is given, so that the streams that form it are not perceptibly swept downwards along the tube towards the positive terminal by the streams from the negative terminal.* The main body of the relief streams has had time to cross the tube and impinge on the glass before the streams from the negative terminal can reach the spot. But when we attempt to produce relief effects in the immediate neighbourhood of the negative terminal the case is far different. The two streams are then on equal terms,

* It would not be wise to attach too much weight to this. The interference of two molecular streams at right angles is not very great when it does not take place in the immediate neighbourhood of the source of one of them.

and the relief-phosphorescence is found to be swept downward along the tube, while the interference between the two streams is greatly increased for the reason that the main parts of both come into collision with each other.

Two other experiments in connexion with interference should also be considered in dealing with this matter. If a shadow of a small piece of conducting material, such as a piece of wire enclosed within the tube, be cast on the side of the tube by relief-phosphorescence and a conductor be brought in contact with the tube near to one end of the wire, the shadow of the whole wire, including the other end of it, will bulge considerably. This bulging is exceedingly black, showing that the whole of the streams that passed close to the wire on their way to produce the relief-phosphorescence have been diverted from their course. Thus the discharge from the wire must have continued during practically the whole of the time that the relief molecular streams were passing it. Now the discharge from the wire must have commenced at the same time as that from the side of the tube, for they were both in response to the advancing positive electricity, and they will presumably last for an equal time; hence the time required for the molecular streams to cross the tube and arrive at the wire is not an important part of the period during which the discharge lasts, for otherwise the relief molecular streams that passed after the discharge from the wire had ceased would probably have shown themselves in the form of phosphorescence on the bulged part of the shadow.

The indications derivable from the second experiment to which we are about to refer are yet more distinct. In the experiment described in Section XXII., in which a phosphorescent image was formed of a small hole in an intermediate terminal (the air-spark being in the positive), it was found that this image was *splayed out* by the finger being placed on the tube. Now a magnet displaced it as a whole without any splaying out (Plate 28, fig. 23). This, then, pointed to a variation in the relative strength of the interfering stream and the stream interfered with, and such variation must have occurred during the period that they were encountering one another, and were moving in the ordinary way of such streams, for it showed itself in a variation in the extent to which the streams from the negative terminal were diverted. We may hence conclude that the time requisite for the molecules to move the length of the tube was decidedly less than that occupied by the discharge, but was sufficiently comparable with it to allow the diminution of intensity of the streams from the side of the tube to make itself visible before the streams from the negative terminal experienced a similar diminution.

Thus far we have only been dealing with positive relief. But the phenomena of negative-special are equally important in the demonstration of the truth of this theory. In that case we know that the impulses that cause discharge arrive at the negative terminal of the tube and at the tinfoil synchronously, for the difference (if any) in the time required to pass from the machine along the wires to the two places is incomparably smaller than any of the quantities with which we have to deal.* And yet we

* This can be seen by the fact that no difference is produced by making the path to the tinfoil longer

obtain the most splendid virtual shadows. Now the discharge from beneath the tinfoil will certainly not last longer than that from the effective negative terminal of the tube, and yet we find that the negative discharge from beneath the tinfoil and its accompanying molecular streams are still in full vigour when the molecular streams from the negative terminal arrive at that part of the tube where they are situated, for they interfere with one another. We need scarcely repeat that there is every evidence that this interference continues for a substantial portion of the period during which these streams respectively last.

These experiments suffice to show that the duration of the negative discharge is not less than the time occupied by the passage of molecular streams along the tube, but is comparable with it. We shall now proceed to show that *the time occupied by the passage of either kind of electricity along the tube is incomparably shorter than that occupied by the emission of these molecular streams, or (which is the same thing) than that occupied by the negative discharge.*

The truth of this proposition, so far as the positive discharge is concerned, is shown by the following experiment:—If a piece of tinfoil be placed near the positive end of a tube through which a discharge with strong positive intermittence is passing, and be connected by a wire along the tube with a similar piece of tinfoil near the negative terminal, we shall obtain, as we have already said, positive relief-effects at the former piece of tinfoil and positive special at the latter. The relief-effects will not be increased by raising the latter piece of tinfoil from the tube and placing it as far from the tube as possible, keeping it at the same distance from the other piece of tinfoil. But a very remarkable exception presents itself. If the tube be of sufficient exhaust or the air-spark sufficiently long to cause the relieving system, formed by the tinfoil and the wire, to give rise to relief-phosphorescence when at a distance from the tube, it will be found that this disappears when the system is again lowered down so as to rest along the tube. The relief-effects, so far as luminosity is concerned, will be unchanged, or, if anything, increased. But the relief-phosphorescence which should have accompanied them will be found to have disappeared.

Extraordinary as this phenomenon appears at first sight, it will become perfectly intelligible if we accept the hypothesis that the electricity, on its discharge into the tube, spreads along it with a rapidity which enables it to pass to the other end in a very small fraction of the time required for a negative discharge, or its attendant system of molecular streams, to pass off from their source. The positive electricity, on bursting into the tube and arriving at the place where the first piece of tinfoil is situated, produces an impulsive electric tension upon it, which makes it summon from the more distant piece a supply of negative electricity. This summons passes along the wire in so short a time that, as we know, it arrives at the further piece of tinfoil before the discharge has arrived at the part of the tube in its immediate neighbour-

or shorter than that to the terminal. Moreover it has been demonstrated in the former part of this section.

hood. It accordingly causes the negative electricity that is desired to be supplied to the first piece of tinfoil, and thereby causes an impulsive electric tension in the second, which causes a positive discharge within the tube beneath it, showing itself in positive luminosity. The negative that rushes to the first piece of tinfoil forms the usual *blank space** beneath it, and would, in the ordinary state of things, cause a negative discharge within the tube. But before such discharge has got beyond the inchoate stage, the discharge in the tube has arrived at the place where the second piece of tinfoil is situated. Its arrival causes a sudden recall of the negative electricity that had left it previously, thus producing a sudden positive impulse at the first piece of tinfoil, and neutralising the tension there which would otherwise have caused the negative discharge and the accompanying molecular streams. Thus before the molecular streams can start they are revoked through the revocation of the negative discharge that would otherwise have caused them.

That the above must be the explanation of the phenomenon is rendered evident by the consideration that the same relieving system when in a position not more favourable to granting instantaneous relief to the first piece of tinfoil (*i.e.*, when extended at right angles to the tube) is found to cause relief-phosphorescence. The only difference between the two cases is that the relief granted in the one case is subsequently revoked, while in the other it is not. And as we find that such revocation is effective in stopping relief-phosphorescence, it is clear that the streams that were to produce it could not have left the side of the glass beneath the first piece of tinfoil before the revocation came. Hence the time occupied by the positive electricity in passing along the tube is of a higher order of smallness than the time occupied by the negative discharge in passing off from the inside of the tube beneath the tinfoil.

It may be suggested that the streams might have started, but have been stopped on their course by the subsequent action. No doubt, as we shall see, some small portion of them may have started, but certainly not all. For as the phenomena of virtual shadows and the bulging of the shadows of conductors within the tube shows, these molecules are capable of passing through the whole length or breadth of the tube, and probably could go much farther during the time occupied by their emission. Hence it is clear that they, or some portion of them, would have got across the tube, or at all events to a finite and considerable distance from the tinfoil, before the revocation came, if it did not arrive until the whole of the streams had started. Under such cir-

* We have greatly felt the need of a word to describe the dark area which is formed round every source of negative discharge. In a striated column it is called "CROOKES' space," the "negative dark space," or a "stria space," according as it is in the first, second, or a later segment (or physical unit) of the discharge, counting from the negative terminal. But in considering the structure of the discharge generally it is necessary to have a term which denotes it whatever be its position, and whether it occurs in a striated discharge or not. We have therefore adopted the term "blank space." It will be found to be as characteristic of a negative terminal as positive luminosity is of a positive terminal, however such terminal be formed.

cumstances it is clearly impossible that the revocation of the relief could be effective in stopping them, and preventing them from producing phosphorescence, seeing that they consist of material particles moving at high velocities; leaving out of consideration the fact that in all probability the largest portion of them would have actually arrived at the opposite side of the tube before the revocation arrived.

It must not be supposed that this theory requires that absolutely the whole of the negative discharge at the first piece of tinfoil (*i.e.*, the one that is nearest to the positive terminal) should be prevented by the revocation. This would not be in accordance with what the luminous phenomena would lead us to expect, and is not countenanced by the theory itself. There has been a discharge of positive electricity from beneath the further piece of tinfoil, and this has left a quantity of negative electricity free on the surface of the glass there. This in itself must render the impulse of the revocation less than that of the original demand. But the two are sufficiently nearly equal to prevent the negative discharge at the first piece of tinfoil having the violent character which would be necessary to produce relief-phosphorescence and to cause the negative discharge there to become a differential effect, and pass off in a gentle and continuous manner.

The existence of this revocation and its efficacy in preventing the emission of molecular streams in the ordinary way is further shown by there being no virtual shadow formed at the first piece of tinfoil. Now the absence of a virtual shadow is a very much sharper test of the absence of molecular streams than the non-appearance of relief-phosphorescence on the opposite side of the tube, for in the one case the molecular streams produce their effect close to the spot whence they proceed, while in the other case they have to force their way across the tube and must then possess sufficient velocity to produce phosphorescence. In order to show the effect of the revocation on the production of a virtual shadow, one end of a narrow slip of tinfoil a few inches in length was cemented to the side of the tube (through which was passing a discharge with positive intermittence) a little way from the positive terminal, and the slip was allowed to hang downward. A clear virtual shadow appeared, starting from the point where the tinfoil was in contact with the tube. The strip of tinfoil was then laid along the tube, stretching towards the positive terminal, and the virtual shadow disappeared. Now we know that, so far as the effects on luminosity at the cemented end were concerned, the two positions must have been practically equivalent. But in the one case there was a revocation which, though it did not come in time to stop the effects of the relieving system upon luminosity, was yet in time to prevent the emission of molecular streams so that no virtual shadow appeared. This experiment, though very interesting, is not so conclusive of the truth of our theory as is the one first given, for it might be objected that the virtual shadows could not be formed until the discharge in the tube had reached the negative terminal and obtained a response in the form of negative discharge, and a sufficient time had then elapsed to permit the molecular streams which accompanied that response to arrive at the first

piece of tinfoil. This would be necessarily later than the revocation which would take place when the discharge arrived at the further piece of tinfoil, *i.e.*, before it arrived at the negative terminal, and *à fortiori* before the molecular streams that issued from the negative terminal in response to it had returned to the first piece of tinfoil so as to suffer interference there.

We have used the above experiments for the purpose of supporting our proposition, both on account of their great interest and the remarkable way in which they illustrate the whole theory of the discharge, and also because of their analogy with an experiment by which we shall prove for the passage of the negative discharge along the tube the same proposition that we have proved above for the positive discharge. But so far as the positive is concerned, the phenomenon of virtual shadows suffices of itself to prove the proposition; for the response from the negative terminal occurs later than the relief-response at any point by a period equal to that taken by the positive discharge in passing to the negative end of the tube. The molecular streams that pass up the tube from the negative terminal find, on their arrival at the place where the relief is taking place, that the molecular streams there are still continuing, for they indicate this by being deflected by them, and thus forming the molecular shadow. Hence the time required by the discharge to pass to the negative end of the tube must have been less than the time during which these relief-molecular streams were being emitted. Simple as this proof is, however, it is inferior to the experimental proofs that we have previously given, inasmuch as it does not point so clearly to the great contrast in magnitude between the two small time-quantities under consideration as does the complete extinction of the relief-phosphorescence in the experiment first described.

So much then for the velocity of positive electricity along the tube. It remains to demonstrate a similar proposition as regards negative electricity. This was done in the following way:—A piece of tinfoil of some considerable size was laid on a high vacuum tube near to the negative end, the discharge being one of strong negative intermittence. A smaller piece of tinfoil was placed near the farther end, and they were connected by a wire as before. We know that this arrangement must give us negative special effects at the farther piece of tinfoil—*i.e.*, negative discharges there. These would, if strong enough, be accompanied under ordinary circumstances by molecular streams, which would cause phosphorescence on the opposite side of the tube. But no such phosphorescence appeared. In order to test whether this was due to the weakness of the negative impulses from the first piece of tinfoil, the second was placed upon a similar tube in an independent circuit (as in the standard-tube arrangement), the distance between the two pieces being retained unchanged. It gave most brilliant phosphorescence, showing that the negative impulses were abundantly strong enough to have caused phosphorescence had they not been prevented from doing so. And it is clear from our previous remarks that the cause which prevented their doing so was the arrival of the negative discharge in the tube at the farther piece of tinfoil before

the negative discharge there with its attendant molecular streams had had time to leave the surface of the glass.

The above conclusions are strongly fortified by an experimental result which at first sight appears to be inconsistent with them. We were engaged in producing the above-described effects in a tube of only moderate exhaust. They were very well manifested, showing that even in such a class of exhausts the durational character of a negative discharge which is violent enough to give phosphorescence is strongly marked. But to our great surprise we found that when we were working with a positive air-spark, the tinfoil nearest to the negative terminal of the tube gave a small but distinct patch of phosphorescence. Now the effects at that piece of tinfoil were positive-special effects—that is to say, there was first a positive discharge, and then, after the discharge within the tube had come up, the negative left behind by the former positive discharge passed off to meet it. This latter discharge, therefore, must have had all the essentials requisite to produce phosphorescence, and that which was produced was a true case of positive-special-phosphorescence. But we have already shown that the phosphorescence of the positive-special is very decidedly weaker than that of positive relief, for the latter can extinguish it and cause phosphorescence on the very spot of the surface of the tube at which the positive-special takes place and from which its molecular streams are pouring. How, then, is it that we can produce the weaker phosphorescence when we fail to produce the stronger? The reason is that, in the case we are now considering, the requisite time is allowed for its production. It is true that the negative discharge at the further tinfoil is far less than the relief-discharge which originally occurred at the nearer tinfoil, but there is no revocation in the case of the former. It is allowed to pour off during the whole of the time that the tube remains charged with positive electricity—that is to say, until the negative electricity pouring in at the negative terminal has reduced the tube to an electrically inert state, or approximately so. This is clearly a duration of the right type, since it suffices for the production of phosphorescence both in the case of relief and in the case of the discharge from the negative terminal. Thus we see that the presence of the requisite time-element will enable a negative discharge to produce phosphorescence in cases where its absence was a sufficient bar to the production of phosphorescence by a discharge of far greater initial violence.*

These experiments reveal to us the necessity of considering two other time-quantities which, but for the information derived from the consideration of the phenomena of

* We may add that all question as to the phosphorescence in this case being due to positive-special was set at rest by slightly raising from the tube the piece of tinfoil nearest the air-spark terminal. This caused the phosphorescence to fade, because when the tinfoil was in that position the positive-special effect was produced by a much feebler inductive action than when the tinfoil was on the tube. As the tinfoil was raised higher the phosphorescence continued to fade for a short time, and then grew bright again, the system having got so far from the tube that it had ceased to produce positive-special effects and had become a relieving system. The effects upon the luminosity bore witness to this series of changes taking place.

revocation, might very easily have been confounded with those of positive and negative discharge. These additional time-quantities are the periods required for the formation of the positive luminosity, and its correlative, the blank-space. These two cannot be considered separately, for they are really two parts of the same phenomenon. Where positive luminosity ends, there the blank space begins; not that there is necessarily a sharp division between them (though such is ordinarily the case), for they may appear to shade into one another, but that they represent the two states in which the effective* electrical field can exist during a discharge; and these two states are mutually exclusive, however sharply or gradually their boundaries may change. It is the time required for bringing the space within the tube into one or other of these states that we are now about to consider.

We are here approaching one of the most difficult problems connected with the electric discharge, and at the same time one that, above all others, is needful to be solved if we would get at the real secrets of its mechanism. In these two phenomena lies the most important portion of that electric dissymmetry from which we may chiefly hope to get light as to the nature of electricity. We do not feel that we are at present sufficiently advanced to treat these questions in so satisfactory a manner as we could wish, but we hope to be able to throw some light upon them even at the present stage.

Taking, first, the question of positive luminosity, it is clear that its formation along the tube cannot be more rapid than the velocity of the discharge itself—of the positive discharge itself. For although no doubt static induction outruns the discharge so much that, as we have said, we treat its advance as absolutely instantaneous, yet the distance by which it effectively precedes the front of the discharge does not increase during its progress. It is not by the rapidity with which static induction moves that the impulsive character of the action of the discharge both within the tube and upon the surrounding space is caused, but by the rapidity with which the free electricity, carrying with it its static induction, is brought locally into the neighbourhood. Thus an inferior limit of the time required for the advance of the positive luminosity is the time required for the advance of the positive discharge.

But this is only the less important part of the matter. The real question is whether this luminosity is produced immediately on the discharge, or whether it is separated from it by an interval of time comparable with the small quantities of which we have been speaking. As to this we think that we must come to the conclusion that it is cotemporaneous with the discharge that causes it. When we get positive-special effects in the way to which we have so frequently referred in the present section, we find that we can form the typical hollow cone and truncated positive column. Now it is clear that this hollow cone must mark a discharge that has really taken place, either

* We use the phrase *effective electrical field* to denote that part of the tube that is really affected by the discharge, and that undergoes the rapid alterations of electric state which accompany it. We shall presently see that it is quite possible for a portion of the tube to be outside this effective electrical field, and thus to participate in no way in the discharge.

wholly or to a considerable extent, before the main discharge has in its passage along the tube arrived at the tinfoil; as there would be otherwise on such arrival a revocation of the impulsive electrical action in the tinfoil, that is the cause of the very discharge that forms the hollow cone. But we are able to get quite clear and typical effects by this method, so that we are entitled to assume that the positive discharge has actually left the quasi-positive terminal beneath the tinfoil and proceeded along the tube with a velocity which we have already considered, forming the positive luminosity on its path.

The phenomenon which shows in the most striking way the rapidity with which positive electricity leaves its source is the production of positive-special in a high tension tube. Attention has already been called to the wide sweep which the thin positive column takes to avoid the spot at which the positive-special is being produced. Now this avoidance is due to the fact that there is negative electricity there which has been left behind by the positive discharge caused by the positive impulses within the tinfoil. Thus the side of the tube beneath the tinfoil behaves precisely as a negative quasi-terminal, showing that the positive must have wholly passed away from the place at the time the advancing positive discharge arrives at the tinfoil, and has left the corresponding negative electricity free to produce its full effect.

The evidence is strengthened if we consider the case of negative relief produced, by the use of a similar arrangement, at the piece of tinfoil that is nearer to the negative terminal. Here we are able to compare directly the two cases of the relieving tinfoil being on the tube, and at a distance from it. And we find that there is no difference in favour of the latter in the luminous phenomena of relief thereby produced. Hence we are fairly entitled to conclude that the positive discharge passes off in a time shorter than that required for positive electricity to advance along the tube, and that in so doing positive luminosity is formed by it as it goes.*

So far we have nothing that would lead us to draw a distinction in time between the emission of electricity and the formation of the luminous phenomena which accompanies that emission. But when we take the correlative phenomena for negative discharge, *i.e.*, the blank-space, a difficulty arises. It not only must, from its nature, be formed in the same time as the positive luminosity which it limits, but all the phenomena of revocation testify to the fact that it is so. But we have already shown that negative discharge does not take place with a rapidity at all comparable with that of positive discharge, so that we can no longer view the formation of the blank-space simply

* It is doubtless owing to this extreme rapidity of discharge and propagation that there is so much luminosity and so little heat in the vacuum discharge. It has been experimentally shown that the temperature of striæ is not greater than 100°. But all measurements of temperature are measurements of average effects, and such a result would be quite consistent with the hypothesis that they are in a state of intense heat for a very small fraction of the total period, and such a hypothesis would account for so large a proportion of the energy imparted to the gas passing off in the state of light inasmuch as the proportion of rays of high refrangibility to those of low refrangibility increases with the temperature.

as a phenomenon cotemporaneous with the negative discharge (for that is a durational phenomenon) in the same way that we have taken the formation of the positive luminosity to be cotemporaneous with the emission of the positive discharge. We are left then to choose between two hypotheses as to the formation of the blank-space; first, that this is formed at the very initiation of the negative discharge, and that although the discharge itself may be durational, it is initiated sufficiently to produce the blank-space in a time which is no longer than that required for the positive discharge; or, secondly, that it does not owe its existence to actual discharge at all but merely to what may be called a state of readiness for discharge—*i.e.*, the presence of a quantity of negative electricity ready and eager to be discharged, but by its nature compelled to pass off slowly.

These two hypotheses are not mutually exclusive, and may be, and probably are, both true. The best conception at which we have been able to arrive is that the blank-space represents a space which is sufficiently near to a source of negative discharge to prevent the presence in it of the peculiar action which causes positive luminosity. Whether or not this means that it is a space whose dimensions are such that it permits of some operation taking place for relieving electric tension in a gaseous medium—some transfusion, as it were, of the two opposing electricities that are gathered on its opposite sides—we are not yet prepared to assert. But its existence between consecutive striæ and round the negative terminal point to its representing *a space through which an operation equivalent in its results to the passage of electricity can take place without causing luminosity;* nay, we may say that they actually demonstrate that such is the case, for in these instances the blank space actually severs the luminosity into segments, the luminosities in which have no connexion one with another.

Leaving this question for future consideration, when our knowledge of the subject shall be more complete, the important conclusion to be drawn from the experiments last referred to is that there is no reason to regard the formation of this blank-space as a phenomenon connected with the emission of molecular streams. On the contrary, it is of a higher order of instantaneity, for while they can be affected by revocation, it cannot. They are, as we have seen, of a decidedly durational character, while it seems to belong at least to the order of instantaneity to which the passage of electricity along the tube belongs, inasmuch as it shapes and modifies the positive luminosity that attends that passage. However we form impulsive negative discharges of the proper type upon the side of a tube, the blank-space appears, showing that it was formed in time to control the positive luminosity of the discharge.

There is a well known phenomenon which will doubtless be thought by many to make strongly against these conclusions. It is that if the molecular streams be turned aside from passing down the tube, either by means of a magnet or by shifting the direction of the face of the negative terminal, the positive luminosity comes up into closer proximity with the negative terminal, as though the molecular streams had been

driving back the column of luminosity until they were diverted. But we think that this is only in appearance, and not in reality; and that no comparison can be made between the portions of the luminosity in the two cases, as the discharge has been affected by the alteration. Moreover, the same effect is produced by a magnet on discharges in low exhausts where there can be no question of molecular streams proceeding right down the tube. We think that the mistake arises from the effect upon the imagination produced by the view of the well-known negative dark space. This seems to separate the region of the negative terminal from that of the positive terminal, and anything that affects this dark space is thought to do so by directly affecting the action of the negative terminal. But in reality the region of the negative terminal is bounded by the negative glow which is the positive end of the physical unit of discharge, of which CROOKES' space is the blank-space and the negative terminal is the negative end. The negative dark space is only the blank-space of the second physical unit of discharge, and the apparent advance of the positive luminosity shows only that this second physical unit of discharge has been affected in some way by the altered circumstances of the discharge. And when we consider that its exceptionally long blank-space is due to its peculiarities of situation and the fact that its gaseous negative terminal (*i.e.*, the haze at the back of the negative glow) is so unlike the gaseous negative terminals of the other stria spaces, it does not appear strange that an alteration which produces a great modification in these matters should affect the length of this exceptionally extended blank-space.

We now come to the question of the time required for the emission of the positive discharge. This we have, in effect, dealt with already, when we considered the question of the dispatch of the positive discharge in positive special before the arrival at the tinfoil of the discharge along the tube. It must be of an order of magnitude not superior to that of the time required for the discharge to pass along the tube, and is probably of a lower order.

The time required for the passage of the electricity along the wire outside the tube is, as we have seen, so short that it cannot be detected by the aid of any of the other phenomena of the discharge. As it is a case of conduction, it is, of course, the same for both electricities.

The order of the small time-quantities of the discharge is therefore as follows; the groups being arranged in descending order of magnitude:—

A. The interval between two discharges.

B. The time occupied by the discharge of the negative electricity from its terminal.

The time occupied by molecular streams in leaving a negative terminal.

The time occupied by the particles composing molecular streams in passing along the tube.

C. The time occupied by positive electricity in passing along the tube.
   The time occupied by negative electricity in passing along the tube.

D. The time occupied by positive discharge.
   The time required for the formation of positive luminosity at the seat of positive discharge.
   The time required for the formation of the blank space at the seat of negative discharge.

E. The time occupied by either electricity in passing along a wire of the length of the tube.

The period occupied by the whole discharge must be of the order B, since it includes a complete negative discharge. The evidence which shows that the time-quantities in D are greater than the time-quantity in E is much weaker than in any of the other cases, but this defect is not of great importance, as there would be little information to be derived from a comparison of them on account of their difference in nature.

XXVII.—*General conclusions as to the electric discharge.*

II. *In vacuum discharges the durational character of the negative as compared with the positive discharge increases with the degree of exhaustion and becomes very marked in extremely high exhausts.*

We have seen in the previous section that the negative discharge occupies a longer time than the positive in leaving a terminal, whether that terminal be one that is formed on the glass by relief or special action or be an effective terminal of the tube. In the case of the positive discharge the time occupied is less than that required for electricity to pass along the tube; while in the case of negative electricity it is longer than the time required by the comparatively slow-going molecules to do so, and is so much longer than the time required by the electricity to pass along the tube that a revocation caused by the arrival of the electricity at a spot near the other end of the tube is to all appearances in time to stop the negative discharge and its accompanying molecular streams before they have fairly commenced. In the present section we propose to show that this durational character of the negative discharge, as contrasted with the positive discharge, increases with the degree of exhaustion of the tube, and becomes very marked in high exhausts.*

It may fairly be remarked that the experiments upon which the proof of this comparatively *durational* character of the negative discharge was based were made in

* This character of the negative discharge was already noticed by Messrs. De La Rue and Müller, see their paper, Phil. Trans., Part I., Vol. 169, p. 90, and also p. 118, where some very interesting details are given.

tubes of a sufficiently high exhaust to give virtual shadows,* so that it can hardly be said to have been proved to exist in tubes of ordinary exhaust. This cannot be denied, and we are not prepared with any equally rigid proof that it does so exist in tubes of low exhaust, although there are many circumstances which point to such being the case. In the first place, we have the fact that all the phenomena obtained with positive intermittence are sharper in character than the corresponding ones that are obtained with negative intermittence, pointing towards a less instantaneous action in the case of the negative discharge. This is a very well marked phenomenon, which will be found to have been noticed by us in our previous paper, and is manifested by tubes of every class of exhaust. Another circumstance pointing in the same direction is a peculiarity that has long been observed in the negative discharge, viz.: its preference for a surface as compared with a point of discharge. This, and its analogue, the possibility of bifurcating the negative current, seem to point to the discharge at a negative terminal being a continuous process, which is facilitated by its having a large number of places from which it can go on at the same time.†

These general considerations are but poor substitutes for the definite experimental proof which we were able to give for the case of tubes of fairly high exhaustion, and it is hoped that they may be supplemented at some future day. But one reason for this is that the contrast between the character of the two discharges is not a strongly marked phenomenon in low exhausts, though we have no doubt that it exists in some degree. The change that we shall in this section prove to take place as we pass from tubes of fairly high exhaust to tubes of extremely high exhaust takes place also as we pass from tubes of low exhaust to tubes of fairly high exhaust. In the former, the two kinds of discharge are not very markedly different in their duration character. As the exhaust increases, the positive discharge becomes more nearly instantaneous, and the negative discharge becomes more durational till we come to the class of exhaust treated of in that portion of the previous section which deals with that question. We shall now proceed to examine the case of tubes of very high exhaust, and consequently very great resistance, and show that the contrast there becomes very much intensified.

The first class of experiments showing that such is the case consist of observations with the standard-tube. A ring of tinfoil was placed round a tube of very high exhaust with a negative air-spark, and a wire was taken from it to a ring of tinfoil round a tube of moderate exhaust carrying an independent continuous current. Negative effects were of course produced. A supplemental ring was placed round the latter tube, and when touched it was expected to give (in accordance with the

* It must not be thought from this that these tubes were all tubes which gave phosphorescence in any marked degree with the continuous current. It was necessary to use an air-spark in most cases if it was desired to produce phosphorescent effects throughout the tube. But they were tubes in which the blank-space was of considerable breadth, and ought fairly to be considered tubes of high exhaust.

† See De La Rue and Müller, Phil. Trans., Part I., Vol. 171, p. 108, and Plate 10, figs. 27 and 28.

usual rule) negative relief-effects. These, however, were not very clearly manifested, but the remarkable peculiarity was observed that when the wire from earth to the latter tinfoil was not allowed actually to touch it, but was held at short sparking distance from it, these negative relief-effects were very marked and clear. It is difficult to interpret this in any way other than by supposing that the negative discharge in the large tube went on accumulating for a certain time instead of instantaneously flashing into its full intensity as in the case of the positive discharge. This rapidly rising negative charge in the tube of high exhaust would of course drive off negative into the other tube in the ordinary way, but this (if the above law be true) would be done with less sharpness than in an ordinary negative discharge in such a tube. We should thus have a discharge in the latter tube which, though intermittent in its nature, yet partook of a continuous character, and the relief-effects would therefore be poorly defined. If, however, the relief were only permitted to be given at the exact moment when the accumulation of negative reached its height, and was then given impulsively (as would be done by allowing positive electricity to spark into the tinfoil from the earth wire), the relief would be given with the requisite sharpness, and would produce definitely marked relief-effects.

It will be seen that this explanation requires that the negative should accumulate in the original tube. In other words, it shows that it is a difficult (but not necessarily a slow) process for negative to get out of the tube as well as a matter of time to get in. This will no doubt be found to be the case, and when the law is perfectly formulated it will probably include some such statement. The present form is taken only as an approximation.

The second set of observations bearing upon this matter is intimately connected with the difference of sharpness of effect which we have already noticed as existing between positive and negative intermittence, even in low tension tubes. In high tension tubes this difference becomes in some respects immensely exaggerated. If the intermittence be positive and the negative terminal be put to earth and the hand placed on the tube, strong shocks will be felt. If, however, the intermittence be negative and the positive terminal be put to earth the shocks are so much more feeble as hardly to be sensible. And the same difference may be made evident in another way. If an earth wire be held near to a piece of tinfoil placed upon a high vacuum tube it will be found that the sparks will stream between them when the intermittence is positive even though they are a considerable distance apart. But if the intermittence be negative it is difficult to get any but the very shortest sparks to pass between the tinfoil and the wire. And if contact be made, the contrast (if the air-spark be considerable) is equally striking. While in the case of the negative intermission no special phenomena are observed, in the case of the positive intermission the electricity streams from all sides of the tinfoil and evidences the most violent alternations of tension. It is difficult to see any explanation of these peculiarities other than that the impulsive change of tension is very much greater

in the positive than in the negative intermittence—for we know that this, like all the other phenomena of intermittence, depends on sudden changes of tension and not on the absolute tension at any moment. And if the impulsive change of tension is less violent with the negative than with the positive intermittence, it would seem to be a necessary conclusion that it is due to the more durational character of the former discharge.

There is a cognate class of experiments which, when taken in connexion with those we have just described, adds very great force to the conclusions we propose to draw from them. If we try the special effects we shall find the characteristics reversed. While the positive intermittence which gave such very violent relief-action only gives very slight action at the tinfoil, the negative intermittence causes the most violent disturbances there. Just as previously in the case of the positive-relief, so now with the negative special; the sparks stream between the wire and the tinfoil if they are made to approach one another, while if contact is made the electricity streams from all the edges of the tinfoil in the most violent manner. This must, we think, be because the negative pulses that arrive at the terminal can only get relief there slowly by pouring into the tube. They therefore press with all their force at the tinfoil seeking and finding like relief, through the inductive discharges to which they give rise. And it is a considerable time before the rise of tension in the tube, rapid though it be, is capable of balancing this pressure of the negative from without in its endeavours to get relief either by entering the tube or otherwise.*

We shall not dwell further on these experiments. It is difficult, we think, to account for the phenomena they present in any other way than by accepting the truth of the theory that is given at the head of this section, viz.: that the contrast between the two types of discharge, as far as the time required for their emission is concerned, becomes greater as the degree of exhaustion increases.

XXVIII. *General conclusions as to the electric discharge.*
III. *On the positive column.*

We purpose in the present section calling attention to some experiments which throw light upon the function of the positive column, and give us a clue to the action which is going on where it appears, and of which it is doubtless the result. A portion of the experimental evidence relating to this has been already given in Section XXIII., and has to some extent been interpreted there. We shall now give further evidence in favour of the views there advanced.

* Lest it should be thought that these phenomena were due simply to the high resistance of the tube, we took a tube whose vacuum was so low (we estimated it at about 2 inches of mercury) that the resistance was even greater than in the high tension tubes. No perceptible difference was found between the sensations caused by discharges of positive and negative intermittence, nor in the sparking distance between the earth wire and the tinfoil.

The fundamental experiment in this matter was made by us in our experiments with the standard-tube. We had found in working with positive intermittence that there is but little difference between the variations of electric action undergone by a piece of tinfoil upon the tube, an intermediate terminal projecting within the tube, and the non-air-spark terminal, so that it was comparatively a matter of indifference to which the tinfoil on the standard-tube was joined. It then occurred to us that we would try the case of a unipolar discharge. Accordingly a tube was taken in which there was an intermediate terminal situated very near to one of the ends of the tube. The other terminals were in the usual position at the two ends of the tube. The two terminals that were very near together were joined to the terminals of the machine, so that the current only traversed the very short portion of the tube lying between them. A positive air-spark was now introduced into the current. As has been described in the previous paper, this caused a pointed tongue of positive luminosity gradually to advance into the unoccupied portion of the tube, forming a positive unipolar discharge. The terminal at the other end was then joined to the tinfoil on the standard-tube. No effect was, however, produced until the air-spark was lengthened (and with it the unipolar column) so much that the unipolar luminosity came up to that terminal and formed the usual blank-space around it. Instantly the most perfect positive effects appeared in the standard-tube. The air-spark was then decreased so as to make the unipolar luminosity retreat from the terminal, and at once the effect ceased. This was repeated several times, and the result was always the same, showing that so local is the effect of the free electricity discharged into the tube that it is only across a true negative blank-space that it exercises upon another terminal any decided influence of the type required to produce interference. The experiment was repeated, a ring of tinfoil round the tube being substituted for the distant terminal. The result was the same. So soon as the tongue of luminosity got within a like distance of this ring so that the blank space was formed either completely or partially the effect appeared in the standard tube, but not otherwise.

We need not point out how strongly this supports the theory suggested in the previous section as to the signification of the blank-space. We wish rather to point out its bearing upon the question of the significance of the positive column. The luminosity in the unipolar discharge is in reality a positive column which doubles back upon itself when it does not find any negative terminal to which it can discharge. It thus may be taken for our present purposes as a representative of an ordinary positive column, and it becomes strictly a positive column when it reaches the further terminal and forms a blank space round it. Regarding it as such, we see that it is unable to produce interference in the standard-tube through the medium of the further terminal, unless it extends to within the breadth of the ordinary blank-space from that terminal. The effect produced upon the standard-tube could only be caused by the presence of free electricity in the immediate neighbourhood of the terminal. This seems to show that the luminosity marks much more clearly the local situation of

the free positive electricity of the discharge and the locus of demand for negative electricity than we could have predicated independently of this experimental evidence. It will be noticed that this experiment showed that the positive luminosity, plus an enveloping shell of the breadth of a blank-space, was the limit of the effective electric field. The rest of the tube was not used in the discharge.

A tube was taken of tolerably good exhaust enclosing a spiral of wire. There was a positive column in the tube sufficiently nearly filling up the whole section of the tube to make it pass through the coils of the spiral, so that the spiral was, as it were, bathed in positive luminosity. So long as the tube was not touched, the wire did not act in any way upon the positive luminosity, but if the finger were brought in contact with the side of the tube at a point where it was touched by the spiral, a blank-space appeared surrounding the wire of the spiral whenever it had been touched by the positive luminosity. This shows that throughout the luminosity there was a demand for negative electricity, which caused the formation of a blank-space around the wire and negative discharge therefrom so soon as the relief afforded to the wire from the finger permitted it to part with its negative electricity.

The drawback to this experiment was that it was not possible to test whether there was negative discharge equally from all parts of the wire, or whether it was only from those parts which were in the midst of the positive luminosity. The only test of the existence of negative discharge in such a tube being the appearance of a blank-space, it is obvious that no conclusive evidence could be obtained of its existence where there was no luminosity out of which the blank-space could be cut and by which it could be bounded. But this want is supplied to some degree when we come to tubes of higher vacua, for the negative discharge is there accompanied by molecular streams; and by projecting relief-phosphorescence upon the conductor within the cylinder, and observing the bulging out of the shadow, we can detect negative discharge even when no luminosity is visible. These experiments have been referred to in Section XXIII., and they show that the thin positive column in high vacua indicates the local presence of a very intense demand for negative electricity greater than that which is experienced generally throughout the tube, since the result of the positive column passing across the end of a wire within the tube is to cause a very considerable bulging out of the shadow even when no relief is given to the wire from without the tube, and therefore when the wire is subjected throughout its whole length to the ordinary electric tension produced by the presence of the discharge in the tube.*

* Since the above was written these conclusions have received striking confirmation from the following experiment. A long tube of high exhaust was taken in which there was a loose wire of considerable length, and straight in its general direction, but bent in one or two places into sharp angles. With a positive air-spark the tube gave the long, thin, pencil-like column of positive luminosity of which we have spoken. When this did not come in contact with the wire no special appearances presented themselves, but when by the approach of the hand or otherwise it was made to move up to and come in contact with one of the angles in the wire, which rested on the inner surface of the tube, a bright patch of phosphorescence of an oval contour appeared, commencing at the angle and extending from it, evidencing incon-

Another proof that the positive column marks an intense local demand for negative electricity is the attendant phosphorescence. We have already remarked upon this in Section XXIII. It assimilates the positive column to a line of centres of positive discharge.

This seems to throw much light upon the meaning of the blank-space surrounding centres of negative discharge. It marks the area through which the centre of negative discharge is capable of exerting such an influence* as to prevent that intensity of demand for negative electricity arising or continuing, which is the condition of the existence of positive luminosity in that particular gaseous medium. We are not as yet able to define what is the nature of this influence, or how it is exercised, but there seems to us to be clear evidence of its existence. In this way we can understand the existence of the blank spaces between striæ. They show the space which is, as it were, protected from intense need of negative electricity by the influence of the gaseous negative terminal composed of the hollow hazy surface of the next stria.†

We have said that the positive column is like a line of centres of positive discharge, and it is attended by phosphorescence in high vacua, just as such a line of centres would be, supposing them to lie close along the surface of the tube. But the positive luminosity can be attracted to the side of the tube by a wire connected with the positive terminal and passing along near the surface of the tube—the air-spark being, of course, in the positive. The true significance of this phenomenon is worthy of remark now that we know the exact meaning of the thin pencil-like column of positive luminosity. The course of reasoning given in Section XXIII. has shown that there is a continual discharge of negative electricity directed towards this positive luminosity from the surrounding portion of the tube, and that it is thus the locus of the centres of excitation throughout the tube. The presence of the wire parallel to and near the tube has, we see, the property of fixing this locus of centres of excitation on the side of the tube nearest to itself. Now we know that all that it can do is to excite by induction

testably the presence of a negative discharge from the angle of the wire. In other words, the presence of the positive luminosity at the angle had caused there so strong a local demand for negative electricity that it caused a discharge to take place from the angle; the remainder of the wire (although subjected to the general demand for negative electricity) that must exist throughout the tube only serving as a reservoir from which the discharge was drawn. [July, 1880.]

* It may be objected that inasmuch as the breadth of the blank-space ordinarily increases with the degree of exhaust, this hypothesis would make the extent of the negative influence greater in tubes of high exhaust than in other tubes: a result which seems startling in view of the enormously greater difficulty that electricity finds in passing through the former. But it must be remembered that it is much more difficult to produce luminosity in such tubes, and it may well be that this more than counteracts the other effect, and enables the influence of the negative terminal in preventing the formation of luminosity to extend through a greater range than in low vacua where a much less intense need suffices to cause it.

† It will be understood that this is not intended to be a further explanation of the genesis or structure of striæ and the blank spaces between them. It leaves the matter where it was left by our previous paper. It is only intended as an illustration of the application of the present theory of the blank-space to a particular and well-known case of its occurrence.

positive discharges from that part of the tube inwards. Not only can it do this, but it certainly does so, for the effect of the impulsive variations of potential in the wire must of necessity produce such discharges. Hence we see that the effect of producing a locus of such discharges in the tube is to cause the original locus to shift till it coincides with the new locus. Thus the *creation of the new locus has made the first unnecessary*, or, in other words, the discharge does not require any special series of positive centres of discharge along its course, but it is content with one series; but this it must have.

Now if we compare this with the phenomenon of positive-special-effect when a ring of tinfoil is used, we shall gain a valuable insight into the mode of propagation of the discharge. In that case we know that the inductive discharge at the tinfoil takes the place of the original discharge, and the latter is satisfied by the negative that is left behind by the former. If we suppose this effect to be more imperfectly produced and spread along a line, passing along the tube longitudinally, instead of surrounding it, we shall get an idea of how the discharge is effected when we use the special-effect in the way described above. And the knowledge that we now have of the identity of the phenomenon presented by this special effect, and the ordinary discharge (save so far as regards the side of the tube which the luminous column prefers) in high vacuum tubes, seems to point to something like the above mode of propagation of the discharge in all cases.

XXIX.—*General conclusions as to the electric discharge.*
IV. *Molecular streams.*

There are a few questions relating to molecular streams which we are in a better position now to consider than we were before the phenomena of phosphorescence in the sensitive current had been examined by us. These we shall shortly indicate.

In Section XVII. we stated that it was our belief that there was no essential difference between the molecular streams of particles of gas which go to produce phosphorescence and the phenomenon of the driving off from the negative terminal of small loose particles of conducting matter, which also occurs in rarefied media. In order to settle this point, we took a tube of fairly good exhaust containing a little of a mixture of sand and lamp-black, the sand being put there to assist in removing the lamp-black from the sides of the tube in case it should adhere thereto. The tube was placed vertically with the negative end downwards, and a current from the large 12-plate HOLTZ machine was passed through it. In a few seconds the sides of the tube were covered with a coating of lamp-black for about two-thirds of its length. The experiment was then varied by the introduction of an air-spark into the circuit. Whether this was placed in the positive or negative portion of the circuit, the effect was the same, the lamp-black was driven with such violence against the sides of the tube that it became caked in some places, so that it was a troublesome matter to get it off.

Seeing, then, that these particles of lamp-black behaved in the same way as particles

of gas would have done (save that they remained fixed to the glass instead of bounding off and causing phosphorescence), we determined to see whether they could be made to show the sensitive effects which we had observed in phosphorescence. Accordingly, a narrow slip of tinfoil was placed round the tube near its middle point, and connected to earth, and a discharge with a sharp positive intermittence was sent through the tube for a very short time. On examining the tube it was found that there was the usual thick coating of lamp-black over the sides of the tube, except where the tinfoil had been. This remained bare of lamp-black. Thus the relief-molecular streams from under the tinfoil had swept away the particles of lamp-black that would otherwise have lodged on the sides of the tube under the tinfoil.*

We next joined the tinfoil metallically to the positive terminal so as to produce positive special. The result was that the deposit was thicker beneath the tinfoil than elsewhere, corresponding to the appearance of phosphorescence on the tinfoil in a like case. Negative relief gave a similar effect, while negative special kept the surface of the tube beneath the tinfoil clear of lamp-black. Thus, in all four cases the streams of lamp-black behaved in all respects as molecular streams would have done.†

Seeing, then, that any small light particles of a conducting substance are capable, under the action of negative discharge, of forming streams similar to the molecular streams that produce phosphorescence, and governed by like laws, the question naturally suggests itself whether these molecular streams have any necessary electrical function to perform in the discharge, or whether the particles of the gas are not driven off, like the particles of lamp-black, just because they happen to be there and are capable of being so driven off. This is a most important and difficult question to decide. At present we have come to no definite conclusion upon it, but we cannot say that we are aware of anything that conclusively shows that they have any definite electrical function to perform in the discharge, while on the other hand there are many things that point to an opposite conclusion.

In the first place, neither the telephone nor the standard-tube recognises their existence. A piece of tinfoil will cause no louder sound in a telephone nor produce any greater effect on a standard-tube with which it is connected because it is on a spot where there is brilliant phosphorescence. Nor do they take notice of its total or partial absence. It makes no difference whether or not the tinfoil is situated in an absolute or a virtual shadow, and therefore protected from the impact of these streams.

* In order to ascertain that it was an electrical effect and not merely the effect of something being placed round the tube which might have the effect of deadening vibration, or producing some other mechanical effect, the experiment was tried with a ring of tinfoil unconnected with any relieving system, and also with an india-rubber band. In neither case was the coating of lamp-black affected.

† It is necessary that the discharge should be brief in duration, for otherwise the whole surface has a tendency to get coated. It must be remembered that the lamp-black accumulates on the surface, so that it is only by so doing that the conditions resemble those under which phosphorescence is produced.

In the next place, it seems abundantly probable that they are quite left behind by the negative discharge when it is in the tube. The time they take to pass along the tube is of the same order of smallness as the time that is occupied by the emission negative discharge that produces them. But the time that negative electricity takes to pass along the tube is, as we have seen, of a higher order of smallness than this, for when the intermittence is negative the discharge can pass along the tube in time to stop the production of phosphorescence at a piece of tinfoil near the positive terminal, which is connected with a piece near the negative terminal. Thus it appears that it out-runs these molecular streams, and cannot depend on them for its propagation.

Further, we now know that a discharge may be effected by positive passing through the tube and deriving its satisfaction by a response from the negative terminal. Or it may be effected by the negative passing throughout the tube and meeting with a response at the positive terminal. Now, on the supposition that these molecular streams are the carriers of the discharge, or that they have any special function to perform in its propagation, it is very difficult to understand the first of these modes of effecting the discharge. Moreover, it is admitted that there is not the slightest necessity that any of these molecular streams should strike or even pass near the positive terminal, so that the latter of the two modes of effecting the discharge seems equally incomprehensible on the above theory.

The most attractive hypothesis relating to their functions is that they officiate at the birth of the discharge and enable it to get into the gaseous medium, where it has modes of propagation which are independent of these molecular streams.

There is much to recommend this theory which views molecular streams as a necessary attendant phenomenon of negative discharge, but having no share in its propagation. We are not in a position to pronounce upon this hypothesis. The fact that resistance rises higher with increased degree of exhaust, after a certain point is passed, seems to favour it, as this law would then admit of the simple explanation that the resistance was greater because of the lack of carriers to carry the electricity into the gaseous medium. But this increase of resistance may come from other causes, and this single consideration does not seem to be sufficient ground for assigning to the molecular streams such a special function in the absence of other evidence that they possess it.

On the whole, then, we are inclined to doubt whether molecular streams have any necessary function in the discharge. This does not, of course, imply that the molecules that compose them are not charged. On the contrary, it seems very probable that they are, as it would otherwise be difficult to account for their being shot off at so great a velocity or for their obeying a magnet. But the fact that in this way some small portion of the negative discharge is convectively carried along the tube would no more entitle them to be looked upon as having a function to perform in the discharge than it would entitle the particles of lamp-black to be looked upon in that light.

PLATES 25–29.

Index to the references to the figures in the text.

XVII. The Bakerian Lecture.—*On the Photographic Method of Mapping the least Refrangible End of the Solar Spectrum.*

By Captain W. de W. Abney, *R.E., F.R.S.*

Received December 17, 1879,—Read January 8, 1880.

[Plates 30–32.]

The research which I commenced some five years ago on a method of photography by which the least refrangible end of the solar spectrum could be mapped has reached such a stage that it seems desirable that I should put on record the results I have obtained, and with it to present a map of the solar spectrum between wave lengths 7600 and 10,750 based upon measurements from a series of photographs which appear to be satisfactory for the purpose.

Action of dyes on silver bromide.

In December, 1873,* Dr. H. C. Vogel, of Berlin, announced that by means of dyed collodion films which contained silver bromide he had been enabled to photograph with the yellow and green rays of the solar spectrum, which had hitherto been supposed to be possessed of but little chemical effect. About the same time I had set myself the task of mapping the ultra-red region of the spectrum, and I was naturally led to examine the method advocated by Dr. H. C. Vogel.

If a spectrum be thrown on an ordinarily prepared photographic plate containing only silver bromide, it is well known that the length of the spectrum impressed varies considerably from that obtained by a plate containing silver iodide or silver bromo-iodide. The commencement of the photographic action in the first case is somewhere near the line B or slightly below, and in the last two near E in the green; in all the three the action extends to the limit of the solar spectrum in the ultra-violet. The relative chemical effects produced by the different rays show themselves by a varying thickness, or what is usually called density, of the metallic silver reduced or precipitated by the action of developing solutions. For the above-named silver compounds the maximum effect is somewhere about the line G; and if we represent the density of the metallic silver at any point by ordinates, it will be found that the area of the curve

* Photographic News, Dec. 12, 1874.

formed by joining the ordinates fairly represents the relative sensitiveness of the compound to the action of white light.

With silver bromide the curve falls gradually towards B, and similarly towards the ultra-violet. With silver iodide and bromo-iodide the curve falls suddenly near G, and the chemical effect from the bottom of this descent to E is but very feeble; whilst for the more refrangible end above G it follows pretty nearly the curve of the bromide.

When collodion films containing silver bromide are dyed, by flowing over them an alcoholic or aqueous solution of certain dyes, and exposed to the spectrum, the resulting curves are modified in a marked manner. They almost exactly correspond to curves compounded of the absorption curve of the dye used and of the absorption curve of the unreduced silver bromide. For example, if we examine a simple bromide film with the spectroscope, it will be seen that an absorption takes place along the whole visible spectrum corresponding with the density curve of this compound. An examination of the spectrum of eosine, supposing we are going to employ that dye to stain the film, gives an absorption band in the green, together with a less marked region of absorption in the blue and violet. If these two absorption curves be combined, we shall find that in the density curve obtained by exposing a stained film to the action of the spectrum we get a reproduction of this compound curve. VOGEL explained this remarkable action by a theory which seemed to be contrary to our present idea of molecular physics; and in my experiments conducted in this direction I had the good fortune to arrive at a more acceptable solution, involving no new laws of the action of radiation on matter. Some of the dyes employed (and most, if not all, of these belonged to the aniline series) could form compounds with silver, and when brought in contact with silver nitrate the action of light on the compounds was perfectly intelligible. VOGEL, however, pointed out that if a bromide film were perfectly freed from all silver nitrate by immersion in potassium bromide, then washed, next treated with tannin, and dyed, a similar density curve was obtained. I need scarcely recount the numerous experiments which I undertook; one of the final ones will be sufficient to show how a reasonable explanation could be offered.

A few granules of the dye were taken and dissolved in normal collodion, a glass plate was coated in the ordinary manner, and the film dried and exposed to the spectrum. It was found that in those regions where the spectrum was absorbed a bleaching of the dyed film was evident. Thus, with eosine a bleaching took place in the green rays corresponding with its absorption band, and to a lesser degree in the yellow and blue rays. If after such an exposure the dyed collodion were coated in the dark room with collodion containing silver bromide in suspension, and an alkaline developing solution applied, it was found that the silver bromide was reduced to metallic silver on those parts of the plate which had been bleached by the action of the spectrum, and the density curve followed the curve of absorption of the dye.

Photographers have been long aware of the fact that some sorts of organic matter on a glass plate will cause the reduction of silver on those portions of the plate on

which it exists. The bleached dyes partake of the nature of this organic matter, and we are forced to conclude that a state of oxidation favours this disposition. From exposure of the dyes in different media to the action of the spectrum it seems that the bleaching is due to a state of oxidation of the dye. If a film containing silver bromide, and dyed, were exposed in hydrogen to the action of the spectrum, the density curve was that due alone to the action of silver bromide, and the dye did not affect it, excepting so far as it acted as a screen to prevent the full intensity of the light falling on the bromide.

The dyes which are most active are those which are known as yielding fugitive colours; a permanent colour produces no effect beyond the above screening the light from the silver bromide. It should be remarked that when a film containing finely-suspended matter is dyed the change effected on the dye is much more rapid than when a continuous film is acted upon, for the dye surrounds these very small particles, and thus a large surface of it is exposed to the air, and renders oxidation comparatively easy. As a result of these experiments I can confidently state that in no case did the addition of the dye cause any chemical effect to be produced by the rays below A of the solar spectrum, nor has VOGEL claimed that they do. I am aware that in the Proceedings of the Royal Society* Major WATERHOUSE has stated that by staining the film with turmeric he obtained evidence of the existence of lines in the solar spectrum a little less refrangible than A, but it must be observed that the lines so shown are, except in one instance, "reversed." That is, absorption lines appear as opaque on the transparent body of the spectrum instead as of a normal character, viz.: transparent on an opaque body. This reversal is a matter to which I have referred in the Proceedings of the Royal Society,† and is dependent on a different action entirely to that which I am now considering.

Preparation of a film sensitive to the infra-red region.

My earliest endeavours were directed to extending this action of organic matter, so that sensitiveness of the compound might be obtained in the ultra-red regions. By weighting the molecules of the silver bromide with gum resins, the spectrum was impressed considerably below A and the absorption lines were unreversed. Measures of the heating effect of the solar spectrum on lampblack, as shown by the thermo-pile and Sir J. HERSCHEL's well-known researches, showed that the lower limit of the prismatic spectrum had not been reached; it therefore seemed advisable to search in a different direction for a more sensitive compound. The salts of silver still seemed the most feasible to work with, and more especially the bromide, and efforts were made to obtain this compound in a different molecular condition to that generally found. I need not detail the different methods of preparation of this compound in collodion that were carried out. In some cases I obtained it in a state which when viewed in a

* Vol. xxiv., p. 186.

† Vol. xxvii., pp. 291 and 451.

film by transmitted light appeared of a sky-blue colour, inclining to a green tint, visibly absorbing the red. In this condition it was sensitive to the whole spectrum, visible and invisible. There was much uncertainty attaching to the preparation; about one batch of the salt suspended in collodion out of five or six fulfilling the requisite conditions. I am now, however, in a position to prepare it without any risk of failure, my experiments of the last nine months having showed the conditions absolutely necessary for success. The following is the mode of preparation. A normal collodion is first made according to the formula below:—

Pyroxyline (any ordinary kind)	16 grains
Ether (·725 Sp.)	4 oz.
Alcohol (·820)	2 oz.

This is mixed some days before it is required for use, and any undissolved particles are allowed to settle, and the top portion is decanted off. 320 grains of pure zinc bromide are dissolved in $\frac{1}{2}$ oz. to 1 oz. of alcohol (·820) together with 1 drachm of nitric acid. This is added to 3 ozs. of the above normal collodion, which is subsequently filtered. 500 grains of silver nitrate are next dissolved in the smallest quantity of hot distilled water, and 1 oz. of boiling alcohol ·820 added. This solution is gradually poured into the bromized collodion, stirring briskly whilst the addition is being made. Silver bromide is now suspended in a fine state of division in the collodion, and if a drop of the fluid be examined by transmitted light it will be found to be of an orange colour.

Besides the suspended silver bromide, the collodion contains zinc nitrate, a little silver nitrate, and nitric acid, and these have to be eliminated. The collodion emulsion is turned out into a glass flask, and the solvents carefully distilled over with the aid of a water bath, stopping the operation when the whole solids deposit at the bottom of the flask. Any liquid remaining is carefully drained off, and the flask filled with distilled water. After remaining a quarter-of-an-hour the contents of the flask are poured into a well-washed linen bag, and the solids squeezed as dry as possible. The bag with the solids is again immersed in water, all lumps being crushed previously, and after half-an-hour the squeezing is repeated. This operation is continued till the wash water contains no trace of acid when tested by litmus paper. The squeezed solids are then immersed in alcohol ·820 for half-an-hour to eliminate almost every trace of water, when after wringing out as much of the alcohol as possible the contents of the bag are transferred to a bottle, and 2 ozs. of ether (·720) and 2 ozs. of alcohol (·805) are added. This dissolves the pyroxyline and leaves an emulsion of silver bromide, which when viewed in a film is essentially blue by transmitted light.

All these operations must be conducted in very weak red light—such a light, for instance, as is thrown by a candle shaded by ruby glass,* at a distance of 20 feet. It

* If a green light of the refrangibility of about half way between E and D could be obtained it would be better than the faint red light transmitted by ruby glass, since the bromide is less sensitive to it than to the latter.

is most important that the final washing should be conducted almost in darkness. It is also essential to eliminate all traces of nitric acid, as it retards the action of light on the bromide, and may destroy it if present in any appreciable quantities. To prepare the plate with this silver bromide emulsion all that is necessary is to pour it over a clean glass plate, as in ordinary photographic processes, and to allow it to dry in a dark cupboard.*

For development after exposure I recommend what is known as the ferrous oxalate developer. This is prepared by dissolving ferrous oxalate in a saturated solution of neutral potassium oxalate, adding the iron salt till no more is taken up. To make up the developing solution, equal parts of this solution of ferrous oxalate and of a solution of potassium bromide, 20 grains to the ounce, are employed. This mixture is placed in a clean developing glass just before development is to take place. The film is first softened by flowing over it a mixture of equal parts of alcohol and water, and is then well washed. The developer is now poured over the plate, taking care to keep the fingers from touching any part of the film. The image will appear gradually, and should have fair density when all action is exhausted.

The intensity can be materially increased by using the ordinary intensifying solutions of pyrogallic acid, citric acid, and silver nitrate. The unreduced silver bromide is removed by a saturated solution of sodium thiosulphite in water, from all traces of which the film should be thoroughly washed before being allowed to dry.†

The operation of development should take place in a very subdued red light, that recommended for the preparation of the emulsion being the safest.‡ It is, however, somewhat remarkable that when the developing action has once been set up a greater quantity of light may be admitted to fall on the plate than before the action commences. The bromide of potassium probably prevents any further action by the light, which may account for it. It should be noted that the image may be developed by the ordinary alkaline method, though not so satisfactorily, a slight veil being usually apparent.

I may here state that by diminishing the amount of nitric acid to one-fourth the amount given in the preparation of the emulsion, it is possible in very cold weather to obtain plates which are sensitive to very low radiations, such as the radiations proceeding from boiling mercury or even boiling water. In summer-time this emulsion, as would naturally be expected, produces what are known as "foggy pictures;" but it can be rendered of use by flooding with hydrochloric acid (see note). In the prepara-

* It has been found advantageous to coat the plate in red light, and then to wash the plate and immerse it in a dilute solution of HCl, and again wash, and finally dry. These last operations can be done in dishes in absolute darkness; the hydrochloric acid gets rid of any silver sub-bromide which may have been formed by the action of the red light.

† It aids cleanness of the film if, before drying, a solution of HCl in water (one to three) is poured over it, and afterwards eliminated by washing. This removes any calcium oxalate that may be formed in the film.

‡ It can be developed in absolute darkness by using dishes for each operation.

tion of such an emulsion the water bath must be kept at a temperature but little above that of the boiling point of ether.

Apparatus employed.

One of the objects I have had in view was to compare the spectra obtained by this photographic method with the thermographs obtained by Sir JOHN HERSCHEL, and also with the energy curves obtained by means of the thermopile. To do this it seemed necessary to work under the same conditions as those under which these results were obtained, and I attempted to do so. After many experiments, however, I found that the absorption lines lay so close together when the beam of light was sent through one prism that I have employed three prisms to obtain greater dispersion. The fine prisms that were actually employed were of white flint glass of an angle of 62°, worked by Mr. ADAM HILGER. The length of the side of the prism is about $2\frac{3}{4}$ inches, and the height $1\frac{3}{4}$. A motion for setting and keeping the prisms at the angle of minimum deviation for all rays was also made for me by the same optician; the fact that rays of very great comparative wave length were to be examined rendered this all the more necessary. The collimator was 20 inches long, and the focus was adjusted for the lowest visible ray. The slit had jaws, both movable, which could be opened to any required extent by means of a differential screw motion. The lens at first attached to the camera had a focus of 18 inches, but subsequently this was abandoned for a lens of 30 inches focus, and of about 3 inches aperture. A rather imperfect heliostat, but one which sufficiently answered my purpose, was used to cast the beam of light on a condensing lens of 6 feet focus and 5 inches aperture, to form a solar image on the slit. The focusing was rather a matter of guesswork, and trial plates had to be exposed to attain really sharp images in any part of the ultra-red. The rapid alteration in the focus of this portion of the spectrum made it impossible to obtain anything except a narrow strip on which the FRAUNHOFER lines were absolutely sharp, and that only on the portion of the spectrum near A, for at the wave length 10,000 the spectrum became so compressed that any collection of fine absorption lines inevitably appeared as more or less shaded bands. A reflecting mirror was finally adopted in lieu of the camera lens, the position and management of which will be more fully described immediately. The difficulty of focussing was thus mitigated, if not altogether surmounted.

For the photography of the diffraction spectrum the same collimator and camera, with its lens, were used, substituting for the prisms a diffraction grating by RUTHERFURD of about 8600 lines to the inch. A large number of photographs have been taken of the ultra-red end of the spectrum with this apparatus, most of which have proved useless, in consequence of the difficulty in obtaining an accurate focus of the absorption lines. During the past year a different arrangement has been adopted, which is better adapted to the diffraction spectrum.

The following is a sketch of the apparatus as employed:—

A collimator (Plate 30, fig. 1), 8 feet in length, was constructed after my design by Mr. HILGER, and instead of attaching a collimating *lens* to it, a collimating mirror (silver on glass) of about 8 feet focus was substituted.

The slit S is $1\frac{1}{2}$ inch long, and was mounted at the side of the tube as shown. The beam of light was directed full on to the slit by the heliostat, and passing through the slit falls on M, a plane and adjustable mirror (also silver on glass). It is then reflected down the tube, and falls on A, the concave mirror already referred to, and the rays pass up the tube in a parallel beam.

At B they fall on a reflection grating of 17,600 lines to the inch, and about $1\frac{3}{4}$ inch in width*; the dispersed beam falls on another concave mirror, B, of about 3 feet focus, and the image of the spectrum is formed on P, the photographic plate, which is attached to a camera in the usual manner. I may mention that a quartz, and also a rock-salt lens of 8-foot focal length is provided for this collimating tube, the mirror A and slit being movable. The reasons for adopting the long collimator and the mirror may be mentioned here, since at first sight there appears to be but little reason for using them. There is no doubt but that glass absorbs the ultra-red rays to an appreciable extent, and I need only refer to the experiments of Sir WILLIAM HERSCHEL, TYNDALL, LAMANSKY, and others in support of this statement. My primary object was therefore to prevent any serious loss by only allowing the beam of light to traverse as small a thickness of this material as I possibly could. That the rays are not totally absorbed there is proof in the diagram of the prismatic spectrum, which I attach to this communication, the prisms employed being of the glass I have already referred to. By adopting the long collimator the beam of direct light reflected from the heliostat and falling on the grating was a circular patch of about $\frac{3}{4}$ inch diameter, and this had traversed no glass except any that might be placed in front of the slit. There was but little loss of light, since to give the same definition the slit could be opened five times wider than with the shorter collimator, with which a condensing lens could be employed. There was another reason for adopting the system of reflection in the camera and also in the collimator, viz.: that as the foci for all rays are coincident, by focussing any visible portion of the spectrum of the second order on the plate, and then cutting this off by a suitable medium, the absorption lines in the dark rays beyond the red of the first order would be equally well defined. It will also be seen that by this plan it was possible to obtain a photograph of the second order on the same plate as one of the first order, by covering up the top half of the plate for the one, and the bottom half for the other. This is of prime importance in making a determination of the wave lengths of the invisible portions, and it has been utilised for settling the approximate values which are given in the map appended (Plate 31).

I have no hesitation in recommending that a mirror in the camera should be substituted for the ordinary lens. It gives most excellent definition, and can be used to

* Ruled by Mr. CHAPMAN, of New York, with Mr. RUTHERFURD's ruling machine.

bring the invisible rays to a focus with the utmost facility. One great difficulty with which I have had to contend in photographing the diffraction spectrum has been to obtain a proper absorbing medium for the more refrangible end of the spectrum which would be placed in front of the slit without interfering with the invisible rays.

In the prismatic as well as in diffraction spectrum this is of immense importance. With the former the prisms become illuminated with white light, and this causes a veil to appear with the image on the plate when developed, if the exposure be at all prolonged. With the latter, the ultra-red of the first order overlaps the violet, blue, and green of the second order, so that it becomes a necessity to use an absorbing medium unless the different orders are separated by means of a prism: a proceeding which may introduce error in calculating the wave lengths, unless very great precautions are taken.

In experimenting on this subject I employed the prismatic arrangement, and found that stained red and ruby glasses were both serviceable, as by interposing either of them I could obtain photographs of the lowest limit of the prismatic spectrum hitherto obtainable without them. Even when employing a glass cell containing an aqueous solution of potassium chromate, the same limit was reached. The times of exposure, however, differed considerably. Thus to obtain an impression of the ultra-red when the absorbing media were stained red and ruby glasses, the times of exposure were as one to two, and with the chromate half as much as that for the stained red.

The latter was very suitable for photographs of the prismatic spectrum, but did not absorb sufficiently in the green to render it a useful adjunct with the diffraction spectrum.

Messrs. CHANCE, of Birmingham, kindly supplied me with some microscopic glass flashed with ruby, and this has been of great service to me, as it cuts off all rays more refrangible than D. It is this which I have principally used in obtaining the diffraction photographs, though a slightly lower limit is reached when separating the spectra by a rock-salt prism. Recent experiments have shown me that a solution of iodine in carbon disulphide, when placed in a rock-salt cell, may also be employed with advantage, as indicated by TYNDALL.

Map of the infra-red diffraction solar spectrum.

In Plate 31 accompanying this paper I have endeavoured to give as accurate a map of the solar spectrum from A, as far as 10,750, as could be consistently obtained from the photographs I have taken with my large grating. A variety of photographs have been taken at different times, and it is by combining the best of these (and when I say the best I mean those which have the greatest definition for particular regions) that the map has the accuracy which I trust it will eventually prove it has. Mr. DE LA RUE kindly placed at my disposal, as often as I required it, the excellent micrometric measuring machine which he had constructed for measuring the series of

photographs of the sun which were taken at Kew. Thanks to Mr. WHIPPLE and his assistants, the arduous task of reading the measurements was materially lightened, which otherwise must have been protracted over a longer time than they have been. The instrument employed measures accurately to the $\frac{1}{1000}$ of an inch and by estimation to half that quantity; but I may say that this delicacy is rather too great for the photographs, and I can only trust my readings to $\frac{1}{500}$ of an inch. The magnifying power of the instrument was slightly too great for some of the finer and also for some of the less transparent absorption lines, and in the first measurements some escaped detection. By carefully going over the whole photographs with a less magnifying power the missing lines have been inserted with very fair accuracy; since most of them were situated close to lines which had been previously measured with the higher magnifying power. I believe that the accuracy of position of these inserted lines is but little, if anything, inferior to the remainder of the lines. After the whole of the photographs had been measured they were all reduced to a common scale by taking fiducial lines and interpolating in the usual manner. Except for a few of the last lines shown near wave length 9700, more than one photograph was measured, and the reason why only one photograph was used for this region was that that one employed had far better definition than the others, though all showed the presence of the absorption phenomena.

In regard to the accuracy of reading I cannot do better than quote the measurements of the A group, to show the accordance between two sets of readings.

	Photo. I.	Photo. II.	Reduced to scale.
A_0	2778·0	2778·0	100·0
A_0	2824·0	2825·0	131·0
A_0	2826·5	2827·0	132·5
A_1	2838·0	2838·0	142·0
A_2	2846·0	2845·0	147·0
A_3	2853·0	2853·5	152·0
A_4	2861·0	2861·5	157·0
A_5	2870·0	2869·5	163·5
A_6	2880·0	2879·5	170·0
A_7	2890·0	2888·5	177·0
A_8	2900·0	2899·5	184·0
A_9	2910·0	2910·0	190·5

The scale of the original map is 40 times the original measurement, 1 inch being represented by 40 inches.* The size of the plates used was 6 inches by $2\frac{1}{2}$ inches wide, and from A to 10,750 occupies nearly that length.

The slit of the collimator was tolerably narrow, but not quite so closed as is usually the case when photographs of the more refrangible end are taken. Thanks to the

* For publication the scale has been reduced to 20 times the original measurement.

wide dispersion of the grating, however, the width of the slit had but little bad effect in giving the finest lines sharply defined; in fact, the finest lines are the most sharply defined.

I have already described the sensitive salt employed, so I need not refer to it further except to say that as the photographs were taken in the height of summer and autumn, the most sensitive salt could not be employed.

I have not thought it expedient in the map to give the wave lengths as being absolutely definite; there may be, and probably is, a little uncertainty regarding them at the present time, but I hope at some future date to be able to make a table of absolute wave lengths which can be applied to the map. At any rate, it is believed that there cannot be any great error, since certain of the lines have been compared with the less refrangible end of the next order of the spectrum as photographed on the same plate, as described in the foregoing pages. It is worthy of remark that in this portion of the spectrum we come to a locality in which it is continuous. At first sight this seemed inexplicable, and many endeavours were made to obtain photographs of such definition as to show absorption lines of some description. I have not succeeded in so doing, nor do I believe that they are to be found, except they be of the very finest description; in fact, much finer than any I have met with below A. In this conclusion I am borne out by a reference to the prismatic spectrum of this portion, in which the same locality exhibits a perfect blank as regards absorption. From smaller photographs taken with the grating I believe it will be found that this region extends beyond wave length 12,000. It might be thought expedient that a reference should be made to the bands of lines which are due to atmospheric absorption. All that can be said is that, whether at a high altitude or a low one, the same intensities of lines exist, except, perhaps, in one or two cases which are very far from being well marked. I have therefore preferred to give all the lines present, leaving a discussion as to what they may be due to a subsequent paper.

Of one thing, however, certainty may be expressed, viz.: that all rays will pass through several inches of glass and through half-an-inch of water; the absorption due to both these substances is evidently one gradually increasing as the wave length increases. I was not at all prepared to find that these long waves could traverse anything like the thickness of either one that they did without being affected. In some cases plates were absolutely exposed whilst immersed in an aqueous solution of potassium nitrite to prevent oxidation of the image, which can be caused by excess of light, as I have already shown in previous communications to the Royal Society, and still a low limit has been obtained. Regarding the time of exposure of the plates for the diffraction spectrum there is much variation; on a bright spring day the same action takes place in 10 minutes which in autumn takes half-an-hour to effect. I have found that when the A line was well visible with ruby glass in front of the slit a short exposure could be given. With the prismatic spectrum five minutes is sufficient exposure

where four or five prisms of flint glass are used and the slit very narrow, the focal length of the mirror being about 30 inches.

Map of the infra-red prismatic solar spectrum.

Plate 30, fig. 2, is a drawing from a photograph of the prismatic spectrum,* showing bands of absorption, but which resolve themselves into lines when the diffraction grating is used. Excepting the bands A, Z, and X, these bands at first are somewhat difficult to recognise in the diffraction spectrum, but by using an artifice they are at once distinguished. If the diffraction photograph be held obliquely from the eye in such a manner that the lines appear to blend one into the other, we have the same appearance of bands as in the prismatic spectrum.† In my paper originally sent in to the Royal Society a remark was incidentally made that the limit of this spectrum was apparently reached; but Professor STOKES pointed out that the supposition was incorrect, as by setting up a curve of $\frac{1}{\lambda^2}$ and producing it, there could be but little doubt that the theoretical limit lay beyond the boundary of the photograph. In the diagram‡ this latter curve has been drawn, and also a curve showing absolute wave lengths.

LAMANSKY'S *thermograph.*

In December, 1871, LAMANSKY published in the 'Monatsberichte der Königl. Akademie der Wissenschaften zu Berlin,' a communication on the heat spectrum of the sun and the lime light, to the original of which I have not had access, but have used a translation which appears in the Philosophical Magazine for April, 1872. LAMANSKY'S thermograph was made by aid of a thermopile, the deflections being a measure of the heating effect on lamp-black of the various parts of the spectrum. In some cases he used a flint-glass prism with which to obtain his spectrum, in others he used a rock-salt prism, and it is from the results with the latter that his diagram is constructed. In making a comparison of this thermograph§ with the photograph of the prismatic spectrum I have had recourse to the method of graphically setting up ordinates $=\frac{1}{\lambda^2}$ on the diagram.||

* The bands $\psi_{,}$ $\psi_{,,}$ were marked from photographs taken on March 25, 1880, a note to that effect having been sent to the Secretary of the Royal Society June 10, 1880.

† Photographs taken with a wide slit also give the same effect.

‡ The amended diagram and description was communicated to the Royal Society January 8, 1880.

§ The first comparison was withdrawn from the paper, owing to the construction of a new diagram showing the curve of wave-lengths and $\frac{1}{\lambda^2}$ as suggested by Professor STOKES.

|| In the copy of LAMANSKY'S diagram in the Philosophical Magazine there is evidently an error, in that the ordinate of the last maximum cuts the descending curve. This has been corrected.

Taking G, F, E, and D as fiducial points as given by LAMANSKY, it was found that the lines joining them lay nearly in a straight line, with the exception of that joining F and G. The lines F, E, and D were taken as correct, and the straight line on which those points most nearly lay was prolonged to meet the line on which the abscissæ were measured. The various bands and lines of absorption in the ultra-red portion of the spectrum, as shown by the photograph, were inserted. It will be seen (Plate 32, fig. 1) that the thermal minima agree with z, π, and τ, and the maxima with X, σ, and ϕ, positions which would be naturally assumed as most probable. There cannot be much doubt as to the correctness of this view, if a reference be made to a passage in LAMANSKY's paper. He says: "These three breaks or bands are not of equal breadth; the first is much more sharply separated from the second than the second from the third. It may easily happen if the movement of the thermo apparatus be not sufficiently delicate that the *second and third appear as one common break.*" A glance at the diagram will show that this is most probable; π, σ, and τ *would* easily blend into one. LAMANSKY asks the question, "Is not the limit of refraction situated at the place where the heat effect of the solar spectrum attains its last maximum?" If the positions assigned to these maxima by myself be correct it is evident that it has not been attained. From a careful perusal of Sir JOHN HERSCHEL's memoir, which is to be referred to, it would seem that the maxima show themselves by his method with greater difficulty the nearer they approach the limit of refraction.

HERSCHEL's *thermograph.*

In the Philosophical Transactions for 1840 is to be found the thermograph of the prismatic spectrum as delineated by Sir JOHN HERSCHEL. The thermograph itself was made by causing an image of the solar disc to focus itself on a sheet of blackened tissue paper moistened with alcohol, after passing through a flint or other prism; the drying of the alcohol in some parts more rapidly than in others gave a figure such as shown in Plate 32, fig. 4, demonstrating that the heating effect of the spectrum was discontinuous. Figs. 3, 4, and 5 are taken from Sir JOHN HERSCHEL's paper, and it should be noted that these are the images obtained by the sun's whole disc, whilst the photograph was taken with a narrow slit.

Lord RAYLEIGH gives an account of a repetition* of HERSCHEL's experiments, and finds that the thermographs he obtained are not comparable with HERSCHEL's, the maxima of heating effect lying in very different positions; he also indicated that the spot ϵ, and probably δ, lay beyond the theoretical limit of the prismatic spectrum. In some of my own experiments with the same method I found that I obtained thermographs which were very similar to those of HERSCHEL with the exception of the spot ϵ – a spot, it may be remarked, the existence of which Sir JOHN HERSCHEL himself did not absolutely insist upon. I have tried to examine the latter's thermograph with

* Phil. Mag., vol. iv. (fifth series).

due regard to my own experiments, and own to meeting serious difficulties. It can hardly be possible to doubt the accuracy of the measures made by HERSCHEL, but it must be remembered that the length of spectrum with which he worked was very small, 4 inches being the extreme limit of the thermograph, whilst not 2 inches was the length of the visible spectrum. It must also be recollected that he fixed the position of the various parts of his thermograph by a reference to the absorption spectrum of a solar image through cobalt glass, using the centre of the yellow solar image so obtained as a starting point for his measurements. It is to this that I wish to draw special attention as being the cause of the probable difficulty in recognising the breaks in the continuity of the heating effect of solar radiation when examined by the aid of a prism.

We may assume that the upper limit of visibility of the spectrum lies somewhere near H, and if, as is shown in the figure (Plate 32, fig. 2), we deduct the semi-diameter of the sun from the place shown as the limit of visibility, we have a very probable position for the H line. Now in the plate accompanying his paper (Philosophical Transactions, 1840) HERSCHEL shows a photographic spectrum taken on what he calls bromuretted paper. Having experimented on similar paper, I found that the lowest part of the spectrum reached by his method of working was near X (Plate 30, fig. 2).

Again, a measurement of the position of the lithium and sodium lines in regard to the position of the red images of the sun seen through cobalt glass (Plate 32, fig. 4) places the centre of it near the line B.

Taking H, B, and X on HERSCHEL'S scale of lengths as fixed by this discussion, and setting up as ordinates $\frac{1}{\lambda^2}$, we find that the lines joining them lie nearly in a straight line, but that the fiducial line through the centre of the yellow solar image (Plate 32, fig. 4)* lies nearer to D than it should do. This point should be nearly half way between D and E, but slightly near D. The difference between the position given it by this graphic way, and that it should occupy according to HERSCHEL'S drawing, is $\frac{1}{15}$ of an inch—a length which is certainly very small. If the bands and lines shown in the photograph are inserted by means of their $\frac{1}{\lambda^2}$ ordinates, it will be seen that the breaks of continuity in the thermograph agree fairly well with the absorption as shown in the photograph on the same plate. I have shown in chain dotted lines the approximate positions that H, E, D, B, A, X, τ, ϕ, and ψ would occupy when the position of the spectrum is fixed by reference to images of the sun seen through the cobalt glass as given in the diagram. It will be noticed that there is still an agreement in the main between the thermograph and the photograph, but (first) that the upper limit of visibility is a long way in the ultra-violet, and (second) that δ and ϵ of the thermograph are beyond the theoretical limit of the spectrum. This last can scarcely be the proper

* Figs. 3, 4, and 5 are taken from Sir J. HERSCHEL'S plate in the Phil. Trans., 1840.

comparison of the thermograph with the photograph, and I am inclined to believe that the first method gives a result which should be fairly accurate though it omits ϵ from the thermograph.

If H be taken as known, and if the fiducial line $\gamma\ \gamma$ be correct, the limit of the spectrum would be beyond ϵ; but it is hard to reconcile this with the fact that the centre of the last red image of the sun seen through cobalt glass would be near C, otherwise the absorptions shown in the photograph would agree with the breaks of continuity in the thermograph.

Theoretical deductions.

It has seemed advisable to keep any theoretical deductions to the concluding portion of this paper, as the first portion contains wholly facts on which no controversy, I can imagine, can arise; at the same time, it would be wanting in candour did I not point out what seems to me to be some evident conclusions which can be drawn from the experiments that I have made over such a long period.

I have pointed out that the form of silver bromide which is sensitive to the red and ultra-red of the spectrum transmits the least refrangible rays to a marked extent, but it does not do so entirely; in fact, we may say it absorbs all rays, the less refrangible the best, the more refrangible much less, and the green rays least of all. A very instructive experiment to repeat is to photograph the spectrum of burning coal-gas with this salt. It will be found that the curve of intensity constructed as already indicated has the following appearance in the prismatic spectrum.

It shows two well-defined maxima, which are situated somewhere about wave lengths 3800 and 7600. It may be merely fortuitous that these occupy the positions they do in the spectrum; but if we are to look at the occurrence of the maxima at intervals, one of which has double the wave length of the other, we cannot but be struck with the idea that a molecule of the silver bromide is responding to harmonic vibrations. In the state in which the silver bromide has only one maximum at about wave length 3800 it seems probable that the molecule exists of only half the weight. The heavier molecule may be well supposed to take up those vibrations which we may say are an octave higher, whilst it would not at all follow that the lighter molecule would respond to the vibrations of the lower octave.

The blue transmitting form of silver bromide can readily be transformed into the red transmitting form by simple friction, in which state it becomes as insensitive to e lower octave as if the molecule had been formed by chemical means, as by the inary methods of preparation. A difference in molecular state for other compound bodies has been suspected by Mr. J. NORMAN LOCKYER,* and this existence of two states of silver bromide lends confirmation to his view.

I have to thank the Astronomer Royal for kindly placing diffraction gratings at my disposal, as also Dr. J. W. DRAPER and Professor LANGLEY for procuring me others, with which the most successful part of my work has been carried out. My assistant, Corporal JACKSON, R.E., has been most invaluable to me in making the various emulsions under my directions and in taking negatives when my official duties prevented me from personally utilising what little sun we have had during the last summer and autumn. To Mr. DICK my acknowledgments are also due for the care he has taken in preparing the map of the diffraction solar spectrum.

Proc. R. S., note 11, page 153, 1874.—'Studies in Spectrum Analyses,' page 120.

XVIII. *On the Photographic Spectra of Stars.*

By WILLIAM HUGGINS, *D.C.L., LL.D., F.R.S.*

Received December 11,—Read December 18, 1879.

[PLATE 33.]

§ I. *Introduction.*

IN the year 1876 I presented to the Royal Society a preliminary note on the "Photographic Spectra of Stars."* I beg now to give an account in greater detail of my methods of work and of the photographs which I have obtained.

The importance of supplementing the observations by the eye of the spectra of stars by photographs of the violet and ultra-violet portions of their spectra was so obvious, that as early as the year 1863 my friend Dr. W. ALLEN MILLER and I made the attempt to obtain such photographs in addition to our eye-measures of star spectra.† With the apparatus then at our command we were not able to get any clear definition of lines, but a dark streak only upon the negative plate.

Other investigations which I then took up prevented me from resuming this line of work. I was also not encouraged to proceed further with photography at that time, as the clock-motion driving the telescope did not work with the accuracy that was necessary.

In the year 1875 Mr. GRUBB replaced the driving clock by a new one, in which there is a secondary control by means of a pendulum in electrical connexion with a standard clock.‡ I am able to speak in terms of high praise of the performance of this new clock.

The early attempts at photography of the spectra of stars were made with the 8-inch refractor by ALVAN CLARK, then in my observatory. On receiving the new clock the refractor of the instrument lent to me by the Royal Society was dismounted and the CASSEGRAIN telescope, with a metallic speculum of 18 inches diameter, was put in its place upon the equatorial stand. After many preliminary trials I adopted the following arrangements of the spectral apparatus and methods of work.

* Proceedings R.S., No. 176, 1876. Since the publication of my preliminary note, Professor H. DRAPER has written two notes on this subject, 'American Journal of Science,' vol. xiii., Jan., 1877, and Nov. 27, 1879, p. 83.

† Phil. Trans. R.S., 1864, p. 428.

‡ Proceedings R. Dublin Society, April 21, 1879.

§ II. *Apparatus.*

In consequence of the very limited amount of light received from the stars, it was obviously of the first importance to spread out the spectrum to the smallest amount that would give a sufficiently visible separation of the principal lines to permit of their being easily recognised and measured. Another point in this connexion which required consideration was whether a slit should be employed. A slit sufficiently narrow to be of use for the purposes presently to be mentioned would allow a portion only of the light, concentrated by the speculum in the star's image, to enter the collimator, and would therefore greatly lengthen the exposure required to obtain a photograph. Notwithstanding this serious drawback I determined to use a slit partly for the sake of a purer spectrum, and partly on account of the facility which a slit would give to obtain a second spectrum for comparison on the same plate with the star's spectrum. The employment of a slit would also make the same apparatus suitable for use upon the moon and planets.

For the material of the prism I selected Iceland spar, as it is very transparent to the ultra-violet rays, and has so much higher a dispersion than quartz that one prism only would be sufficient. The prism has a refracting angle of 60°, and is cut in a plane perpendicular to the axis of the crystal. Such a prism in any one position gives single refraction for light of one refrangibility only, but practically the separation of the two images through the range of the spectrum which is photographed is too small sensibly to affect the results.* The prism is fixed in a position of minimum deviation for H. The lenses are of quartz, cut perpendicular to the axis and plano-convex in form. The lens of the collimator is $1\frac{1}{2}$ inch diameter, 10 inches focal length. The lens placed after the prism to form the image on the plate is of the same diameter, $6\frac{1}{2}$ inches focal length.†

The form of construction of the spectrum apparatus is shown in the accompanying diagram (fig. 1).

The wooden frame which receives the photographic "backs" is made to tilt so as to allow the plate to be brought into a position in which the rays of different refrangibility shall be, as nearly as is possible, in focus together upon the plate. This

* Professor STOKES has permitted me to add the following note, dated January 23, 1875 :—

"I have worked out the deviations for a prism of 60° of calcareous spar, the axis perpendicular to the bisecting plane of the prism, the line H at minimum deviation and therefore seen single. I have worked out the deviations for B with the results :

Deviation for H ordinary, extraordinary	54°	37′·76
Deviation for B ordinary	51°	32′·49
" " extraordinary	51°	32′·36

The difference comes smaller than I had expected, only 0′·13 or 8″, the spectrum from B to H being over 3°. For a line half way between B and H the difference would be only a quarter of that, or 2″. The difference comes out practically insensible."

† The prism and lenses were cut for me by A. HILGER.

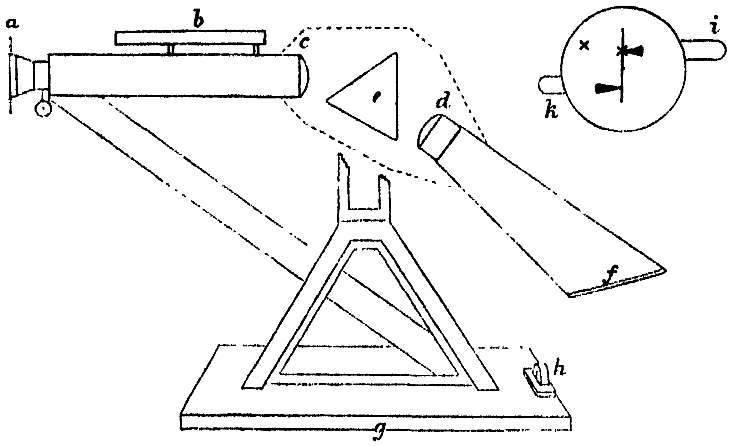
position was previously found by means of solar light, and the frame was then firmly fixed in position before the apparatus was mounted in the telescope.

The photographic plates are $1\frac{1}{2}$ inch long by $\frac{1}{2}$ inch wide, and the length of the photographic spectrum between the lines G and P in the ultra-violet about $\frac{1}{2}$ inch.

The definition is so good that the photographs can be examined with advantage under a low-power microscope, and notwithstanding their small size, about fourteen lines may be counted between the lines H and K.

The apparatus combines very successfully a sufficiently defined separation of the parts of the spectrum with a moderate diminution only of the intensity of the star's light.

Fig. 1.



a. Slit plate.
b. Tube for collimation.
c, *d*. Quartz lenses.
e. Prism of Iceland spar.
f. Photographic plate.
g. Bevelled edge.
h. Screw for adjustment in focus of mirror.
i, *k*. Shutters of slit.
l. Silver plate with slit.

The width of the slit which was finally adopted was based on a compromise. The very narrow slit which gave the best photograph of the solar spectrum was found to diminish too seriously the light of the stars, and the slit was then opened until the interval between the edges was about $\frac{1}{850}$th of an inch. When the slit is of this width of opening the solar lines are still well defined, but the number of lines to be counted between H and K is reduced to about seven. I found it was not possible to work with a narrower slit.

The base plate of the apparatus is bevelled at the edges and slides in the grooves of a second plate, which is firmly screwed to a wooden platform which is attached to the end of the telescope tube (fig. 2). The small convex speculum was removed from the CASSEGRAIN telescope, and the spectrum apparatus fixed at the end of the tube, as already described, was so adjusted that the slit was brought exactly to the position

of the principal focus of the large speculum. The grooves in which the apparatus slides are graduated, so that the apparatus after removal can be replaced exactly in its former position. A final determination of this position was made from the definition of photographs of star spectra.

For the adjustment of the collimator of the spectrum apparatus in the optical axis of the large speculum a tube six inches long with cross wires at both ends was fixed on the collimator and parallel to it. Advantage was then taken of the hole in the large speculum. The eye-tube of the CASSEGRAIN was removed and a small GALILEAN telescope, magnifying 16 diameters, was fixed in its place. The spectroscope was then so adjusted by suitable screws that the cross wires at the first end of the small tube were made to coincide, when viewed through the GALILEAN telescope, with those at the farther end of the tube.

§ III. *Methods of Work.*

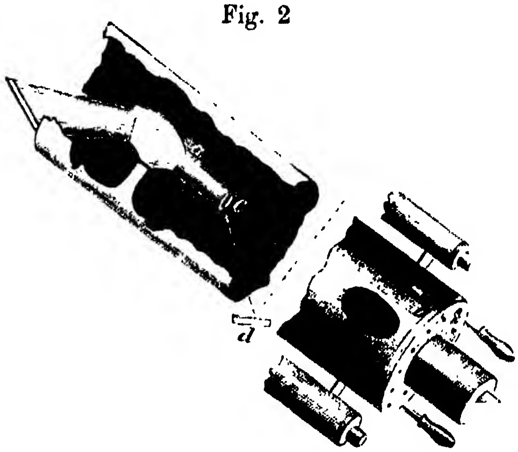
Two requirements at once presented themselves of such primary importance that success was seen to depend upon the perfection of the method adopted to meet them. It was necessary to have a method of bringing the image of star readily and with exactness upon any desired part of the slit. Further, it was necessary to have a convenient method of continuously watching the star's image when upon the slit during the whole time of the photographic exposure, in order to correct by hand any small inequalities of the motion of the telescope which might throw the star's image off the slit. In a large equatorial there are other sources of small inequalities of motion besides those due to want of uniformity in the clock itself. When it is remembered that the star's image must remain steadily upon a slit of $\frac{1}{350}$th an inch in width for perhaps an hour, it will be seen how necessary is the power of continuous supervision and of instant control by hand. The following methods were perfectly successful.

A round thin plate of polished silver (*l*, fig. 1; *c*, fig. 2), $1\frac{1}{2}$ inch in diameter, with a narrow opening in the middle rather longer and wider than the slit itself, was fixed over the slit of the spectroscope. This plate forms a plane mirror, and when the telescope has been brought approximately into position by its finders, the bright image of the star is seen somewhere upon the plate by looking into the small GALILEAN telescope fixed in the place of the eye-piece of the CASSEGRAIN telescope. Now if at the same time artificial light is thrown upon the plate, it becomes itself visible, and then the opening in it and the slit within the opening can be distinctly seen at the same time as the image of the star as a bright point upon it. By the aid of this arrangement there is no difficulty in bringing the star's image by the slow motion handles of the equatorial readily and with precision upon any part of the slit.

As the position of the star's image even upon the slit itself can be seen, the image being somewhat wider than the slit and therefore not wholly lost within it, it is

possible to keep the star in view upon the slit during the whole time the photograph is being taken, and to correct instantly by hand any small departure of the star's image from its proper place upon the slit.

Fig. 2



I will now describe how the necessary breadth was given to the spectrum without the employment of a cylindrical lens. As the star's image is not a point, its linear spectrum has a small breadth, but not more than about half the breadth which is necessary for the lines to be well seen. After the exposure had proceeded sufficiently to produce a linear spectrum, the image of the star was moved upon the slit in the direction of its length, through a space equal to about the apparent diameter of the star's image. The exposure was then continued for a period of about the same length. In this way a photographic spectrum can be obtained of the breadth that is desired by the union of two or more linear spectra.

The artificial light was thrown upon the silver plate by a small mirror fixed on the side of the telescope tube opposite to the end of the declination axis (*d*, fig. 2). This axis is hollow and the light passes through it from a lamp suspended at the end. The precaution was taken of making this light pass through a plate of yellow glass.

§ IV. *Photography.*

At the early stages of these experiments I used wet collodion, but I soon found how great would be the advantages of using dry plates. Dry plates are not only more convenient for astronomical work, being always ready for use, but they possess the great superiority of not being liable to stains from draining and partial drying of the plates during the long exposures which are necessary even with the most sensitive plates. I then tried various forms of collodion emulsions, but finally gave up these in favour of gelatine plates, which can be made more sensitive. The development was sometimes by the ferrous oxalate process, at others by the ordinary pyrogallic method.

Positives were taken from the original negatives by placing the negative plate upon a similar dry plate and exposing to a gas flame for two or three seconds. Some of the

negatives were enlarged, but it was found that a more satisfactory determination of doubtful points could be made from the original small negative or the positive taken from it when viewed under a microscope of low power than from an enlarged copy.

In the negatives the dark lines are represented by transparent spaces where the light has not acted. When these spaces are rather broad there may be some uncertainty in placing the wire of the micrometer exactly upon the middle of the transparent space. In the positives these spaces become dark lines, in which the middle part is usually the darkest. In nearly all cases, therefore, it was found desirable to confirm the measures of the lines made on the negatives by measures of the same lines in the positives taken from them.

In some cases the negatives were intensified by the usual methods, but they were not varnished.

§ V. *Spectra for Comparison.*

It has been stated that one of the reasons for using a slit was that spectra for comparison might be taken on the same plate.

The slit is furnished with two sliding shutters (*i* and *k*, fig. 1), each of which closes one-half of the length of the slit. When a star was photographed, one-half only of the slit was in use, the other half being closed by its shutter.

If the moon or one of the brighter planets was situated conveniently for the purpose, the shutter which had remained closed was withdrawn, and the other shutter pushed in over the half of the slit which had been in use for the star. The telescope was then directed to the moon or planet. In this way the star's spectrum was obtained, together with that of solar light from the moon or planet.

If this method was not available, after exposure to the star's light the shutter was closed, and the apparatus with both shutters pushed in was allowed to remain until the next day, when by means of a small hand mirror, direct sun light or light from the sky, was reflected through the hole in the large speculum, so as to fall upon the slit in the direction of the axis of the collimator.

More recently advantage was taken of those stars, the spectra of which had been compared with solar light. A spectrum of one of these stars was taken through the second half of the slit to serve as a fiducial spectrum of comparison. This method has the advantage of permitting the development of the plate to be performed the same evening.

The spectra of the planets were obtained on the same plate with the lunar or solar spectrum. When, however, circumstances permitted, the plan employed by Dr. MILLER and myself in our earlier eye observations was preferentially adopted. We wrote in 1864 :—

"The length of the opening of the slit is much greater than the diameter of the telescopic image of the planet (Jupiter) even after elongation by the cylindrical lens. If therefore at the time of observation the light from the sky is sufficiently intense to

form a visible spectrum, the spectrum of the sky is seen in the instrument, together with the spectrum of Jupiter, and much exceeding it in breadth. When the period is so chosen that the degree of illumination of the sky is suitable in proportion to the intensity of the light of Jupiter, the solar lines and those due to our atmosphere are well seen in close contiguity with the lines in the spectrum of Jupiter, and occupying exactly similar relative positions. The sky-spectrum is seen under precisely similar conditions of altitude and of state of atmosphere. To the light of Jupiter under these circumstances of observation is added the light reflected from the small area of sky immediately between the observer and the planet. This light is, however, too faint in proportion to that of Jupiter to become a source of error."*

Under similar circumstances, both shutters being withdrawn, spectra of the planets Jupiter and Venus were taken upon the broader spectrum of the sky. The solar lines are thus strictly comparable with those of the planetary spectra, since they were photographed under the same conditions of altitude and of terrestrial atmosphere.

When it was desired to obtain spectra of terrestrial substances for comparison, the spectroscope, as a whole, was drawn out of the grooves which hold it in its place at the end of the telescope, and was then fixed upon a kind of optical bench, on which also slide two lenses of quartz, and an apparatus to hold electrodes and tubes. These are so arranged that an image of the spark or tube is formed upon the slit. In this way photographs were taken, which are comparable with those of the stars, and could serve for the purpose of comparison when any known line was common to both spectra to form a fiducial point of measurement. As all the stellar photographs contain the line H, calcic chloride or metallic calcium was introduced into the spark, and the line of calcium corresponding to H_1 was used for this purpose.

There would have been no serious difficulty in so arranging the electrodes that a spectrum of the induction spark should be taken immediately after the star upon the same plate, but in actual practice there was some inconvenience in this arrangement. Two spectra on the same plate were not found to be satisfactory for comparison unless the "back" containing the plate had remained in its place. If it was removed, some difficulty was found in replacing it with the necessary accuracy.

§ VI. *Determination of Wave Lengths.*

The map of M. CORNU of the solar spectrum from *h* to O†, together with M. MASCART's determinations of the wave lengths of the lines of cadmium in the ultra-violet,‡ were used for the reduction of the measures to wave lengths.

* Phil. Trans., 1864, p. 422.

† 'Annales de l'École Normale,' 2e série, tom. 3, pl. 1.

‡ MASCART's "Recherches sur la détermination des longeurs d'onde," 'Annales de l'École Normale,' tom. 4, p. 1; also CORNU's "Détermination des longeurs d'onde des radiations très réfrangibles du Magnesium, du Cadmium, du Zinc et de l'Aluminium," 'Archives des Sciences Physique et Naturelle,' 15 Juillet, 1879.

The photographic spectra of the brighter stars can be traced upon the plate from about b to beyond S, but in the accompanying map I have limited myself to the portion of the spectrum between the line of hydrogen (γ) near G and O in the ultra-violet.

An admirable wire micrometer by DOLLOND, attached to a microscope furnished with a two-inch objective, was used to measure the photographs. The readings of the micrometer head give 2·947 hundreths of a revolution for each ·0000001 m.m. of wave length at the position of H.

By means of photographs of the solar spectrum, and of those of the spectra of iron, cadmium, calcium, and magnesium, a curve on a sufficiently large scale was laid down on paper ruled in millimetres connecting the measures of the micrometer with the intervals of wave length. Great care was taken by cross measurements in different ways to make this curve as accurate as possible. The positions of the lines as determined in wave lengths were afterwards confronted with solar lines by actual measurement under the microscope. I do not think the probable error of the determination in wave lengths exceeds in any case ± 2 ten millionths of a millimetre. For most of the lines I think it is less than half of this amount.

§ VII. *Results.*

It need hardly be mentioned that only nights of great atmospheric clearness are suitable for stellar photography. The unusual prevalence of unfavourable weather during the time this work has been in hand has greatly limited the number of successful photographs I have been able to obtain. The remarkable circumstance of the apparent absence of the line K in one of my earlier photographs of Sirius, made me select, in the first instance, other stars belonging to the same class.

In the accompanying map I have given of this class of stars the spectra of Sirius, Vega (α Lyræ), α Cygni, α Virginis, η Ursæ Majoris, and α Aquilæ, and representing a different class of stars the spectrum of Arcturus. In addition to these stars I have obtained photographs of β Pegasi, Betelgeux, Capella, α Herculis, and α Pegasi; but as these are more or less incomplete, in consequence of the unfavourable state of the atmosphere when they were taken, I prefer to reserve any description of their spectra for the present.

I have obtained good spectra of the planets Venus and Jupiter, taken together with a broader spectrum of daylight for comparison, and also of Mars.

Numerous photographs of limited areas of the moon's surface have been taken under different conditions of illumination, and also of the moon during a partial eclipse of that body.

Besides the above objects there are several directions in which celestial spectrum photography could doubtless be applied with great advantage. One of these, which the bad weather alone has prevented me from attempting, was to supplement my former eye observations of the spectra of gaseous nebulæ by photography. As the

light of these bodies is distributed among a few lines only, it seems by no means hopeless to obtain on the very sensitive gelatine plates which may now be made, photographs of any lines which may exist in the violet and ultra-violet portions of their spectra.

Another class of bodies to which the application of photography might give us much new knowledge are comets. The form of apparatus described would make it possible to obtain photographic spectra of the light from different parts of these bodies.

We may entertain some hope from photographic spectra of obtaining information of the condition of things under which the increase and diminution of light occurs in those stars which are periodically variable. It is not improbable that modifications may be discovered in the photographic portion of the spectrum, even when none are seen by the eye.

This same form of apparatus, with some obvious modification, would be useful in obtaining photographic spectra of the different portions of a sun-spot.

The photographic method may also be of use in the determination of the relative motion of two stars in the line of sight. The photographs I have obtained of the spectra of two stars on the same plate do show a very small relative shift; but in an inquiry of so great delicacy some special arrangements, which I need not here describe, would be necessary to ensure the photographs from some causes of possible minute instrumental displacement. Also photographic spectra of opposite limbs of the sun on the same plate may give evidence of the sun's rotation.

§ VIII. *White Stars.—Sirius, Vega* (*α Lyræ*), *α Aquilæ, α Virginis, α Cygni, α Virginis.*

The photographic spectra of all these stars possess very strong characteristic features in common; indeed, the differences between their spectra must be regarded as modifications of a typical spectrum common to the whole class.

In our eye observations of stars of this class, Dr. W. ALLEN MILLER and myself called attention to the intensely strong lines of hydrogen corresponding to C and F. Under favourable conditions of atmosphere we were able to see also, in stars of this class, very fine lines corresponding to the principal lines of sodium, magnesium, and iron, though in some of these stars the least refrangible line only of *b* was seen. We remarked of these stars: "It is worthy of notice that in the case of Sirius and a large number of white stars, at the same time that the lines of hydrogen are abnormally strong as compared with the solar spectrum, all the metallic lines are very faint."*

The photographs present a spectrum of twelve very strong lines (of these seven were given in my preliminary note). Beyond these lines a strong continuous spectrum can be traced as far as S, but without any further indication of lines. The least refrangible of these lines is coincident with the line (γ) of hydrogen near G. The next line in order of greater refrangibility agrees in position with *h* of the solar spectrum. The third

* Phil. Trans., 1864, pp. 427, 428, 429.

line is H. K, if present at all, is thin and inconspicuous.* The nine lines which follow do not appear to be coincident with any of the stronger lines of the solar spectrum. These lines appear to be common to all the stars of this class, though it may be that some of the more refrangible lines are sometimes absent.

For the sake of convenience of reference I have distinguished these nine lines by the letters of the Greek alphabet in the order of their refrangibility.

I would wish to call attention to the remarkable arrangement in position in the spectrum of these lines. As the refrangibility increases the lines diminish in breadth, and the distance between any two adjacent lines is less. The group possesses a distinctly symmetrical character. The suggestion presents itself whether these lines are not intimately connected with each other, and present the spectrum of one substance.† It is also of importance to notice that the spectrum does not end with the group. Beyond the last line, between M and N, the continuous spectrum runs on far beyond S in the ultra-violet. The wave lengths of these lines will be found under Vega. The spectrum of Vega may be taken conveniently as typical of the whole class of white stars, so that the distinctive features of the other stars of this class may be regarded as modifications or departures from this common typical form. There are principally three directions in which the changes take place:

1. In the breadth and greater or less marginal diffuseness of the typical lines.

2. In the presence or absence of K, and if present in its breadth and intensity relatively to H.

3. In the number and distinctness of the other lines of the spectrum.

* In 1876, Mr. LOCKYER suggested that photographs of the spectra of the brighter stars might show modifications of this character of the lines of the calcium spectrum, and that such modifications would confirm his views as to the dissociation of this substance. (Proc. R.S., No. 168, 1876.) Mr. LOCKYER gives a fuller statement of his views on this and other points in connexion with the different classes of the stars in Proc. R.S., Dec., 1878, see fig. 1.

† [There is a high probability that this substance is hydrogen. The two lines in the visible part of the spectrum C and F forming part of the same group belong to hydrogen. Also, as stated above, the first two lines of the photographic group correspond to the line of hydrogen near G and to that at the position of h. Dr. H. W. VOGEL has called my attention to a paper of his "On the Spectrum of Hydrogen" in the 'Monatsbericht der Königl. Academie der Wissenschaften zu Berlin,' July 10, 1879. Among other lines he gives the following, which agree in position with four of the typical lines:—

VOGEL's numbers.	My numbers.	
λ 3968	λ 3968	H
3887	3887·5	α
3834	3834	β
3795	3795	γ

Dr. VOGEL says in his letter (February 18, 1880): "In one of my last photographs I have another line λ 3769, your next line is 3767·5."

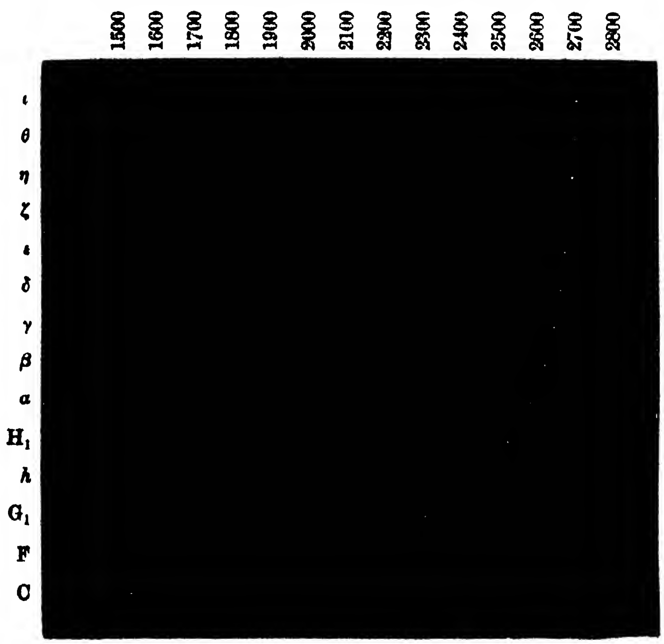
January 24, 1880, I received the following note from Mr. JOHNSTONE STONEY, F.R.S.:—

"There can remain very little doubt that your typical lines are due to hydrogen. The evidence of their all being members of one physical system is made very plain when their positions are plotted down

One of these modifications which possesses great suggestiveness, consists of the absence or difference of character presented by the line K. In all the stars of this class this line is either absent or is very thin as compared with its appearance in the solar spectrum, at the same time that H remains very broad and intense. In the spectrum of Arcturus, a star which belongs to another class, which includes our sun, this line K has passed beyond the condition in which it occurs in the solar spectrum, and even exceeds the solar K in breadth and intensity.

The spectra of these stars may therefore be arranged in a continuous series, in which first we find this line to be absent. Then it appears as an exceedingly thin line. We then pass to another stage in which it is distinct and defined at the edges; in the solar spectrum it becomes broad and winged; and lastly in Arcturus there is further progress in the same direction, and the line, now a broad band, exceeds in intensity H.

as in the following diagram, for it there becomes more conspicuous that they lie on, or very near, a definite curve, which could not happen by chance.



This question of whether they lie actually on, or only near, a definite curve is, if I mistake not, of very great significance in the theory. If they lie *on* a curve obeying any exact mathematical law, their connexion must, I think, be attributed to their corresponding to the *consecutive* partial tones of some vibrating system (like those of an elastic rod or bell, for example). If, on the other hand, they lie near but not on the curve this circumstance would support the hypothesis (which seems to accord with other facts) that the visible lines are members of harmonic series, most of the members of which are invisible, those only being seen whose positions chance nearly to fulfil a definite condition—a state of things which I have shown to exist in some acoustic arrangements, and which wherever it prevails exalts the intensity of the harmonics whose positions nearly fulfil the requisite condition.

To ascertain which of the two foregoing alternatives is the true account of your typical lines, I converted the wave lengths as you determined them into wave frequencies (the reciprocals of the wave

Now the lines H and K agree in position with two strong lines in the spectrum of calcium, and are therefore usually believed to be produced by the vapour of that substance. It was therefore of some interest to ascertain if any of the other strong lines of the typical spectrum were concident in position with a pair of strong lines in the calcium spectrum at 3736·5 and 3705·5. The calcium line 3736·5 falls nearly half-way between ϵ and ζ. The stellar line θ is indeed very near the calcium line 3705·5, but does not coincide with it, its position being 3707·5.

I prefer in the present paper not to enter into any discussion of the physical

lengths) and made the following table of their first and second differences. Assuming that the irregularities in the second differences cannot be referred to errors of observation, I think that the accuracy of your work gives evidence which must be accepted that the second alternative is the true one, viz., that the lines do not lie on but near a definite curve.

W.—Wave length in air. n.—Wave frequency in air.
Δn.—First difference. $\Delta^2 n$.—Second difference.

	W.	n.	Δn.	$\Delta^2 n$.
C	6562·1	1523·9		
F	4860·7	2057·3		
Hγ	4340·1	2304·1		
			134·2	
h	4101·2	2438·3		52·4
			81·8	
H_1	3968·1	2520·1		29·6
			52·2	
α	3887·5	2572·3		16·3
			35·9	
β	3834	2608·2		9·1
			26·8	
γ	3795	2635·05		7·5
			19·3	
δ	3767·5	2654·3		3·7
			15·6	
ϵ	3745·5	2669·9		4·5
			11·1	
ζ	3730	2681·0		2·1
			9·	
η	3717·5	2690		1·8
			7·2	
θ	3707·5	2697·2		1·0
			6·2	
ι	3799	2703·4		

This so far goes to show that the typical lines are not consecutive members of one series, but the members of one or more series whose positions lie near the curve. This appears to be corroborated by finding that H_1 and the hydrogen line near G are connected harmonically, these rays being exactly the 35th and 32nd harmonics of a vibration whose fundamental is $\frac{\tau}{72 \cdot 003}$ (τ being the time in which light

conditions corresponding to these variations in the line K,* nor to many important suggestions which naturally present themselves when we study the modifications of what it is convenient to regard as the most typical form of spectrum. Do these modifications not represent some of the stages through which our sun has passed? I hope

travels a millimetre in air). In fact, taking their wave frequencies in air I find as follows, the differences being wholly insensible.

	n by calculation.	n by ÅNGSTRÖM's observations.
For line near G	$32 \times 72{\cdot}003 = 2304{\cdot}096$	2304·09
" " H	$35 \times 72{\cdot}003 = 2520{\cdot}105$	2520·10

The remaining typical lines do not belong to either this series or that of which C, F, and h are members; and to include them we must suppose two other motions at least to exist in hydrogen.

Possibly six of these lines may be harmonics of $\frac{\tau}{9{\cdot}0572}$ for I find:

					Calculated.	Observed.	Outstanding differences.
α	284	×	9·0572	=	2572·2	2572·3	+0·1
β	288	×	"	=	2608·5	2608·2	−0·3
γ	291	×	"	=	2635·6	2635·05	−0·55
δ	293	×	"	=	2653·8	2654·3	+0·5
ζ	296	×	"	=	2680·9	2681·0	+0·1
η	297	×	"	=	2690·0	2690·0	0

and possibly the others, viz.: ϵ, θ, and ι may be harmonics of $\frac{\tau}{6{\cdot}845}$ for

					Calculated.	Observed.	Outstanding differences.
ϵ	390	×	6·845	=	2669·6	2669·9	+3
θ	394	×	"	=	2696·9	2697·2	+3
ι	395	×	"	=	2703·8	2703·4	−4

I do not attribute much weight to the last two series, for I fancy the computed positions of γ and δ are too divergent from your observed positions. The calculation puts these lines 1 degree of wave frequency scale (=1·4 degree of ÅNGSTRÖM's scale) nearer together than your determination."

Mr. LOCKYER, in a "Note on the Spectrum of Hydrogen" (Proceedings Royal Society, December 17, 1879), describes a line in his photographs of hydrogen coincident with H in the solar spectrum.

In many of my own photographs this line and also *fine* lines coincident with most of the typical lines are seen, but I reserve for the present any further description of my experiments. The line H in such stars as Vega must be ascribed, chiefly at least, to hydrogen. To what extent in cooler stars this line may be due also to calcium we do not know.—March 10, 1880.]

[Messrs. DEWAR and LIVEING state that the calcium line K is more easily reversed than the calcium line at H (Proceedings Royal Society, February 20, 1879). This fact should be considered in connexion with the presence of a line of hydrogen at H in any explanation that may be attempted of the phenomena presented in the stars.—March 30, 1880.]

* Professors DEWAR and LIVEING have permitted me to witness some of their experiments in which analogous changes of relative intensity of K to H occur in the *emission* spectrum of calcium. They are of opinion that these variations and similar changes in the absorption spectrum, such as those shown in the stars, naturally follow from the known laws of emission and absorption. They state that the line of calcium K is more easily reversed than the line at the position of H.

to supplement my eye observations of 1864 of the gaseous nebulæ by photographs of the more refrangible parts of their spectra. Such photographs, taken together with those described in this paper, and combined with our knowledge of the visible spectra of these bodies, would help us probably to a better understanding of the typical changes in the order of time through which a star passes.

In the hope of throwing some light on these and other questions which suggest themselves, I have taken for comparison a number of terrestrial spectra under various physical conditions. As I am still pursuing this inquiry I prefer at present to reserve an account of this part of my work.

§ IX. *Vega* (*α Lyræ*).

The photographic exposure with sensitive gelatine plates was from 15 minutes to 30 minutes. Recently, with more sensitive plates, these times have been reduced. The photographs of this star show with great distinctness the twelve strong typical lines. There is a thin line at the position of K. In one photograph of this star I suspected the presence of a very delicate line between the lines H and K, but as I cannot be sure of its existence I have not inserted this line in the map. The line, if present, would be about λ 3945. A circumstance of great importance is the entire absence of any lines in the spectrum beyond ι, λ 3698. The spectrum, which then becomes continuous, is strong, and extends beyond S in the ultra-violet. In solar photographs taken with the same apparatus the lines of this region are well-defined for some distance beyond S, and therefore this abrupt cessation of lines cannot be referred to an instrumental cause. All the lines are broad, and winged at the edges. After H the lines become less intense, and also better defined in the order of refrangibility.

Lines.

	W.L.			W.L.
H	4340		δ	3767·5
h	4101		ε	3745·5
H_1	3968		ζ	3730
K	3933	very thin.	η	3717·5
α	3887·5		θ	3707·5
β	3834		ι	3699
γ	3795			

§ X. *Sirius*.

In the photographs of this star we have a spectrum very similar to that of α Lyræ. I am not able to detect any line at the position of K, but as the altitude of the star is low the definition in the photograph is not quite so good as that of Vega. It is probably due to this cause that I have not been able to be sure of any lines beyond δ.

I incline to believe that they would be detected probably in a more perfect spectrum. It may be, however, that ϵ, ζ, η, θ, ι are really absent in the spectrum of Sirius.

LINES.

	W.L.			W.L.
H	4340		α	3887·5
h	4101		β	3834
H_1	3968		γ	3795
K		probably absent.	δ	3767·5

[See Addendum.]

§ XI. η *Ursæ Majoris.*

The spectrum of this star is very similar to the typical spectrum of Vega. When the two spectra are seen together on the same plate it is at once perceived that the lines are rather less winged and broad than those of Vega. Eleven lines have been measured. The existence of a twelfth line ι is doubtful, and therefore I have not inserted it in the map. As to K, any suspicion of a line here is far too doubtful to justify its insertion. As the spectrum is beautifully defined in the photograph, I think there is a strong presumption that it is absent.

A strong continuous spectrum extends beyond S. In this star we may mark a first step in the direction of a spectrum containing fine lines in the photographic portion of the spectrum. Four fine lines are inserted in the map.

LINES.

H	4340		β	3820	very thin.
	4087·5	thin and faint.	γ	3795	
	4137·5	thin, but distinct.	δ	3767·5	
h	4101		ϵ	3745·5	
	4021	thin, distinct.	ζ	3730	
H_1	3968		η	3717·5	
K		probably absent.	θ	3707·5	
α	3887·5		ι		probably present.
β	3834				

§ XII. α *Virginis.*

In this spectrum we find ourselves advancing towards a condition in which the twelve lines are narrower and defined at the edges. At the same time a greater number of fine lines have appeared. I suspect a thin line at the position of K, and I have indicated this probability by a dotted line in the map. There is no doubt of line between H and K. In this spectrum I have not been able to measure lines beyond η, though the continuous spectrum is strong and extends to about S.

LINES.

H	4340		α	3887·5	
	4137·5	thin.	β	3834	
	4120	thin.		3816·1	thin
h	4101		γ	3795	
	4022·5	thin.	δ	3767·5	
	4004·5	thin.	ϵ	3745·5	
H_1	3968		ζ	3730	
		thin.	η	3717·5	
K	3933	probably present as a thin line.	θ		doubtful
	3920	thin.	ι		doubtful

§ XIII. α *Aquilæ*.

All the lines are narrower than in Vega, and are well-defined at the edges. In this spectrum we have numerous lines, besides the twelve strong lines, and the spectrum may be regarded as changed considerably in the direction of stars of the solar type. The line K is now strong, though still inferior to H. Six lines have been measured in the spectrum beyond ι, and possibly there may be lines still more refrangible. In addition to these, seventeen fine lines have been measured between the strong typical lines.

LINES.

H	4230		β	3816
	4172·5			3807·5
	4131	pair of lines.	γ	3795
	4120	pair of lines.	δ	3767·5
h	4101			3757·5
	4072		ϵ	3745·5
	4022·5		ζ	3730
	4000	pair.	η	3717·5
	3997	pair.	θ	3707·5
H_1	3968		ι	3698
K	3933			3690
	3915			3677·5
α	3887·5			3656
	3862·5			3654
	3854			3637·5
β	3834			

§ XIV. α *Cygni*.

If we consider only the breadth of K and the narrowness and defined character of the lines, this spectrum is much altered in the direction of the solar type. On the other hand, few lines beyond the typical ones are present. The photograph is not strong, and I have not been able to measure the two most refrangible of the typical lines θ and ι.

LINES.

H	4340		β	3834
h	4101		γ	3795
H_1	3968		δ	3767·5
K	3933	a strong line, nearly as strong as H_1.		3757·5
α	3887·5		ϵ	3745·5
	3862·5		ζ	3730
			η	3717·5

§ XV. *Arcturus.*

In this spectrum we have to do with a different order of suns, and have now entered upon the solar class of stars. The line K is very broad and winged, more so than H and is stronger than it is in the solar spectrum.

In the eye observations by Dr. MILLER and myself we said: "This is a red star, the spectrum of which somewhat resembles that of the sun. In this we have measured upwards of thirty lines and ascertained the existence of a double sodium line at D."* The triple line of magnesium coincident with b is so well seen in the spectrum of this star that I made use of these lines in my determination of the star's motion in the line of sight in 1871.

In the photographic spectrum a great many lines are seen in the part of the spectrum which is less refrangible than that included in my map, namely, from about b to G. The whole photographic spectrum is crowded with lines similar in general characters to those of the solar spectrum. Twenty-one of the stronger of these lines have been measured and are given in the map. Several of these agree in position with similar lines in the solar spectrum.

On the more refrangible side of h the appearance of a bright band is seen which suggests a bright line. After a careful examination of the two negatives which I have of this star, and of positives taken from them, I have come to the conclusion that this appearance is really due to the absence of the finer lines which probably crowd the other parts of the spectrum, though they are too fine and close to be seen separately in the photographs.

Beyond K we have a strong contrast presented in the character of the lines. Here the lines are much broader and more intense, and arranged more or less in triple and other forms of grouping with finer lines between.

The stronger lines only of the crowded spectrum have been measured and inserted in the map. There are lines corresponding to some of the lines of the Vega class.

The dissimilarity of this spectrum from the class of white stars is further seen in the circumstance that as far as the spectrum can be traced upon the plate it is crowded

* Phil. Trans. 1864, p. 428.

with lines, as is the case in the solar spectrum. The portion of the spectrum beyond H is unlike the solar spectrum in character, as will be at once apparent upon an inspection of the map.

LINES.

	W. L.	
H_1	4340	as in solar spectrum.
	4325	doubtless the group G
	4307·5	clearly multiple.
	4289	
	4271	stronger.
	4252·5	
	4237·5	
	4227·5	
	4214	
	4201	
	4195	
	4185	thin.
	4176	
	4170	
	4150	probable group.
	4141	
	4132·5	thin rather.
	4112	
	4099	
	4075	
	4064	
	4055	
	4045	
	4034	
	4040	
	3995	
	3980	
H_1	3968	
H_1	3933	
	3920	
	3905	
	3900	
	3887·5	
	3881	
	3870	
	3859	
	3856	
	3850	
	3838	
	3835	
	3832·5	

	W. L.
α	3822·5
	3815
	3814·5
	3810
	3805
	3798
γ	3795
	3789
	3775
	3762·5
	3755
ϵ	3745·5
	3732·5
ζ	3730
η	3717·5
θ	3707·5
	3702·5
	3690
	3682·5
	3672·5
	3662·5
	3657·5
	3641
	3637·5
	3625
	3610
	3602·5
	3592·5
	3585
	3575
	3560
	3551
	3515
	3507·5
	3504·5
	3487
	3482
	3475
	3467
	3457

§ XVI. *The Planets.*

Venus.—Several photographs of this planet have been taken, together with a broad daylight spectrum. In the most perfect of these photographs, the FRAUNHOFER lines can be distinctly seen from *b* to S in the ultra-violet, and any differences, even if very slight between the planetary spectra and the daylight spectrum, could be at once recognised. I cannot, however, discover any additional absorption lines, nor any modifications of the solar light. In our early eye observations, Dr. MILLER and myself failed to detect any change due to the atmosphere of this planet. The photograph shows even no strong general absorption of the blue and violet region.

Jupiter and Mars.—Similar photographs have been taken of these planets, but they fail to show any planetary modification of the solar light in the photographic region. In the visible region of the spectra of these planets, Dr. MILLER and myself observed lines due to the atmospheres of these planets.*

§ XVII. *The Moon.*

During the last two years a large number of photographs of the light from limited areas of the lunar surface have been taken under very different conditions of illumination, and also during partial eclipses of the moon.

Most of these photographs present great differences in the relative general intensity of the ultra-violet region, but I have not been able to detect any indications of selective absorption. I am inclined to think that the differences of intensity of the more refrangible part of the spectrum which I have mentioned are not greater than may be accounted for on the ground of differences of intensity of the reflected light as a whole, and cannot therefore be taken as an evidence of the existence of a lunar atmosphere.

THE MAP.

M. CORNU'S map of this region of the spectrum is placed at the top and bottom of the map. The portion from G to H is on the same scale, and for this part ÅNGSTRÖM'S map of the solar spectrum has been made use of. An attempt has been made to give, approximately, the relative intensity and character of the stellar lines. The lines have been carefully laid down, but for any purposes requiring great accuracy, use should be made of the tables of wave lengths.

* Phil. Trans., 1864, pp. 421, 423. For a discussion of the observations of other astronomers on the visible spectra of the planets, see VOGEL'S, 'Über die spectra der Planeten.'

ADDENDUM.

(Added March 10, 1880.)

Since this paper was sent to the Royal Society the following observations have been made :—

Sirius.—A photograph was taken January 2, 1880, which possesses better definition than those taken previously. In this photograph a fine line at the position of K is seen, of about the same intensity as the line in the spectrum of Vega. The typical lines are in a small degree broader and more diffused at the edges than is the case in the spectrum of Vega.

I cannot see with certainty more than ten of the typical lines. I am unable to say if the remaining lines θ and ι are really absent or very faint.

Rigel.—Photograph taken January 3, 1880. All the typical lines are seen. They are rather broader than in the spectrum of α Cygni, but not quite so broad as in α Virginis. In the arrangement I have adopted in the map, Rigel should be placed between these two stars. There is a thin defined line at the position of K. I have a suspicion of lines beyond the typical group, and also of a line between α and β at λ 3862·5, and a line between β and γ.

Betelgeux.—Photographs were obtained of this star February 17, 1877, but in a photograph taken February 17, 1880, the spectrum is better defined. It is difficult to obtain a photograph of this star. An exposure forty times greater than would have been necessary for a good spectrum of Sirius gave but a faint spectrum of limited extent of Betelgeux. The photographic impression is strongest about G. On the less refrangible side it can be traced to F ; on the other side it appears to end abruptly at H, but by careful illumination a faint trace of the spectrum can be traced to a short distance beyond H.

Of the visible spectrum of this star, Dr. MILLER and myself remark (Phil. Trans. 1864, p. 425, and Plate 9) : "The light of this star has a decided orange tinge. None of the stars we have examined exhibits a more complex or remarkable spectrum. Strong groups of lines are visible in the red, the green, and blue portions." The measures are given of about eighty lines. At that time we were not able to see the lines of hydrogen at the positions C and F.

Later (Proceedings of the Royal Society, 1872, p. 388) I remark on this point : "I was able with the more powerful instruments at my command to see a narrow defined line in the red apparently coincident with Hα, and a similar line at the position of Hβ. The line Hα falls on the less refrangible side of a small group of strong lines. Hβ occurs in the space between two groups of strong lines, where the lines are faint."

In the photograph there is a line apparently coincident with Hγ (near G) but it is

not strong. The spectrum about H is too faint for any certainty as to the characters of the lines H and K, which I believe are present. I give the wave lengths of some of the most conspicuous lines between G and H.

4340	4145
4319	4132
4298·5	4099
4252	4075
4226	4025
4171	

Aldebaran.—The light of this star is of a pale red. We described the visible spectrum with some minuteness (Phil. Trans. 1864, p. 424, Plate 11). This star requires a very long exposure. An exposure fifty times greater than would have been necessary for Sirius, gives a spectrum extending from about F to H, with a faint trace of the more refrangible portion. The part from F to H is strongly photographed and well defined. It is crowded with lines. About fifty of the stronger lines could be measured without much difficulty, but unfortunately, from clouds coming on, the spectrum of Capella, which was taken on the same plate, is too weak to give with accuracy a fiducial point from which to take the measures. The less refrangible part of the photographed portion of the spectrum (roughly from F to G) is brighter (darker in the negative) from fewer lines of absorption. In the other portion (from about G to H) the lines are more numerous, and exhibit a different character, being broader, more intense, and probably more diffused at the edges.

Capella.—The spectrum of this star was photographed by Dr. MILLER and myself in February, 1863. It is a white star, and exhibits a visible spectrum closely resembling that of our sun.

The photographs recently obtained exhibit a spectrum from F to beyond S, which so closely resembles the Solar spectrum that a photograph of this star would, at first sight, be taken for a solar one. This close general resemblance is even maintained on closer scrutiny. The lines G, H, and K are of about the same intensity and breadth as in the solar spectrum. Beyond H several of the more distinctive groupings of the solar lines are clearly seen in the spectrum of this star. I have not attempted to measure the lines in detail, for the task would be as great as the measuring of the corresponding parts of the solar spectrum.

The great interest of this star in connexion with the researches contained in this paper is that it appears to be a sun in the same stage as that in which our sun is.

Whether the order of change from the more simple typical spectrum in which these researches show that the stars may be arranged, also indicates some of the successive

stages of their life changes through which they pass is a point on which we know nothing certainly. On this hypothesis the stars which have been observed would have to be arranged approximately in the following order :—*

Sirius. Vega.
η Ursæ Majoris.
α Virginis.
α Aquilæ.
Rigel.
α Cygni.

Capella. The Sun.
Arcturus.
Aldebaran.
Betelgeux ?

* According to the reasoning of Mr. JOHNSTONE STONEY, the changes of the stars in time would be in the inverse order of the arrangement I have suggested in the text. (See Proc. Roy. Soc., vol. xvii., pp. 47–51.)

XIX. *On the Electromagnetic Theory of the Reflection and Refraction of Light.*

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Communicated by G. J. STONEY, *M.A., F.R.S., Secretary of the Queen's University of Ireland.*

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IN the second volume of his 'Electricity and Magnetism' Professor J. CLERK MAXWELL has proposed a very remarkable electromagnetic theory of light, and has worked out the results as far as the transmission of light through uniform crystalline and magnetic media are concerned, leaving the questions of reflection and refraction untouched. These, however, may be very conveniently studied from his point of view.

If we call W the electrostatic energy of the medium, it may be expressed in terms of the electromotive force and the electric displacement at each point as is done in Professor MAXWELL's 'Electricity and Magnetism,' vol. ii., part iv., ch. 9. I shall adopt his notation and call the electromotive force $\mathfrak{E}$ and its components P, Q, R, and the electric displacement $\mathfrak{D}$ and its components f, g, h. As several of the results of this paper admit of a very elegant expression in Quaternion notation I shall give the work and results in both Cartesian and Quaternion form, confining the German letters to the Quaternion notation. Between these quantities then we have the equation

$$W=-\tfrac{1}{2}\iiint S\mathfrak{E}\mathfrak{D}.dxdydz=\tfrac{1}{2}\iiint(Pf+Qg+Rh)dxdydz$$

Similarly the kinetic energy T may be expressed in terms of the magnetic induction, $\mathfrak{B}$, and the magnetic force, $\mathfrak{H}$, or their components a, b, c and α, β, γ by the equation

$$T=-\frac{1}{8\pi}\iiint .S\mathfrak{B}\mathfrak{H}.dxdydz=\frac{1}{8\pi}\iiint(a\alpha+b\beta+c\gamma)dxdydz$$

I shall at present assume this to be a complete expression for T and return to the case of magnetized media for separate treatment, as Professor MAXWELL has proposed additional terms in this case in order to account for their property of rotatory polarisation. I shall throughout assume the media to be isotropic as regards magnetic induction, for the contrary supposition would enormously complicate the question and be, besides, of doubtful physical applicability. For the present I shall not assume them to be electrostatically isotropic. Hence $\mathfrak{E}$ is a linear vector and self-conjugate function of $\mathfrak{D}$, and

consequently P, Q, R linear functions of f, g, h, so that we may write in Quaternion notation

$$\mathfrak{E}=\phi\mathfrak{D}$$

and if we call U the general symmetrical quadratic function of f, g, h we may assume

$$U=Pf+Qg+Rh$$

and consequently

$$W=-\tfrac{1}{2}\iiint S\mathfrak{D}\phi\mathfrak{D}.dxdydz=\tfrac{1}{2}\iiint U dxdydz$$

As the medium is magnetically isotropic we have

$$\mathfrak{B}=\mu\mathfrak{H} \text{ or } a=\mu\alpha,\ b=\mu\beta,\ c=\mu\gamma$$

where μ is the coefficient of magnetic inductive capacity, and consequently the electrokinetic energy may be written

$$T=-\frac{\mu}{8\pi}\iiint\mathfrak{H}^2.dxdydz=\frac{\mu}{8\pi}\iiint(\alpha^2+\beta^2+\gamma^2)dxdydz$$

Now I shall assume the mediums to be nonconductors, and although this limits to some extent the applicability of my results, and notably their relation to metallic reflection, yet it is a necessity, for otherwise the problem would be beyond my present powers of solution. With this assumption, and using NEWTON's notation of $\dot{x}$ for $\frac{dx}{dt}$, we have the following equations (see 'Elect. and Mag.,' vol. ii., § 619)

$$4\pi\dot{\mathfrak{D}}=V\nabla\mathfrak{H}$$

using ∇ for the operation

$$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}$$

or the same in terms of its components, namely,

$$4\pi\dot{f}=\frac{d\gamma}{dy}-\frac{d\beta}{dz}$$

$$4\pi\dot{g}=\frac{d\alpha}{dz}-\frac{d\gamma}{dx}$$

$$4\pi\dot{h}=\frac{d\beta}{dx}-\frac{d\alpha}{dy}$$

Assuming now a quantity $\mathfrak{R}$ with components ξ, η, ζ, such that

$$\mathfrak{R}=\int\mathfrak{H}dt$$

and consequently

$$\dot{\mathfrak{R}}=\mathfrak{H}$$

or in terms of the components

$$\dot{\xi}=\alpha,\ \dot{\eta}=\beta,\ \dot{\zeta}=\gamma$$

we may evidently write

$$4\pi\mathfrak{D}=V\nabla\mathfrak{R}$$

i.e.,

$$4\pi f=\frac{d\zeta}{dy}-\frac{d\eta}{dz},\ 4\pi g=\frac{d\xi}{dz}-\frac{d\zeta}{dx},\ 4\pi h=\frac{d\eta}{dx}-\frac{d\xi}{dy}$$

so that we have

$$W=-\frac{1}{.32\pi^2}\iiint S(V\nabla\mathfrak{R}.\phi V\nabla\mathfrak{R})dxdydz$$

$$T=-\frac{\mu}{8\pi}\iiint\dot{\mathfrak{R}}^2dxdydz=\frac{\mu}{8\pi}\iiint(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)dxdydz$$

LAGRANGE's equations of motion may often be very conveniently represented as the conditions that $\int(T-W)dt$ should be a minimum, or in other words that

$$\delta\int(T-W)dt=0$$

and this method, from its symmetry, is particularly applicable to the methods of Quaternions.

Proceeding on this method we obtain immediately the equation

$$0=-\int\left[\mu\iiint S\dot{\mathfrak{R}}\delta\dot{\mathfrak{R}}.dxdydz-\frac{1}{4\pi}\iiint S(V\nabla\delta\mathfrak{R}.\phi V\nabla\mathfrak{R})dxdydz\right]dt$$

or in Cartesian notation

$$0=\int\left[\frac{\mu}{4\pi}\iiint(\dot{\xi}\delta\dot{\xi}+\dot{\eta}\delta\dot{\eta}+\dot{\zeta}\delta\dot{\zeta})dxdydz-\tfrac{1}{2}\iiint\left(\frac{dU}{df}.\delta f+\frac{dU}{dg}.\delta g+\frac{dU}{dh}.\delta h\right)dxdydz\right]dt$$

Now we may evidently integrate the terms in $\delta\dot{\mathfrak{R}}$ and $\delta\dot{\xi}$, $\delta\dot{\eta}$, $\delta\dot{\zeta}$ with reference to the time, and the terms depending on the limits of the time must vanish separately, and we are not at present concerned with them, so that the equation reduces to

$$\iiiint\left[\mu S\ddot{\mathfrak{R}}\delta\mathfrak{R}+\frac{1}{4\pi}S(V\nabla\delta\mathfrak{R}.\phi V\nabla\mathfrak{R})\right]dxdydzdt=0$$

or

$$\iiiint\left[\frac{\mu}{4\pi}(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}.\delta\zeta)+\tfrac{1}{2}\left(\frac{dU}{df}.\delta f+\frac{dU}{dg}.\delta g+\frac{dU}{dh}\delta h\right)\right]dxdydzdt=0$$

I shall now proceed to integrate this by parts relatively to x, y, z, and in order to express the result conveniently I shall assume ds to be an element of the surface of the medium and $\mathfrak{N}$, with components l, m, n, to be a unit normal to this element of surface, when we evidently obtain

$$\int\left[\frac{1}{4\pi}\iint \mathrm{S}(\mathfrak{N}.\phi V\nabla\,\mathfrak{R}.\delta\mathfrak{R})ds+\iiint\left\{\mu\mathrm{S}\ddot{\mathfrak{R}}\delta\mathfrak{R}+\frac{1}{4\pi}\mathrm{S}(\mathrm{V}\nabla\,\phi V\nabla\,\mathfrak{R}.\delta\mathfrak{R})\right\}dxdydz\right]dt=0$$

or

$$\mu\iiint(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}\delta\zeta)dxdydz$$
$$+\tfrac{1}{2}\iiint\left[\left(\frac{d}{dy}\cdot\frac{d\mathrm{U}}{dh}-\frac{d}{dz}\cdot\frac{d\mathrm{U}}{dg}\right)\delta\xi+\left(\frac{d}{dz}\cdot\frac{d\mathrm{U}}{df}-\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dh}\right)\delta\eta+\left(\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dg}-\frac{d}{dy}\cdot\frac{d\mathrm{U}}{df}\right)\delta\zeta\right]dxdydz+dt=0$$
$$+\tfrac{1}{2}\iint\left[\left(n\frac{d\mathrm{U}}{dg}-m\frac{d\mathrm{U}}{dh}\right)\delta\xi+\left(l\frac{d\mathrm{U}}{dh}-n\frac{d\mathrm{U}}{df}\right)\delta\eta+\left(m\frac{d\mathrm{U}}{df}-l\frac{d\mathrm{U}}{dg}\right)\delta\zeta\right]ds$$

It is evident that the superficial and general integrals must vanish separately, and as $\delta\mathfrak{R}$ or $\delta\xi$, $\delta\eta$ and $\delta\zeta$, is arbitrary in the general integrals, we must evidently have

$$4\pi\mu\ddot{\mathfrak{R}}+V\nabla\,\phi V\nabla\,\mathfrak{R}=0$$

or

$$\mu\ddot{\xi}+\tfrac{1}{2}\left(\frac{d}{dy}\cdot\frac{d\mathrm{U}}{dh}-\frac{d}{dz}\cdot\frac{d\mathrm{U}}{dg}\right)=0$$

$$\mu\ddot{\eta}+\tfrac{1}{2}\left(\frac{d}{dz}\cdot\frac{d\mathrm{U}}{df}-\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dh}\right)=0$$

$$\mu\ddot{\zeta}+\tfrac{1}{2}\left(\frac{d}{dx}\cdot\frac{d\mathrm{U}}{dg}-\frac{d}{dy}\cdot\frac{d\mathrm{U}}{df}\right)=0$$

As it will not take long I will deduce the ordinary equations for the transmission of plane-waves from these before proceeding to discuss the superficial condition. In the first place, as our axes of coordinates are perfectly arbitrary I shall assume z to be normal to the plane of the wave, and consequently ξ, η, ζ to be functions of z and t only, so that $\nabla=k\frac{d}{dz}$ and $\frac{d}{dx}=\frac{d}{dy}=0$, and consequently $\ddot{\zeta}=0$ and $h=0$ and

$$V\nabla\,\mathfrak{R}=i\frac{d\eta}{dz}-j\frac{d\xi}{dz}$$

so that if

$$\phi\rho=\lambda\mathrm{S}i\rho+\mu\mathrm{S}j\rho+\nu\mathrm{S}k\rho$$

$$\phi V\nabla\,\mathfrak{R}=\mu.\frac{d\xi}{dz}-\lambda\frac{d\eta}{dz}$$

and consequently

$$4\pi\mu\ddot{\mathfrak{R}}=\mathrm{V}k\mu.\frac{d^2\xi}{dz^2}-\mathrm{V}k\lambda.\frac{d^2\eta}{dz^2}$$

Now if we had originally assumed

$$U=Af^2+Bg^2+Ch^2+2F.gh+2G.hf+2H.fg$$

we should have

$$\lambda=Ai+Hj+Gk,\ \mu=Hi+Bj+Fk$$

so that we obtain

$$4\pi\mu(i\ddot{\xi}+j\ddot{\eta}+k\ddot{\zeta})+i\left(B.\frac{d^2\xi}{dz^2}-H.\frac{d^2\eta}{dz^2}\right)+j\left(A\frac{d^2\eta}{dz^2}-H\frac{d^2\xi}{dz^2}\right)=0$$

so that if we assume i and j parallel to the axes of the section of $S\rho\phi\rho=1$ by the wave-plane we evidently have got $H=0$, and consequently $\ddot{\zeta}=0$ and

$$4\pi\mu\ddot{\xi}=B.\frac{d^2\xi}{dz^2},\ 4\pi\mu\ddot{\eta}=A\frac{d^2\eta}{dz^2}$$

and as A and B are inversely proportional to the squares of the axes of the section of $S\rho\phi\rho=1$ or $U=1$ by the wave-plane, we get M'CULLAGH's result that each component in these directions is propagated independently and with a velocity inversely proportional to the perpendicular axis of the section of $S\rho\phi\rho=1$ by the wave-plane.

This comes out at once from the Cartesian equations, for as $h=0$, U reduces to

$$U=Af+2H.fg+Bg^2$$

and by choosing x and y parallel to the axes of this section, we have $H=0$, so that

$$\frac{dU}{df}=2A.f \qquad \frac{dU}{dg}=2B.g$$

$$\text{and } 4\pi f=-\frac{d\eta}{dz} \text{ and } 4\pi g=\frac{d\xi}{dz}$$

and the result of putting these in is evidently the same as before.

Returning now to the superficial conditions, I shall follow M'CULLAGH, and assume that at each point of the surface of separation of two media the values of the elements of the integrals must be equal for the two media, and, indeed, I think it is pretty evident that if the original integrals are to express the whole state of affairs in the case of a motion propagated from one medium into another, the superficial integrals must vanish when the limits introduced in them are the functions corresponding to the two contiguous media.

Using the suffix $_0$ for one medium and $_1$ for the other, and assuming the normal to the surface as k, we get, as δR is evidently arbitrary and the same for both media at the surface of separation,

$$\therefore Vk\phi_0 V\nabla\mathfrak{R}=Vk\phi_1 V\nabla\mathfrak{R}$$

or the two equations

$$Si\phi_0 V\nabla\mathfrak{R} = Si\phi_1 V\nabla\mathfrak{R}$$
$$Sj\phi_0 V\nabla\mathfrak{R} = Sj\phi_0 V\nabla\mathfrak{R}$$

which are the same as can be got at once by putting $l=1$ $m=n=0$ in the Cartesian superficial equations, and taking $\delta\xi$ and $\delta\eta$ as arbitrary and independent, when we get

$$\frac{dU}{df^0}=\frac{dU}{df^1}, \quad \frac{dU}{dg^0}=\frac{dU}{dg^1}$$

If now we assume that the axis of x is the line of intersection of the plane of incidence and the surface, we may evidently assume $\mathfrak{R}_0$ to be the resultant of an incident and reflected ray, and $\mathfrak{R}_1$ to be the resultant of the two refracted rays, so that we may write

$$\mathfrak{R}_0 = j_0\eta_0 + j'_0\eta'_0 \quad \mathfrak{R}_1 = j_1\eta_1 + j'_1\eta'_1$$

when j_0, j'_0, j_1, j'_1 are respectively unitvectors parallel to the direction of magnetic displacement in the incident, reflected and each refracted ray. Similarly calling k_0, k'_0, k_1, k'_1 the unitvectors normal to these wave-planes, and z_0, z'_0, z_1, z'_1 the variable distance along these, evidently we may assume $\nabla_0 = k_0\frac{d}{dz_0}$, $\nabla'_0 = k'_0\frac{d}{dz'_0}$, &c., and substituting these we at once obtain

$$\frac{d\eta_0}{dz_0}.Si\phi_0 i_0 + \frac{d\eta'_0}{dz'_0}.Si\phi_0 i'_0 = \frac{d\eta_1}{dz_1}.Si\phi_1 i_1 + \frac{d\eta'_1}{dz'_1}.Si\phi_1 i_1$$
$$\frac{d\eta_0}{dz_0}.Sj\phi_0 i_0 + \frac{d\eta'_0}{dz'_0}.Sj\phi_0 i'_0 = \frac{d\eta_1}{dz_1}.Sj\phi_1 i_1 + \frac{d\eta'_1}{dz'_1}.Sj\phi_1 i'_1$$

In these I shall now assume $\phi_0=1$ as it is convenient to suppose this medium to be isotropic and the velocity of propagation in it unity, and i_0 and i'_0 are the directions in the wave-plane perpendicular to the magnetic displacement; so that if α_0, β_0, γ_0, α'_0, β'_0, γ'_0 be the direction angles of these lines referred to the superficial axes

$$Si\phi_0 i_0 = \cos\alpha_0, \quad Si\phi_0 i'_0 = \cos\alpha'_0, \quad Sj\phi_0 i_0 = \cos\beta_0, \quad Sj\phi_0 i'_0 = \cos\beta'_0,$$

and if s^{-1} be that axis of the section of $S\rho\phi\rho=1$ by one of the refracted wave-planes which is perpendicular to the direction of magnetic displacement, it is evidently the velocity of propagation of this wave in this medium, and we may write

$$i_1 = Us = s^{-1}.Ts. \therefore \phi i_1 = \phi s^{-1}.Ts = vTs$$

when v^{-1} is the perpendicular on the corresponding tangent plane, and consequently

$$Si\phi i_1 = TvTs.\cos\alpha_1$$

when α_1 is the angle between v and the axis of x, and evidently Ts represents the velocity of this wave, while Vvj is the direction of the ray. Hence our equations may be written

$$\cos\alpha_0\frac{d\eta_0}{dz_0}+\cos\alpha'_0\frac{d\eta'_0}{dz'_0}=\mathrm{T}v.\mathrm{T}s.\cos\alpha_1\frac{d\eta_1}{dz_1}+\mathrm{T}v'\mathrm{T}s'.\cos\alpha'_1.\frac{d\eta'_1}{dz'_1}$$

$$\cos\beta_0.\frac{d\eta_0}{dz_0}+\cos\beta'_0\frac{d\eta'_0}{dz'_0}=\mathrm{T}v.\mathrm{T}s.\cos\beta_1\frac{d\eta_1}{dz_1}+\mathrm{T}v'\mathrm{T}s'.\cos\beta'_1\frac{d\eta'_1}{dz'_1}$$

These may be readily deduced from the Cartesian equations as follows: U_0 being isotropic it can be written $=\mathrm{A}_0(f_0^2+g_0^2+h_0^2)$, and is evidently unaltered by transformation, and if $l:m:n$ and $l_1:m_1:n_1$ be the direction cosines of x and y to any arbitrary axis, we get

$$\mathrm{A}_0f_0=l\frac{d\mathrm{U}_1}{df}+m\frac{d\mathrm{U}_1}{dg}+n\frac{d\mathrm{U}_1}{dh}$$

$$\mathrm{A}_0g_0=l_1\frac{d\mathrm{U}_1}{df}+m_1\frac{d\mathrm{U}_1}{dg}+n_1\frac{d\mathrm{U}_1}{dh}$$

and now, as these are linear, we may suppose the superficial disturbance in one medium to be due to an incident and reflected wave, and in the other to two refracted rays, so that we can write these as

$$\mathrm{A}_0(f_0+f'_0)=\left(l\frac{d\mathrm{U}_1}{df}+m\frac{d\mathrm{U}_1}{dg}+n\frac{d\mathrm{U}_1}{dh}\right)+\left(l'\frac{d\mathrm{U}'_1}{df}+m'\frac{d\mathrm{U}'_1}{dg}+n'\frac{d\mathrm{U}'_1}{dh}\right)$$

$$\mathrm{A}_0(g_0+g'_0)=\left(l_1\frac{d\mathrm{U}_1}{df}+m_1\frac{d\mathrm{U}_1}{dg}+n_1\frac{d\mathrm{U}_1}{dh_1}\right)+\left(l'_1\frac{d\mathrm{U}'_1}{df_1}+m'_1\frac{d\mathrm{U}'_1}{dg_1}+n'_1\frac{d\mathrm{U}'_1}{dh}\right)$$

and as U_1 and U'_1 are supposed to be referred to arbitrary axes, we may suppose U_1 to be referred to such that η_1 is the only component—*i.e.*, to such that the direction of the magnetic force is the axis of y, that of z being normal to the wave-plane, and similarly η'_1 being the only component in U'_1 while the z axis in this case is normal to its wave-plane, and of course this η_1 will be a function of z_1 only and η'_1 of z'_1 only, these being the corresponding ordinates. We thus obtain

$$4\pi f_1=-\frac{d\eta_1}{dz_1}\quad 4\pi f'_1=-\frac{d\eta'_1}{dz'_1}$$

$$g_1=g'_1=h_1=h'_1=0$$

so that

$$\frac{d\mathrm{U}}{df_1}=\mathrm{A}_1f_1\quad\frac{d\mathrm{U}'}{df'_1}=\mathrm{A}'_1f'_1\quad\frac{d\mathrm{U}_1}{dg}=\mathrm{H}_1f_1\quad\frac{d\mathrm{U}'_1}{dg}=\mathrm{H}'_1f'_1\quad\frac{d\mathrm{U}_1}{dh}=\mathrm{G}_1f_1\quad\frac{d\mathrm{U}'_1}{df'_1}=\mathrm{G}'_1f'_1$$

and the equations become

$$A_0(f_0+f'_0)=(lA_1+mH_1+nG_1)\frac{d\eta'}{dz_1}+(l'A'_1+m'H'_1+n'G'_1)\frac{d\eta'_1}{dz'_1}$$

$$A_0(g_0+g'_0)=(l_1A_1+m_1H_1+n_1G_1)\frac{d\eta_1}{dz_1}+(l'_1A'_1+m'_1H'_1+n'_1G'_1)\frac{d\eta'_1}{dz'_1}$$

Now if we consider A_1, H_1, G_1 we see that they are proportional to the direction cosines of the perpendicular on the tangent plane to U where it is met by $g=0$, $h=0$ —*i.e.*, it is the direction conjugate to the electric displacement corresponding to η_1. Now if we call A_0s the velocity of propagation of this wave and A_0s' of the other, we see at once that s^{-1} is an axis of the section of U=a constant by one wave-plane, and s'^{-1} is an axis of the section by the other wave-plane, and if v^{-1} and v'^{-1} be the corresponding conjugate directions, and α_1 and α'_1, β_1 and β'_1 the angles between these latter and the superficial x and y axes, and if α_0, β_0, α'_0, β'_0 be the corresponding angles between η_0 and η'_0 and these same superficial axes, we get exactly the same equations as before, and write

$$\frac{d\eta_0}{dz_0}\,.\cos\alpha_0+\frac{d\eta'_0}{dz'_0}\,.\cos\alpha'_0=vs\,.\cos\alpha_1\frac{d\eta_1}{dz_1}+v's'\,.\cos\alpha'_1\frac{d\eta'_1}{dz'_1}$$

$$\frac{d\eta_0}{dz_0}\,.\cos\beta_0+\frac{d\eta'_0}{dz'_0}\,.\cos\beta'_0=vs\,.\cos\beta_1\frac{d\eta_1}{dz_1}+v's'\,.\cos\alpha'_1\frac{d\eta'_1}{dz'_1}$$

In order to reduce these further we assume

$$\eta_0=T_0\,.\cos\frac{2\pi}{\lambda}\,.\,(t-z_0)\qquad \eta'_0=T'_0\cos\frac{2\pi}{\lambda'_0}(t-z'_0)$$

$$\eta_1=T_1\cos\frac{2\pi}{\lambda_1}\,(st-z_1)\qquad \eta'_1=T'_1\cos\frac{2\pi}{\lambda'_1}(s't-z'_1)$$

and if our superficial axes are so placed that the axis of x is the intersection of the plane of incidence and the surface, and if i_0, i'_0, i_1, i'_1 be the angles the respective wave normals make with the normal to the surface, we evidently may write

$$z_0=z\cos i+x\sin i\qquad z'_0=z\cos i'+z\sin i'$$

$$z_1=z\cos i_1+x\sin i_1\qquad z'_1=z\cos i'_1+z\sin i'_1$$

as it is easy to convince oneself that any terms involving y would be inadmissible, as they could not by hypothesis occur in z_0, and the terms involving the time could not vanish out of our equations if they occurred in the others. Hence these wave normals are all in the same plane. When $z=0$ our equations must evidently be true independently of the time and x, from which we see that no change of phase is possible in

reflection. Hence these equations cannot explain metallic reflection. Indeed, this question of change of phase seems to be one of a higher order than I am here dealing with, and requires a discussion of the nature of the transition from one medium to another, which, of course, cannot be abrupt, as our equations suppose, nor indeed probably, in these cases, even very small compared with the vibrations. In order that these terms involving the time should disappear from the equations when $z=0$ we must have

$$\lambda_0 : \lambda'_0 : \lambda_1 : \lambda'_1 :: \sin i_0 : \sin i'_0 : \sin i_1 : \sin i'_1 :: 1 : 1 : s : s'$$

which involves that the angle of incidence should be equal to the angle of reflection; and if the second medium were an ordinary one, so that $s=\frac{1}{\rho}=s'$ we should have that the ratio of the sines of the angles of incidence and refraction was constant. Putting in these values, our equations reduce to

$$\mathrm{T}_0 \cos \alpha_0 + \mathrm{T}'_0 \cos \alpha'_0 = v\mathrm{T}_1 \cos \alpha_1 + v'\mathrm{T}'_1 \cos \alpha'_1$$
$$\mathrm{T}_0 \cos \beta_0 + \mathrm{T}'_0 \cos \beta'_0 = v\mathrm{T}_1 \cos \beta_1 + v'\mathrm{T}'_1 \cos \beta'_1$$

together with the condition that when $z=0$ the superficial displacements should be the same to whichever medium they belong, namely,

$$\xi=\xi',\ \eta=\eta',\ \zeta=\zeta'$$

As each of these is a resolved part of the vibrations η_0, η'_0 and η_1, η'_1 we get three additional equations, the last of which, however, is the same as the second of the former ones and there result, consequently, but four equations from which the four quantities, namely, the three intensities T'_0, T_1, T'_1, and the azimuth of T'_0 are to be determined. It is remarkable that whether we assumed or no that $\zeta=\zeta'$ it is here introduced. That is, however, no proof that it is wrong to omit it, as in FRESNEL'S method of obtaining the intensities of the reflected and refracted rays, for the fact of its turning up independently shows that there is something at least debateable about it, and as I shall have cause to omit this equation as leading to inconvenient results in a subsequent part of my paper, I thought it well to mention that there is something curious about it even here. These equations are those long ago given by M'CULLAGH in the 'Transactions of the Royal Irish Academy,' vol. xxi., to solve the problem of crystalline reflection and refraction, and from which he deduces his beautiful theorem of the polar plane and thus marvellously simplifies an extremely complicated problem. Indeed, as the forms into which I have thrown T and W are identical with his expressions for what are practically the same quantities, my whole investigation so far is but a modification of his. In the simple case of the second medium being also isotropic we have that $v=v'=s=s'$, and if we in the first place suppose η_0 to be in the plane of incidence, we have at once $\alpha_0=90°$, and consequently $\alpha'_0=\alpha'=90°$, and T'_1

vanishes, or at least may be supposed to do so, the medium being isotropic and $s=s'$, so that the two refracted waves coincide. Also $\beta_0=0$ and $\beta'_0=\beta_1=0$ likewise, as can easily be seen by assuming that the reflected and refracted waves have components out of the plane of incidence and then trying to satisfy the equations. Hence our equations reduce to

$$T_0+T'_0=\frac{1}{\rho^*}T_1$$

and $\xi_0=\xi'_0$ becomes

$$(T_0-T'_0)\sin i=T_1\cos r$$

i being the angle of incidence and r of reflection. The first of these is

$$(T_0+T'_0)\sin i=T_1\sin r$$

and solving them we get

$$T'_0=-T_0\frac{\sin(i-r)}{\sin(i+r)}\quad T_1=T_0\cdot\frac{\sin 2i}{\sin(i+r)}$$

If η_0 be perpendicular to the plane of incidence we obtain $\alpha_0=i$, $\beta_0=0$, and our equations become

$$T_0+T'_0=T_1$$

and

$$(T_0-T'_0)\sin i\cos i=T_1\sin r\cos r$$

which give

$$T_0=-T'_0\cdot\frac{\tan(i-r)}{\tan(i+r)}\quad T'_1=T_0\frac{\sin 2i}{\sin(i+r)\cos(i-r)}$$

Having now deduced the already known laws of reflection and refraction of light at crystalline and ordinary surfaces, I shall proceed to consider the case that Mr KERR'S wonderfully beautiful experiments have made so interesting—namely, the case of reflection from magnetic surfaces. In order to do this I shall assume, with Professor J. CLERK MAXWELL (see 'Electricity and Magnetism,' vol. ii., § 824), that the kinetic energy of the medium contains a term depending on the displacement of certain supposed vortices, and that it may be expressed by an equation of the form (§ 826)

$$T=-C\iiint S\left(\frac{d\mathfrak{R}}{d\theta}\cdot V\nabla\dot{\mathfrak{R}}\right)dxdydz=4\pi C\iiint\left(\frac{d\xi}{d\theta}\cdot\dot{f}+\frac{d\eta}{d\theta}\cdot\dot{g}+\frac{d\zeta}{d\theta}\dot{h}\right)dxdydz$$

where

$$\frac{d}{d\theta}=\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}$$

and α, β, γ are now the components of the vortex $\mathfrak{M}$.

I shall assume the medium to be isotropic, so that taking $\phi=\frac{4\pi}{K}$ the electrostatic energy of the medium may be expressed as

* When ρ is the refractive index.

$$W=-\frac{1}{8\pi K}\iiint(V\nabla\mathfrak{R})^2.dxdydz$$

$$=+\frac{1}{8\pi K}\iiint\left[\left(\frac{d\xi}{dy}-\frac{d\eta}{dz}\right)^2+\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)^2+\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)^2\right]dxdydz$$

While the complete value of T is

$$T=-\frac{1}{8\pi}\iiint\left[\mu\dot{\mathfrak{R}}^2+8\pi CS\left(\frac{d\mathfrak{R}}{d\theta}.V\nabla\dot{\mathfrak{R}}\right)\right]dxdydz$$

$$=\frac{1}{8\pi}\iiint\left[\mu(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)+8\pi C\left\{\frac{d\xi}{d\theta}\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\frac{d\eta}{d\theta}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\frac{d\zeta}{d\theta}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\right\}\right]dxdydz$$

Working as before, and obtaining our equations as the condition that

$$\int(T-W)dt$$

should be a minimum, we have to satisfy the equations

$$\iiiint\left[\mu S\dot{\mathfrak{R}}\delta\dot{\mathfrak{R}}+4\pi CS\left(\frac{d\delta\mathfrak{R}}{d\theta}.V\nabla\dot{\mathfrak{R}}\right)+4\pi CS\left(\frac{d\mathfrak{R}}{d\theta}.V\nabla\delta\dot{\mathfrak{R}}\right)-\frac{1}{K}S(V\nabla\mathfrak{R}.V\nabla\delta\mathfrak{R})\right]dxdydzdt=0$$

or in Cartesian coordinates

$$\iiiint\left[\begin{aligned}&\mu(\dot{\xi}\delta\dot{\xi}+\dot{\eta}\delta\dot{\eta}+\dot{\zeta}\delta\dot{\zeta})\\&+4\pi C\left\{\frac{d\delta\xi}{d\theta}\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\frac{d\delta\eta}{d\theta}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\frac{d\delta\zeta}{d\theta}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\right\}\\&+4\pi C\left\{\frac{d\xi}{d\theta}\left(\frac{d\delta\dot{\zeta}}{dy}-\frac{d\delta\dot{\eta}}{dz}\right)+\frac{d\eta}{d\theta}\left(\frac{d\delta\dot{\xi}}{dz}-\frac{d\delta\dot{\zeta}}{dx}\right)+\frac{d\zeta}{d\theta}\left(\frac{d\delta\dot{\eta}}{dx}-\frac{d\delta\dot{\xi}}{dy}\right)\right\}\\&-\frac{1}{K}\left\{\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dy}\right)\left(\frac{d\zeta}{dy}-\frac{d\eta}{dy}\right)+\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right.\\&\qquad\left.+\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\left(\frac{d\eta}{dx}-\frac{d\zeta}{dy}\right)\right\}\end{aligned}\right]dxdydzdt=0$$

Now of course the terms depending on C are the only additional ones, and it will be necessary to integrate them both relatively to the time and also relatively to θ. The integration with respect to the time is easily performed as it merely consists in removing the dot from one of the terms under the integral to the other and changing the sign, so that, neglecting the terms depending on the limits of the time with which we are not concerned, our equation becomes

$$\iiiint\left[\mu S(\ddot{\mathfrak{R}}\delta\mathfrak{R})-4\pi CS\frac{d\delta\mathfrak{R}}{d\theta}V\nabla\dot{\mathfrak{R}}+4\pi CS\left(\frac{d\dot{\mathfrak{R}}}{d\theta}V\nabla\delta\mathfrak{R}\right)+\frac{1}{K}S(V\nabla\mathfrak{R}.V\nabla\delta\mathfrak{R})\right]dxdydzdt=0$$

or in Cartesian coordinates

$$\iiiint \left\{ \begin{aligned} &\mu(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}\delta\zeta) \\ &-4\pi C\left\{\frac{d\delta\xi}{d\theta}\cdot\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\frac{d\delta\eta}{d\theta}\cdot\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\frac{d\delta\zeta}{d\theta}\cdot\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\right\} \\ &+4\pi C\left\{\frac{d\dot{\xi}}{d\theta}\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)+\frac{d\dot{\eta}}{d\theta}\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)+\frac{d\dot{\zeta}}{d\theta}\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\right\} \\ &+\frac{1}{K}\left\{\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right. \\ &\qquad\left.+\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{d\eta}\right)\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\} \end{aligned} \right\} dxdydzdt=0$$

Integrating these now by parts relatively to x, y, z, and calling $l : m : n$ the direction cosines of $\mathfrak{N}$ the normal to the surface of separation, and $\alpha l+\beta m+\gamma n=S\mathfrak{M}\mathfrak{N}=\epsilon$ when $\mathfrak{M}$ is the vortex whose components are α, β, γ, we get

$$\int\left[\iint\left\{S.\left[4\pi C\left\{V\left(\mathfrak{N}.\frac{d\dot{\mathfrak{R}}}{d\theta}\right)-\epsilon.V\nabla\mathfrak{R}\right\}+\frac{1}{K}V(\mathfrak{N}.V\nabla\mathfrak{R})\right]\delta\mathfrak{R}\right\}ds\right.$$
$$\left.+\iiint S.\left\{\left(\mu\ddot{\mathfrak{R}}+8\pi CV\nabla\frac{d\dot{\mathfrak{R}}}{d\theta}+\frac{1}{K}V\nabla V\nabla\mathfrak{R}\right)\delta\mathfrak{R}\right\}dxdydz\right]dt=0$$

or in Cartesian coordinates

$$\int\left\{\begin{aligned} &\iint\frac{1}{K}\left[\left\{n\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-m\left(\frac{d\eta}{dz}-\frac{d\xi}{dy}\right)\right\}\delta\xi+\left\{l\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)-n\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\right\}\delta\eta\right. \\ &\qquad\left.+\left\{m\left(\frac{d\xi}{dy}-\frac{d\eta}{dz}\right)-l\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right\}\delta\zeta\right]ds \\ &+4\pi C\iint\left\{\frac{d}{d\theta}(n\dot{\eta}-m\dot{\zeta}).\delta\xi+\frac{d}{d\theta}(l\dot{\zeta}-n\dot{\xi}).\delta\eta+\frac{d}{d\theta}(m\dot{\xi}-l\dot{\eta}).\delta\zeta\right\}ds \\ &-4\pi C\iint(l\alpha+m\beta+n\gamma)\left\{\left(\frac{d\zeta}{dy}-\frac{d\dot{\eta}}{dz}\right)\delta\xi+\left(\frac{d\dot{\xi}}{dz}-\frac{d\zeta}{dx}\right)\delta\eta+\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\delta\zeta\right\}ds \\ &+\iiint\mu(\ddot{\xi}\delta\xi+\ddot{\eta}\delta\eta+\ddot{\zeta}\delta\zeta)dxdydz \\ &+8\pi C\iiint\left\{\frac{d}{d\theta}\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right).\delta\xi+\frac{d}{d\theta}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)\delta\eta+\frac{d}{d\theta}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\delta\zeta\right\}dxdydz \\ &+\frac{1}{K}\iiint\left\{\left[\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)-\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right]\delta\xi+\left[\frac{d}{dz}\left(\frac{d\xi}{dy}-\frac{d\eta}{dz}\right)-\frac{d}{dx}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right]\delta\eta\right. \\ &\qquad\left.+\left[\frac{d}{dx}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\right]\delta\zeta\right\}dxdydz \end{aligned}\right\}dt=0$$

I shall not consider the general equations of wave propagation in the Cartesian form as they are the same as those given by MAXWELL in a more general form, for he assumes that the static energy of the medium is expressed more generally than I have assumed, but as the Quaternion investigation is not long I shall give it.

The equation of motion is

$$\mu\ddot{\mathfrak{R}}+8\pi \mathrm{C}\mathrm{V}\nabla\frac{d\mathfrak{R}}{d\theta}+\frac{1}{\mathrm{K}}\mathrm{V}\nabla\mathrm{V}\nabla\mathfrak{R}=0$$

and if z be the normal to a wave-plane, we may evidently assume $\mathrm{V}=k\frac{d}{dz}$, and as $\mathfrak{R}$ is in the wave-plane we have

$$\mathfrak{R}=ai\sin\frac{2\pi}{\lambda}(vt-z)+bj\cos\frac{2\pi}{\lambda}(vt-z)$$

and as $\frac{d}{d\theta}=\gamma\frac{d}{dz}$ our equation becomes

$$\mu\ddot{\mathfrak{R}}+8\pi \mathrm{C}\gamma.\mathrm{V}k\frac{d^2\mathfrak{R}}{dz^2dt}+\frac{1}{\mathrm{K}}.\mathrm{V}k\frac{d}{dz}\mathrm{V}k\frac{d\mathfrak{R}}{dz}=0$$

Substituting for $\mathfrak{R}$, and equating the coefficients of i and j separately to zero, we get

$$a\mu v^2-\frac{16\pi^2\mathrm{C}\gamma}{\lambda^2}.bv-\frac{a}{\mathrm{K}}=0,\quad b\mu v^2-\frac{16\pi^2\mathrm{C}\gamma}{\lambda}.av-\frac{b}{\mathrm{K}}=0$$

from which it is easy to see that $a=\pm b$ and that v is determined by the equation

$$\mu v^2\mp\frac{16\pi^2\mathrm{C}\gamma}{\lambda^2}.v-\frac{1}{\mathrm{K}}=0$$

which gives of course two different values of v, one for each circularly polarised ray, or disregarding the solutions for waves going in the negative direction, we have approximately

$$v_1=\frac{1}{\sqrt{\mu\mathrm{K}}}+\frac{8\pi^2\mathrm{C}\gamma}{\mu\lambda^2},\quad v_2=\frac{1}{\sqrt{\mu\mathrm{K}}}-\frac{8\pi^2\mathrm{C}\gamma}{\lambda^2\mu}$$

and of course C can easily be determined from these by observing the rotation produced, but there is nothing except experiment to prove that C may not be a function of λ, as we know K to be, to some extent at least; so that using these formulæ in order to obtain the laws of dispersion of rotatory polarisation seems to approach towards deducing the known from the unknown. All we can be sure of is that C is in general extremely minute.

Returning to the superficial equations, and using / as a sign of substitution, we get

$$/\frac{1}{\mathrm{K}}.\mathrm{V}(\mathfrak{M}.V\nabla\mathfrak{R})+4\pi\mathrm{C}\left(\mathrm{V}\mathfrak{M}\frac{d\dot{\mathfrak{R}}}{d\theta}-\epsilon\mathrm{V}\nabla\dot{\mathfrak{R}}\right)=0$$

As I intend only to work out the results in Cartesian coordinates, I shall confine myself to them, and for simplification assume the axes to be z normal to and x and y in the surface, and consequently $l=m=0$, $n=1$, and $\epsilon=\gamma$. C may also be supposed to vanish for one of the media for which K_1 is the dielectric inductive capacity. I shall also assume that $\delta\zeta=0$ as the vanishing of its coefficient leads to inconvenient results, and the assumption may be to some extent justified by considering that as these $\delta\xi$, $\delta\eta$, $\delta\zeta$ are superficial values, no virtual displacement out of the surface, as this would be, is admissible. From the other two, $\delta\xi$ and $\delta\eta$, we evidently get

$$\frac{1}{\mathrm{K}_1}\left(\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}\right)=\frac{1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+4\pi\mathrm{C}\frac{d\dot{\eta}}{d\theta}-4\pi\mathrm{C}\gamma\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right.$$

$$\frac{1}{\mathrm{K}_1}\left(\frac{d\zeta_1}{dy}-\frac{d\eta_1}{dz}\right)=\frac{1}{\mathrm{K}}.\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+4\pi\mathrm{C}\frac{d\dot{\xi}}{d\theta}+4\pi\mathrm{C}\gamma.\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right.$$

A remarkable point about these equations is that they admit of being integrated with regard to the time as far as their right-hand members are concerned, so that in addition to the values which satisfy them as they stand, and which are the only ones of much interest, there are other periodic values of ξ, η, ζ independently of ξ_1, η_1, ζ_1 which satisfy these superficial conditions, and which may consequently be looked upon as a sort of free vibration of the surface of the medium. But even if this could be propagated into the rest of the ether it is improbable that the resultant vibrations would be of such a period as to be visible, though some energy might be expended on them.

I shall now further assume that the axis of x is the intersection of the surface with the plane of incidence of a plane-wave, and that consequently none of my quantities are functions of y, which reduces these equations to

$$\frac{1}{\mathrm{K}_1}\left(\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}\right)=\frac{1}{\mathrm{K}}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+4\pi\mathrm{C}\left(\frac{d\dot{\eta}}{d\theta}+\gamma_1\frac{d\dot{\eta}}{dz}\right)$$

$$\frac{1}{\mathrm{K}_1}\frac{d\eta_1}{dz}=\frac{1}{\mathrm{K}}\frac{d\eta}{dz}-4\pi\mathrm{C}\left\{\frac{d\dot{\xi}}{d\theta}+\gamma\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)\right\}$$

In order still further to simplify the problem I shall now consider separately the cases in which the magnetisation of the medium is normal and that in which it is in the surface. In the first place, it is evident that if it were all in y we should have $\alpha=\gamma=0$, $\beta=\mathfrak{M}$, and the coefficient of C would vanish in both equations because

$\frac{d}{d\theta}=\mathfrak{M}\frac{d}{d\eta}$, and hence we get Mr. KERR's result ('Phil. Mag.,' March, 1878, p. 174) that when the plane of incidence is normal to the lines of magnetic force the magnetising of the mirror produces no change in the reflected light.

Assuming, then, first that the magnetisation is normal to the surface, we have $\alpha=\beta=0$, $\gamma=\mathfrak{M}$, and $\frac{d}{d\theta}=\mathfrak{M}\frac{d}{dz}$, so that calling $4\pi CK_1\mathfrak{M}=\nu$ our equations become

$$\left.\begin{aligned}\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}&=\frac{K_1}{K}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+2\nu\frac{d\dot{\eta}}{dz}.\\ \frac{d\eta_1}{dz}&=\frac{K_1}{K}\cdot\frac{dx}{dz}-\nu\left(2\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\end{aligned}\right\} \quad \text{(I)}$$

while if it be all in x—*i.e.*, in the intersection of the plane of incidence and the surface, $\alpha=\mathfrak{M}$, $\beta=\gamma=0$, and $\frac{d}{dh}=\mathfrak{M}\frac{d}{dx}$, so that our equations are

$$\left.\begin{aligned}\frac{d\xi_1}{dz}-\frac{d\zeta_1}{dx}&=\frac{K_1}{K}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\nu\frac{d\dot{\eta}}{dx}\\ \frac{d\eta_1}{dz}&=\frac{K_1}{K}\cdot\frac{d\eta}{dz}-\nu\frac{d\dot{\xi}}{dx}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

I shall now proceed to solve these two systems of equations each for the two cases of waves whose magnetic forces $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ are first in the plane of incidence, and secondly at right angles to it. From the forms of the equations it is evident that we cannot assume either the reflected or refracted rays to be similarly polarised, but it is easy to convince oneself that there can be no difference of phase introduced in this case any more than in the former one of ordinary reflection, and, as I then remarked, it is evidently a question of greater complication than to be capable of being deduced from the simple assumption that the alteration of the nature of the medium in going from one into another is abrupt. Until more is known of the nature and extent of this change I fear we must be content with theories which only partially represents the facts.

As before, I shall assume ξ_1, η_1, ζ_1 to be due to the incident and reflected rays, while ξ, η, ζ are due to the refracted ray. The incident ray may be taken to depend upon the angle

$$\phi_1=\frac{2\pi}{\lambda_1}(t-z\cos i-x\sin i)$$

calling the velocity in this medium unity and the reflected wave on

$$\phi'_1=\frac{2\pi}{\lambda_1}(t+z\cos i-x\sin i),$$

as it is easy to see that for these to vanish from the equations when $z=0$ we must have the sines of the angles of incidence and reflection equal, while the refracted wave must be taken to depend upon

$$\phi=\frac{2\pi}{\lambda_1}(st-z\cos r-x\sin r)$$

and here we must have, as before, $\frac{s}{\lambda}=\frac{1}{\lambda_1}=\frac{\sin i}{\lambda_1}=\frac{\sin r}{\lambda}$, and this involves, as before, the ordinary laws of refraction, as s is constant and in general

$$s=\sqrt{\frac{\mu_1 K_1}{\mu K}}\ \therefore \frac{\mu_1 K_1}{\mu K}=s^2=\frac{\sin^2 v}{\sin^2 i}$$

Taking then first the case of the magnetisation being all normal to the surface and the incident displacements ξ, η, ζ in the plane of incidence, we must evidently assume

$$\left.\begin{array}{ll}\xi_1=a_1\cos i\cos\phi_1-b_1\cos i\cos\phi'_1 & \xi=a\cos r\,.\cos\phi\\ \eta_1=c_1\sin\phi'_1 & \eta=c\sin\phi\\ \zeta_1=-a_1\sin i\cos\phi_1-b_1\sin i\,.\sin\phi'_1 & \zeta=-a_1\sin r\,.\cos\phi\end{array}\right\}\quad .\ .\quad \text{(A)}$$

in order to satisfy the equations

$$\frac{d\xi_1}{dz}-\frac{d\zeta}{dx}=\frac{K_1}{K}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+2\nu\frac{d\eta}{dz}$$

$$\frac{d\eta_1}{dz}=\frac{K_1}{K}\cdot\frac{d\eta}{dz}-\nu\left(2\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)$$

$$\xi_1=\xi\qquad \eta_1=\eta$$

As it leads to inconsistent results I, in accordance with FRESNEL, omit the third equation, $\zeta_1=\zeta$. It is not easy to justify this omission, I fear, and the result must do so, as well as the consideration that probably if we had a better insight into the nature of the change from one medium into another our equations would be so modified as not to present these anomalies.

From the last equation it is manifest that

$$c=c_1$$

and as ν is very small it is obvious that c will always be small, so that $\nu\eta$ may be omitted in the first equation. Putting in the values of ξ, η, ζ, &c., and remembering that when $z=0$, $\phi_1=\phi'_1=\phi$, these equations evidently reduce to

$$(a_1+b_1)\sin i=\frac{\mu}{\mu_1}a\sin r$$

$$c_1 . \sin i\cos i=-\frac{\mu}{\mu_1}c_1 . \sin r . \cos r-\frac{2\pi s\nu}{\lambda}a(1+\cos^2 r)\sin i$$

$$(a_1-b_1)\cos i=a\cos r$$

and from these we must solve for a, b, and c in terms of a_1. However, c_1, the component introduced at right angles to the original plane of vibration by reflection, is the only one of much interest, though the alteration produced in the values of the reflected component by assuming $\frac{\mu}{\mu_1}$ to differ from unity are noteworthy. Calling T the period of vibration of the wave we are considering, we may put $\frac{s}{\lambda}=\frac{1}{T}$, and if we call $\frac{\mu}{\mu_1}=\chi$, we get for c the equation

$$c_1=-\frac{4\pi\nu}{T}a_1\frac{(1+\cos^2 r)\sin i\sin 2i}{(\sin 2i+\chi\sin 2r)(\sin i\cos r+\chi\cos i\sin r)}$$

which if $\chi=1$ reduces to

$$c_1=-\frac{4\pi\nu}{T} . a_1 . \frac{(1+\cos^2 r)\sin^2 i\cos i}{\sin^2(i+r).\cos(i-r)}$$

In the second case, when the direction of the vibration of the incident ray is perpendicular to the plane of incidence, we must evidently assume

$$\left.\begin{array}{ll}\xi_1=-c_1\cos i\sin\phi'_1 & \xi=c\cos r\sin\phi \\ \eta_1=a_1\cos\phi_1+b_1\cos\phi'_1 & \eta=a\cos\phi \\ \zeta_1=-c_1\sin i . \sin\phi'_1 & \zeta=-c\sin r . \sin\phi\end{array}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

and, as before, it is evident that c is generally very small so that $\nu\xi$ and $\nu\zeta$ may be omitted, and our equations become

$$\frac{d\xi_1}{dz}-\frac{d\xi_1}{dx}=\frac{K_1}{K}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+2\nu\frac{d\eta}{dz}$$

$$\frac{d\eta_1}{dz}=\frac{K_1}{K}\cdot\frac{d\eta}{dz}$$

$$\xi_1=\xi \qquad \eta_1=\eta$$

and when the values above are introduced into them we obtain

$$-c_1\cos i=c\cos r$$

$$(a_1-b_1)\sin i\cos i=a\chi . \sin r\cos r$$

$$a_1+b_1 \qquad\qquad =a$$

$$-c_1=-\chi\cdot\frac{\sin r}{\sin i}c+\frac{4\pi\nu}{T}a\frac{\sin i\cos r}{\cos r}$$

Hence we get solving for c_1 in terms of a_1

$$c_1=-\frac{8\pi\nu}{T}a_1\frac{\sin 2i\sin^2 i\cos^2 r}{\sin r(\sin 2i+\chi\sin 2r)(\sin i\cos r+\chi\sin r\cos i)}$$

which when $\chi=1$ reduces to

$$c_1=-\frac{4\pi\nu}{T}a_1\frac{\sin 2i\,.\sin^2 i\cos^2 r}{\sin r\sin^2(i+r)\cos(i-r)}$$

Turning now to the case where the magnetisation is in the surface, I shall first suppose the incident vibration to be in the plane of incidence, when we evidently assume, as before, equations (A) for ξ_1, η_1, ζ_1, ξ, η, ζ, and these must satisfy the equations (II), and in addition

$$\xi_1=\xi \qquad \eta_1=\eta$$

As before, we evidently get $c=c_1$, and from the second equation of (II) see that c must be small, as ν is, and consequently we may omit $\nu\eta$ in the first equation, and so the first equation is the same as before, and we have

$$(a_1+b_1)\sin i=a\chi\,.\sin r$$
$$(a_1-b_1)\cos i=a\cos r$$
$$c_1\sin i\cos i=-c_1\chi\sin r\,.\cos r-\frac{2\pi\nu}{T}a\cos r\,.\sin^2 i$$

which when solved for c_1 in terms of a_1 gives

$$c_1=-\frac{4\pi\nu}{T}a_1\frac{\sin 2i\sin^2 i\cos r}{(\sin 2i+\chi\sin 2r)(\sin i\cos r+\chi\cos i\sin r)}$$

which when $\chi=1$ reduces to

$$c_1=-\frac{2\pi\nu\,.a_1}{T}\cdot\frac{\sin 2i\,.\sin^2 i\cos r}{\sin^2(i+r)\,.\cos(i-r)}$$

Finally, supposing the incident vibration to be perpendicular to the plane of incidence, we have ξ_1, η_1, ζ_1, ξ, η, ζ determined by the equation (B), which when substituted in equations (II), neglecting $\nu\xi$ as before on account of its smallness, give

$$c_1=-\frac{4\pi\nu}{T}a_1\frac{\sin 2i\sin^2 i\cos r}{(\sin 2i+\chi\sin 2r)(\sin i\cos r+\chi\cos i\sin r)}$$

which when $\chi=1$ reduces to

$$c_1=-\frac{2\pi\nu}{T}a_1\,.\frac{\sin 2i\,.\sin^2 i\,.\cos r}{\sin^2(i+r)\,.\cos(i-r)}$$

It is remarkable that in this case the two components introduced by reflection are the same, whether the vibration be in, or perpendicular to, the plane of incidence.

Comparing this method of obtaining the effects of reflection from a magnetised surface with that given by me in the 'Proceedings of the Royal Society for 1876,' No. 176, it is to be observed that my equations (I) and (II) are unaltered if the signs of ν and η are both reversed together, or if those of ν and ξ and ζ are all reversed, showing that a circularly polarised ray in one direction is reflected according to the same laws when the magnetisation is one way as the oppositely circularly polarised ray would be if the magnetisation were reversed, and hence my former method of dividing the incident plane polarised ray into two opposite circularly polarised ones, each of which was reflected according to its own laws, is justified.

In comparing these expressions with the results of Mr. KERR's admirable experiments, it is necessary to observe, as I mentioned before, that the introduction of a difference of phase between the reflected components is a question of a different order from that here discussed, and probably to some extent at least depends on the want of abruptness in the change from one medium to the other. For instance, my expressions give no change of plane of polarisation when light is reflected normally from the end of a magnet, but they would lead one to expect that the only effect was a slight elliptic polarisation, the major axis of the ellipse being in the same plane as the original plane of polarisation. Now Mr. KERR's experiments show that there is some rotation of this plane by reflection, and a supposition similar to one long ago proposed to explain the known elliptic polarisation of metallic reflection—namely, that the efficient reflecting surface has some depth—may easily be shown to lead to Mr. KERR's result. On this hypothesis the reflected ray is the resultant of the rays reflected from a small thickness at the surface of separation of the media; and in the case of normal reflection from the end of the pole of a magnet, each of these components would be slightly turned from its original plane of polarisation owing to having passed through a very small thickness of a very powerful rotatory polarising substance—namely, this superficial layer of the magnet—hence it is evident that their resultant would no longer be polarised in the same plane as the incident ray. I only give this as an instance of how this question of a difference of phase affects the results, and how the hypotheses that have been framed to explain it might be used to bring my results into complete accord with Mr. KERR's experiments. I hardly think it worth while going into this more fully, as it is treading so closely upon unknown ground—namely, the connexion between matter and ether—that our hypotheses are to a great extent merely conveniences.

Another question is the extent to which χ affects ordinary reflection from a powerfully magnetic substance like iron. I have never come across any experiment tending to show that the reflection from iron was at all peculiar. This may be owing to the electrostatic inductive capacity being a characteristic of the ether in the matter, while magnetic inductive capacity is a characteristic of the matter, and so only affects the

wave-propagation in a secondary degree—*i.e.*, only to the extent to which ether motions are transformed into currents in or motions of the matter. The same applies to the opacity of the medium, which is similarly due to an exchange of energy between the ether and the matter. The function Professor MAXWELL assumes to represent the effect of magnetisation on wave-propagation is an expression of the same hypothesis. In comparing my equations with Mr. KERR's results I shall consequently assume $\chi=1$.

Mr. KERR's most elaborate experiments have been made with the lines of magnetic force in the intersection of the surface of the medium and the plane of incidence,* and I shall confine my attention to this case. My equations give that the principal effect of reflection is to introduce a component perpendicular to the original plane of polarisation whose amplitude is represented by the equation

$$c=-\frac{2\pi\nu}{\mathrm{T}}\frac{\sin 2i\sin^2 i\cos r}{\sin^2(i+r)\,.\cos(i-r)}$$

when the amplitude of the incident vibration is unity.

The following table represents the values of the variable part of this for the several incidences mentioned by Mr. KERR in his paper. This table is calculated on the assumption that 75° is the polarising angle for iron.

$i=$	0	30°	45°	60°	65°	75°	80°	85°	90°
$\frac{\sin 2i\sin^2 i\cos r}{\sin^2(i+r)\,.\cos(i-r)}=$	0	·6272	·8646	1·0007	1·0058	·9000	·7554	·4904	0

This table shows that the intensity of this introduced component vanishes for the two limiting incidences 0° and 90°, and attains a maximum value of 1·0067 at about 63° 20′. On comparing this with Mr. KERR's results there is a most striking correspondence. Summing up the results of his paper on reflection from a surface which is magnetised so that the lines of force are in the surface and the plane of incidence ('Phil. Mag.,' March, 1878, § 23), he says: "When the vibration reflected from the unmagnetised mirror is either parallel or perpendicular to the plane of reflection, the effect of magnetisation is to introduce a new and very small component vibration in a direction perpendicular to the primitive vibration." This is what I have called c. He goes on to define its direction relatively to the Amperean currents which are supposed to produce the magnetic force; but as the sign of c depends on the sign of ν, and thus on that of C, which cannot be certainly determined for iron otherwise than by these very experiments, any confirmation founded upon it such as I mentioned in my former paper is to a certain extent illusory; but the same arguments as are there used would

* 'Phil. Mag.,' March, 1878.

lead to the same conclusion here—namely, that iron, if transparent, would behave like those ferromagnetic substances observed by VERDET, which rotate the plane of polarisation in the opposite direction to the Amperean currents, as is of course most probable. He further mentions, in § 24, that the phase of this component is always nearly the same as that of the component of the reflected ray polarised in the plane of incidence. This question, however, of difference of phase is one I am not at present prepared to explain, and, as I have mentioned already, must await a more comprehensive theory. Considering next the intensity, of course no very accurate measures were possible on account of the minuteness of the phenomenon ; but he several times remarks that all effects of magnetisation vanished at normal and grazing instances, while his maximum effects were always obtained at incidences of about 60° or 65°. The anomalies observed connected with the incidence 75° all belong to the question of the difference of phase between the components, which my investigation does not touch. On the whole then I think that my results, as far as they go, are in complete accordance with Mr. KERR's experiments.

This investigation is put forward as a confirmation of Professor MAXWELL's electromagnetic theory of light, in which, though there are some points requiring further investigation, nevertheless the foundation has certainly been laid of a very great addition to our knowledge, and if it induced us to emancipate our minds from the thraldom of a material ether might possibly lead to most important results in the theoretic explanation of nature.

XX. *On the Secular Changes in the Elements of the Orbit of a Satellite revolving about a Tidally distorted Planet.*

By G. H. DARWIN, *F.R.S.*

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TABLE OF CONTENTS.

Introduction.

THE following paper treats of the effects of frictional tides in a planet on the orbit of its satellite. It is the sequel to three previous papers on a similar subject.*

The investigation has proved to be one of unexpected complexity, and this must be my apology for the great length of the present paper. This was in part due to the fact that it was found impossible to consider adequately the changes in the orbit of the satellite, without a reconsideration of the parallel changes in the planet. Thus some of the ground covered in the previous paper on "Precession" had to be retraversed; but as the methods here employed are quite different from those used before, this repetition has not been without some advantage.

* "On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus," Phil. Trans., Part I., 1879.

"On the Precession of a Viscous Spheroid, and on the remote History of the Earth," Phil. Trans., Part II., 1879.

"On Problems connected with the Tides of a Viscous Spheroid," Phil. Trans., Part II., 1879.

These papers are hereafter referred to as "Tides," "Precession," and "Problems" respectively.

There is also a fourth paper, treating the subject from a different point of view, viz.: "The Determination of the Secular Effects of Tidal Friction by a Graphical Method," Proc. Roy. Soc., No. 197, 1879. And lastly a fifth paper of more recent date, "On the Analytical Expressions which give the History of a Fluid Planet of Small Viscosity, attended by a Single Satellite," Proc. Roy. Soc., No. 202, 1880.

It will probably conduce to the intelligibility of what follows, if an explanatory outline of the contents of the paper is placed before the reader. Such an outline must of course contain references to future procedure, and cannot therefore be made entirely intelligible, yet it appears to me that some sort of preliminary notions of the nature of the subject will be advantageous, because it is sometimes difficult for a reader to retain the thread of the argument amidst the mass of details of a long investigation, which is leading him in some unknown direction.

Part VIII. contains a general review of the subject in its application to the evolution of the planets of the solar system. This is probably the only part of the paper which will have any interest to the general reader.

The mathematical reader, who merely wishes to obtain a general idea of the results, is recommended to glance through the present introduction, and then to turn to Part VII., which contains a summary, with references to such parts of the paper as it was not desirable to reproduce. This summary does not contain any analysis, and deals more especially with the physical aspects of the problem, and with the question of the applicability of the investigation to the history of the earth and moon, but of course it must not be understood to contain references to every point which seems to be worthy of notice. I think also that a study of Part VII. will facilitate the comprehension of the analytical parts of the paper.

Part I. contains an explanation of the peculiarities of the method of the disturbing function as applied to the tidal problem. At the beginning there is a summary of the meaning to be attached to the principal symbols employed. The problem is divided into several heads, and the disturbing function is partially developed in such a way that it may be applicable either to finding the perturbations of the satellite, or of the planet itself.

In Part II. the satellite is supposed to move in a circular orbit, inclined to the fixed plane of reference. It here appears that the problem may be advantageously subdivided into the following cases: 1st, where the permanent oblateness of the planet is small, and where the satellite is directly perturbed by the action of a second large and distant satellite such as the sun; 2nd, where the planet and satellite are the only two bodies in existence; 3rd, where the permanent oblateness is considerable, and the action of the second satellite is not so important as in the first case. The first and second of these cases afford the subject for the rest of this part, and the laws are found which govern the secular changes in the inclination and mean distance of the satellite, and the obliquity and diurnal rotation of the planet.

Part III. is devoted to the third of the above cases. It was found necessary first to investigate the motion of a satellite revolving about a rigid oblate spheroidal planet, and perturbed by a second satellite. Here I had to introduce the conception of a pair of planes, to which the motions of the satellite and planet may be referred. The problem of the third case is then shown to resolve itself into a tracing of the secular changes in the positions of these two "proper" planes, under the influence of tidal

friction. After a long analytical investigation differential equations are found for the rate of these changes.

Part IV. contains the numerical integration of the differential equations of Parts II. and III., in application to the case of the earth, moon, and sun, the earth being supposed to be viscous.

Part V. contains the investigation of the secular changes of the eccentricity of the orbit of a satellite, together with the corresponding changes in the planet's mode of motion.

Part VI. contains a numerical integration of the equations of Part V. in the case of the earth and moon. The objects of Parts VII. and VIII. have been already explained.

In the abstract of this paper in the Proceedings of the Royal Society,* certain general considerations are adduced which throw light on the nature of the results here found. This general reasoning is not reproduced here, because it is incapable of leading us to definite results, and it was only used there as a substitute for analysis.

I.

THE THEORY OF THE DISTURBING FUNCTION.

§ 1. *Preliminary considerations.*

In the theory of disturbed elliptic motion the six elements of the orbit may be divided into two groups of three.

One set of three gives a description of the nature of the orbit which is being described at any epoch, and the second set is required to determine the position of the body at any instant of time. In a speculative inquiry like the present one, where we are only concerned with very small inequalities which would have no interest unless their effects could be cumulative from age to age, so that the orbit might become materially changed, it is obvious that the secular changes in the second set of elements need not be considered.

The three elements whose variations are not here found are the longitudes of the perigee, the node, and the epoch; but the subsequent investigation will afford the materials for finding their variations if it be desirable to do so.

The first set of elements whose secular changes are to be traced are, according to the ordinary system, the mean distance, the eccentricity, and the inclination of the orbit. We shall, however, substitute for the two former elements, viz.: mean distance and eccentricity, two other functions which define the orbit equally well; the first of these is a quantity proportional to the square root of the mean distance, and the second is the ellipticity of the orbit. The inclination will be retained as the third element.

The principal problem to be solved is as follows:—

* No. 200, 1879.

A planet is attended by one or more satellites which raise frictional tides (either bodily or oceanic) in their planet; it is required to find the secular changes in the orbits of the satellites due to tidal reaction.

This problem is however intimately related to a consideration of the parallel changes in the inclination of the planet's axis to a fixed plane, and in its diurnal rotation.

It will therefore be necessary to traverse again, to some extent, the ground covered by my previous paper "On the Precession of a Viscous Spheroid."

In the following investigation the tides are supposed to be a bodily deformation of the planet, but a slight modification of the analytical results would make the whole applicable to the case of oceanic tides on a rigid nucleus.* The analysis will be such that the results may be applied to any theory of tides, but particular application will be made to the case where the planet is a homogeneous viscous spheroid, and the present paper is thus a continuation of my previous ones on the tides and rotation of such a spheroid.

The general problem above stated may be conveniently divided into two :—

First, to find the secular changes in mean distance and inclination of the orbit of a satellite moving in a circular orbit about its planet.

Second, to find the secular change in mean distance, and eccentricity of the orbit of a satellite moving in an elliptic orbit, but always remaining in a fixed plane.

As stated in the introductory remarks, it will also be necessary to investigate the secular changes in the diurnal rotation and in the obliquity of the planet's equator to the plane of reference.

The tidally distorted planet will be spoken of as the earth, and the satellites as the moon and sun.

This not only affords a useful vocabulary, but permits an easy transition from questions of abstract dynamics to speculations concerning the remote history of the earth and moon.

§ 2. *Notation—Equation of variation of elements.*

The present section, and the two which follow it, are of general applicability to the whole investigation.

For reasons which will appear later it will be necessary to conceive the earth to have two satellites, which may conveniently be called Diana and the moon. The following are the definitions of the symbols employed.

The time is t, and the suffix 0 to any symbol indicates the value of the corresponding quantity initially, when $t=0$. The attraction of unit masses at unit distance is μ.

For the earth, let—

$M=$ mass in ordinary units; $a=$ mean radius; $w=$ density, or mass per unit volume, the earth being treated as homogeneous; $g=$ mean gravity; $\mathfrak{g}=\frac{2}{5}g/a$;

* Or, as to Part III., on a nucleus which is sufficiently plastic to adjust itself to a form of equilibrium.

C, A= the greatest and least moments of inertia of the earth; if we neglect the ellipticity they will be equal to $\frac{2}{5}Ma^2$; $n=$ angular velocity of diurnal rotation; $\psi=$ longitude of autumnal equinox measured along the ecliptic from a fixed point in the ecliptic—the ecliptic being here a name for a plane fixed in space; $i=$ obliquity of ecliptic; χ the angle between a point fixed on the equator and the autumnal equinox; ρ the radius vector of any point measured from the earth's centre.

For Diana, let—

$c=$ mean distance; $\xi=(c/c_0)^{\frac{1}{2}}$; $\Omega=$ mean motion; $e=$ eccentricity of orbit; $\eta=$ ellipticity of orbit; $\varpi=$ longitude of perigee; $j=$ inclination of orbit to ecliptic; $N=$ longitude of node; $\epsilon=$ longitude of epoch; $m=$ mass; $\nu=$ ratio of earth's mass to Diana's or M/m; $l=$ true longitude measured from the node; $\theta=$ true longitude measured from the autumnal equinox; $\tau=\frac{3}{2}\mu m/c^3$, so that $\tau=3\Omega^2/2(1+\nu)$, also $\tau=\tau_0/\xi^6$; r the radius vector measured from earth's centre.

Also $\lambda=\Omega/n$; $\mathfrak{m}$ the ratio of the earth's moment of momentum of rotation to that of the orbital motion of Diana (or the moon) and the earth round their common centre of inertia.

For the moon let all the same symbols apply when accents are added to them.

Where occasion arises to refer merely to the elements of a satellite in general, the unaccented symbols will be employed.

Let R be the disturbing function as ordinarily defined in works on physical astronomy.

Other symbols will be defined as the necessity for them arises.

Then the following are the well-known equations for the variation of the mean distance, eccentricity, inclination, and longitude of the node.

$$\frac{dc}{dt}=\frac{2\Omega c^2}{\mu(M+m)}\frac{dR}{d\epsilon} \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$\frac{de}{dt}=\frac{\Omega c}{\mu(M+m)}\left[\frac{1-e^2}{e}\frac{dR}{d\epsilon}-\frac{\sqrt{1-e^2}}{e}\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right)\right] \quad \ldots \ldots \quad (2)$$

$$-\frac{dj}{dt}=\frac{\Omega c}{\mu(M+m)}\frac{1}{\sqrt{1-e^2}}\left[\frac{1}{\sin j}\frac{dR}{dN}+\tan\tfrac{1}{2}j\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right)\right] \quad \ldots \quad (3)$$

$$\sin j\frac{dN}{dt}=\frac{\Omega c}{\mu(M+m)}\frac{1}{\sqrt{1-e^2}}\frac{dR}{dj} \quad \ldots \ldots \ldots \ldots \quad (4)$$

The last of these equations will only be required in Part III.

Now let $R=WC(M+m)/Mm$; then if we substitute this value for R in each of the equations (1–4), it is clear that the right hand side of each will involve a factor $\Omega cC/\mu Mm$.

Then let

$$k=\frac{\mathrm{C}}{\mu Mm}\Omega_0 c_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(For a homogeneous earth $\frac{\mathrm{C}}{\mu Mm}=\frac{2\nu}{5g}$, and $\Omega_0 c_0=\left[(ga^2)\frac{1+\nu}{\nu}\right]^{\frac{1}{3}}\Omega_0^{\frac{1}{3}}$. Thus if we put

$$s=\tfrac{2}{5}\left[\left(\frac{a\nu}{g}\right)^2(1+\nu)\right]^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$k=s\Omega_0^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$s^{\frac{3}{4}}$ is a time, being about $3^{\mathrm{hrs.}}\ 4\frac{1}{2}^{\mathrm{min.}}$ for the homogeneous earth. k is also a time, being about 57 minutes, with the present orbital angular velocity of the moon, and the earth being homogeneous).

Then since $\Omega=\Omega_0\xi^{-3}$, $c=c_0\xi^2$, therefore

$$\frac{\mathrm{C}}{\mu Mm}\Omega c=\frac{k}{\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Again, since $(c/c_0)^{\frac{1}{2}}=\xi$, therefore

$$\frac{1}{c}\frac{dc}{dt}=\frac{2}{\xi}\frac{d\xi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and since $\eta=1-\sqrt{1-\mathrm{e}^2}$, therefore

$$\frac{d\eta}{dt}=\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}\frac{d\mathrm{e}}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Then substituting for R in terms of W in the four equations (1–4), and using the transformations (8–10), we get,

$$\frac{d\xi}{dt}=k\frac{d\mathrm{W}}{d\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{d\eta}{dt}=-\frac{k}{\xi}\left(\eta\frac{d\mathrm{W}}{d\epsilon}+\frac{d\mathrm{W}}{d\varpi}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and if the orbit be circular, so that $\mathrm{e}=0$, $d\mathrm{W}/d\varpi=0$,

$$-\frac{dj}{dt}=\frac{k}{\xi}\left(\frac{1}{\sin j}\frac{d\mathrm{W}}{dN}+\tan\tfrac{1}{2}j\frac{d\mathrm{W}}{d\epsilon}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\sin j\frac{dN}{dt}=\frac{k}{\xi}\frac{d\mathrm{W}}{dj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

These are the equations of variation of elements which will be used below. The last two (13) and (14) will only be required in the case where the orbit is circular.

The function W only differs from the ordinary disturbing function by a constant factor, and so W will be referred to as the disturbing function.

I will now explain why it has been convenient to depart from ordinary usage, and will show how the same disturbing function W may be used for giving the perturbations of the rotation of the planet.

In the present problem all the perturbations, both of satellites and planet, arise from tides raised in the planet.

The only case treated will be where the tidal wave is expressible as a surface spherical harmonic of the second order.

Suppose then that $\rho=a+\sigma$ is the equation to the wave surface, superposed on the sphere of mean radius a.

Then the potential V of the wave σ, at an external point ρ, must be given by

$$V=\tfrac{4}{5}\pi\mu wa\left(\frac{a}{\rho}\right)^3\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Here w is the density of the matter forming the wave; in our case of a homogeneous earth, distorted by bodily tides, w is the mean density of the earth. (If we contemplate oceanic tides, the subsequent results for the disturbing function must be reduced by the factor $\frac{2}{11}$, this being the ratio of the density of water to the mean density of the earth.)

Now suppose the external point ρ to be at a satellite whose mass, radius vector, and mean distance are m, r, c. Then if we put $\tau=\frac{3}{2}\mu m/c^3$, and observe that $C=\frac{8}{15}\pi wa^5$, we have

$$V=\frac{C}{m}\tau\left(\frac{c}{r}\right)^3\frac{\sigma}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Where σ is the height of tide, where the wave surface is pierced by the satellite's radius vector.

But the ordinary disturbing function R for this satellite is this potential V augmented by the factor $(M+m)/M$, because the planet must be reduced to rest. Hence our disturbing function

$$W=\tau\left(\frac{c}{r}\right)^3\frac{\sigma}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where σ is the height of tide at the place where the wave surface is pierced by r.

Now let us turn to the case of the planet as perturbed by the attraction of the same satellite on the same wave surface. The whole force function of the action of the satellite on the planet is, by (16), clearly equal to

$$m\left[\frac{M}{r}+\frac{C}{m}\tau\left(\frac{c}{r}\right)^3\frac{\sigma}{a}\right]$$

The latter term of this expression will give the perturbing couples; it is equal to C W.

In the accompanying fig. 1 let X Y Z be axes fixed in space, and (adopting the phraseology for the case of the earth) let X Y be the ecliptic; let A B C be axes fixed in the planet; let χ be the angle A N or B C D; i the obliquity of the ecliptic; ψ the longitude of the autumnal equinox from the fixed point X in the ecliptic.

Fig. 1.

Now suppose W to be expressed in terms of χ, i, ψ.

Then the perturbing couples, which act on the planet, are

$$C\frac{dW}{di} \text{ about N, tending to increase } i.$$

$$C\frac{dW}{d\psi} \text{ about Z, tending to increase } \psi.$$

$$C\frac{dW}{d\chi} \text{ about C, tending to increase } \chi.$$

Now let $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ be the perturbing couples acting about A, B, C respectively. Then must

$$C\frac{dW}{d\psi}=-\mathfrak{L}\sin i\sin\chi-\mathfrak{M}\sin i\cos\chi+\mathfrak{N}\cos i$$

$$C\frac{dW}{di}=-\mathfrak{L}\cos\chi+\mathfrak{M}\sin\chi$$

$$C\frac{dW}{d\chi}=\mathfrak{N}$$

Whence

$$\frac{\mathfrak{L}}{C}=\frac{1}{\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\sin\chi-\frac{dW}{di}\cos\chi$$

$$\frac{\mathfrak{M}}{C}=\frac{1}{\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\cos\chi+\frac{dW}{di}\sin\chi$$

$$\frac{\mathfrak{N}}{C}=\frac{dW}{d\chi}$$

But if ω_1, ω_2, ω_3 be the component angular velocities of the planet about A, B, C respectively, and if we may neglect (C−A)/A compared with unity, the equations of motion may be written

$$\frac{d\omega_1}{dt}=\frac{\mathfrak{L}}{C}, \frac{d\omega_2}{dt}=\frac{\mathfrak{M}}{C}, \frac{d\omega_3}{dt}=\frac{\mathfrak{N}}{C}$$

as was shown in section (6) of my previous paper on "Precession."

Then since $\chi=nt$, we have by integration,

$$\omega_1=-\frac{1}{n\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\cos\chi-\frac{1}{n}\frac{dW}{di}\sin\chi$$

$$\omega_2=\frac{1}{n\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\sin\chi-\frac{1}{n}\frac{dW}{di}\cos\chi$$

Then substituting these values in the geometrical equations,

$$\frac{di}{dt}=-\omega_1\cos\chi+\omega_2\sin\chi$$

$$\sin i\frac{d\psi}{dt}=-\omega_1\sin\chi-\omega_2\cos\chi$$

We have finally,

$$\left.\begin{aligned} n\sin i\frac{di}{dt}&=\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\\ n\sin i\frac{d\psi}{dt}&=\frac{dW}{di}\\ \frac{dn}{dt}&=\frac{dW}{d\chi}\end{aligned}\right\} \qquad (18)$$

These are the equations which will be used for determining the perturbations of the planet's rotation.

We now see that the same disturbing function W will serve for finding both sets of perturbations.

It is clear that it is not necessary in the above investigation that σ should actually be a tide wave; it may just as well refer to the permanent oblateness of the planet. Thus the ordinary precession and nutations may be determined from these formulas.

§ 3. *To find spherical harmonic functions of Diana's coordinates with reference to axes fixed in the earth.*

Let A, B, C be rectangular axes fixed in the earth, C being the pole and AB the equator.

Fig. 2.

Let X, Y, Z be a second set of rectangular axes, XY being the plane of Diana's orbit.

Let M be the projection of Diana in her orbit.

Let $i_{\prime}=ZC$, the obliquity of the equator to the plane of Diana's orbit.

$\chi_{\prime}=AX=BCY$.

$l_{\prime}=MX$, Diana's longitude from the node X.

Let $\left.\begin{aligned} M_1 &= \cos MA \\ M_2 &= \cos MB \\ M_3 &= \cos MC \end{aligned}\right\}$ Diana's direction-cosines referred to A, B, C.

Then

$$\left.\begin{aligned} M_1 &= \cos l_{\prime} \cos \chi_{\prime} + \sin l_{\prime} \sin \chi_{\prime} \cos i_{\prime} \\ M_2 &= -\cos l_{\prime} \sin \chi_{\prime} + \sin l_{\prime} \cos \chi_{\prime} \cos i_{\prime} \\ M_3 &= \sin l_{\prime} \sin i_{\prime} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We may observe that M_2 is derivable from M_1 by writing $\chi_{\prime}+\frac{1}{2}\pi$ in place of $\chi_{\prime}$.

These expressions refer to the plane of Diana's orbit, but we must now refer to the ecliptic.

Fig. 3.

In fig. 3, let A be the autumnal equinox, B the ascending node of the orbit, C the intersection of the orbit with the equator, being the X of fig. 2, and let D be a point fixed in the equator, being the A of fig. 2.

Then if we refer to the sides and angles of the spherical triangle A B C by the letters a, b, c, A, B, C as is usual in works on spherical trigonometry, we have

$A=i$, the obliquity of the ecliptic.

$B=j$, the inclination of the orbit.

$\pi-C=i_{\prime}=ZC$ of fig. 2.

$c=N$, the longitude of the node measured from A, for at present we may suppose $\psi=0$, without loss of generality.

Then let $\chi=\text{DA}$, and we have

$$\chi-\text{b}=\text{DC}=\chi_{\prime}$$

Again, if M be Diana in her orbit, $\text{MB}=l$, and since $\text{MC}=l_{\prime}$, therefore

$$l+\text{a}=l_{\prime}$$

Whence

$$\begin{aligned}
\cos\chi_{\prime}&=\cos\chi\cos\text{b}+\sin\chi\sin\text{b}\\
\sin\chi_{\prime}&=\sin\chi\cos\text{b}-\cos\chi\sin\text{b}\\
\cos l_{\prime}&=\cos l\cos\text{a}-\sin l\sin\text{a}\\
\sin l_{\prime}&=\sin l\cos\text{a}+\cos l\sin\text{a}
\end{aligned}$$

Substituting these values in the first of (19) we have

$$\begin{aligned}
\text{M}_1=&\cos\chi\cos l(\cos\text{a}\cos\text{b}-\sin\text{a}\sin\text{b}\cos i_{\prime})+\sin\chi\cos l(\cos\text{a}\sin\text{b}+\sin\text{a}\cos\text{b}\cos i_{\prime})\\
&-\cos\chi\sin l(\sin\text{a}\cos\text{b}+\cos\text{a}\sin\text{b}\cos i_{\prime})-\sin\chi\sin l(\sin\text{a}\sin\text{b}-\cos\text{a}\cos\text{b}\cos i_{\prime})
\end{aligned}$$

Now $\cos i_{\prime}=-\cos\text{C}$, and

$$\begin{aligned}
&\cos\text{a}\cos\text{b}+\sin\text{a}\sin\text{b}\cos\text{C}=\cos\text{c}=\cos N\\
&\cos\text{a}\sin\text{b}-\sin\text{a}\cos\text{b}\cos\text{C}=\sin\text{a}[\cot\text{a}\sin\text{b}-\cos\text{b}\cos\text{C}]=\sin\text{a}\cot\text{A}\sin\text{C}\\
&\qquad=\cos i\sin N\\
&\sin\text{a}\cos\text{b}-\cos\text{a}\sin\text{b}\cos\text{C}=\sin\text{b}[\cot\text{b}\sin\text{a}-\cos\text{a}\cos\text{C}]=\sin\text{b}\cot\text{B}\sin\text{C}\\
&\qquad=\cos j\sin N\\
&\sin\text{a}\sin\text{b}+\cos\text{a}\cos\text{b}\cos\text{C}=\sin\text{a}\sin\text{b}+\cos\text{c}\cos\text{C}-\sin\text{a}\sin\text{b}\cos^2\text{C}\\
&\qquad=\sin\text{a}\sin\text{b}\sin^2\text{C}+\cos\text{c}(-\cos\text{A}\cos\text{B}+\sin\text{A}\sin\text{B}\cos\text{c})\\
&\qquad=\sin\text{A}\sin\text{B}\sin^2\text{c}+\sin\text{A}\sin\text{B}\cos^2\text{c}-\cos\text{A}\cos\text{B}\cos\text{c}\\
&\qquad=\sin i\sin j-\cos i\cos j\cos N
\end{aligned}$$

Then substituting in the expression for M_1,

$$\begin{aligned}
\text{M}_1=&\cos\chi\cos l\cos N+\sin\chi\cos l\sin N\cos i-\cos\chi\sin l\sin N\cos j\\
&-\sin\chi\sin l(\sin i\sin j-\cos i\cos j\cos N)
\end{aligned}$$

Let $P=\cos\frac{1}{2}i$, $Q=\sin\frac{1}{2}i$, $p=\cos\frac{1}{2}j$, $q=\sin\frac{1}{2}j$

Then

$$\begin{aligned}
\text{M}_1=&(P^2+Q^2)(p^2+q^2)\cos\chi\cos l\cos N+(P^2-Q^2)(p^2+q^2)\sin\chi\cos l\sin N\\
&-(P^2+Q^2)(p^2-q^2)\cos\chi\sin l\sin N+(P^2-Q^2)(p^2-q^2)\sin\chi\sin l\cos N\\
&-4PQpq\sin\chi\sin l\\
=&P^2p^2\cos(\chi-l-N)+P^2q^2\cos(\chi+l-N)+Q^2p^2\cos(\chi+l+N)\\
&+Q^2q^2\cos(\chi-l+N)+2PQpq[\cos(\chi+l)-\cos(\chi-l)]. \qquad (20)
\end{aligned}$$

Since M_2 is derivable from M_1 by writing $\chi_{,}+\frac{1}{2}\pi$ for $\chi_{,}$, therefore it is also derivable by writing $\chi+\frac{1}{2}\pi$ for χ. Hence $-M_2$ is the same as M_1, save that sines replace cosines

Again $M_3=\sin l_{,} \sin i_{,}=\sin l \cos \mathrm{a} \sin i_{,}+\cos l \sin \mathrm{a} \sin i_{,}$

But $\sin \mathrm{a} \sin i_{,}=\sin i \sin N=2PQ \sin N$

And $\cos \mathrm{a} \sin i_{,}=\sin i \cot \mathrm{a} \sin \mathrm{c}=\sin i\,(\cot \mathrm{A} \sin \mathrm{B}+\cos \mathrm{c} \cos \mathrm{B})$

$$=\cos i \sin j+\sin i \cos j \cos N$$
$$=2pq(P^2-Q^2)+2PQ(p^2-q^2) \cos N$$

Therefore

$$M_3=2PQ\left[p^2 \sin (l+N)-q^2 \sin (l-N)\right]+2pq(P^2-Q^2) \sin l \quad . \quad . \quad . \quad (21)$$

For the sake of future developments it will be more convenient to replace the sines and cosines in the expressions for the M's by exponentials, and for brevity the $\sqrt{-1}$ will be omitted in the indices.

Then

$2M_1=e^{\chi-l-N}[Pp-Qqe^{N}]^2+e^{\chi+l+N}[Qp+Pqe^{-N}]^2+$ the same with the signs of the indices of the exponentials changed,

$-2M_2\sqrt{-1}=$ the same with sign of second line changed,

$M_3\sqrt{-1}=e^{l+N}\,[Pp-Qqe^{-N}]\,[Qp+Pqe^{-N}]-$ same with signs of the indices of the exponentials changed.

Now let

$$\left.\begin{array}{l}\varpi=Pp-Qqe^{N}, \quad \kappa=Qp+Pqe^{N} \\ \underline{\varpi}=Pp-Qqe^{-N}, \quad \underline{\kappa}=Qp+Pqe^{-N}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From these definitions it appears that ϖ and κ are two imaginary functions, which oscillate between the real values $\cos \frac{1}{2}(i+j)$ and $\cos \frac{1}{2}(i-j)$, and $\sin \frac{1}{2}(i+j)$ and $\sin \frac{1}{2}(i-j)$ as the node of the orbit moves round.

Also let $\theta=l+N$, the true longitude of Diana measured from the autumnal equinox. Strictly speaking, when longitudes are measured from a fixed point in the ecliptic $\theta=l+N-\psi$, but in the present investigation nothing is lost by regarding ψ as zero; in § (12), and in Part III., we shall have to introduce ψ.

Then

$$\left.\begin{array}{rl}2M_1= & \varpi^2 e^{\chi-\theta}+\underline{\kappa}^2 e^{\chi+\theta}+\underline{\varpi}^2 e^{-\chi+\theta}+\kappa^2 e^{-\chi-\theta} \\ 2M_2\sqrt{-1}= & -\varpi^2 e^{\chi-\theta}-\underline{\kappa}^2 e^{\chi+\theta}+\underline{\varpi}^2 e^{-\chi+\theta}+\kappa^2 e^{-\chi-\theta} \\ M_3\sqrt{-1}= & \underline{\varpi}\underline{\kappa} e^{\theta}-\varpi\kappa e^{-\theta}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The object of the present investigation is to find the following spherical harmonic functions of the second degree of M_1, M_2, M_3, viz.:

$$M_1^2 - M_2^2,\ 2M_1M_2,\ 2M_2M_3,\ 2M_1M_3,\ \tfrac{1}{3} - M_3^2$$

Then by adding the squares of the first and second of (23), we have

$$\begin{aligned} 2(M_1^2 - M_2^2) = & \ \varpi^4 e^{2(\chi-\theta)} + 2\varpi^2\underline{\kappa}^2 e^{2\chi} + \underline{\kappa}^4 e^{2(\chi+\theta)} \\ & + \underline{\varpi}^4 e^{-2(\chi-\theta)} + 2\underline{\varpi}^2\kappa^2 e^{-2\chi} + \kappa^4 e^{-2(\chi+\theta)} \quad . \quad . \quad . \quad . \quad . \quad (24) \end{aligned}$$

From (20) we know that M_1 has the form $\Sigma A \cos(\chi + B)$, and $-M_2$ the form $\Sigma A \sin(\chi + B)$; therefore $(M_1 + M_2)2^{-\frac{1}{2}}$ has the form $\Sigma A \cos(\chi + \frac{1}{4}\pi + B)$, and $(M_1 - M_2)2^{-\frac{1}{2}}$ the form $\Sigma A \sin(\chi + \frac{1}{4}\pi + B)$. Hence if we write $\chi - \frac{1}{4}\pi$ for χ in $M_1^2 - M_2^2$, we obtain $-2M_1M_2$. Therefore from (24) we obtain

$$\begin{aligned} -4M_1M_2\sqrt{-1} = & \ \varpi^4 e^{2(\chi-\theta)} + 2\varpi^2\underline{\kappa}^2 e^{2\chi} + \underline{\kappa}^4 e^{2(\chi+\theta)} \\ & - \underline{\varpi}^4 e^{-2(\chi-\theta)} - 2\underline{\varpi}^2\kappa^2 e^{-2\chi} - \kappa^4 e^{-2(\chi+\theta)} \quad . \quad . \quad . \quad . \quad . \quad (25) \end{aligned}$$

The $\sqrt{-1}$ appears on the left hand side because $e^{\frac{\pi}{2}} = -(-1)^{-\frac{1}{2}}$, $e^{-\frac{\pi}{2}} = (-1)^{-\frac{1}{2}}$. It is also easy to show that,

$$\begin{aligned} 2M_2M_3 = & -\varpi^3\kappa e^{\chi-2\theta} + \varpi\underline{\kappa}(\varpi\underline{\varpi} - \kappa\underline{\kappa})e^{\chi} + \underline{\varpi}\underline{\kappa}^3 e^{\chi+2\theta} \\ & - \varpi^3\underline{\kappa} e^{-(\chi-2\theta)} + \underline{\varpi}\kappa(\underline{\varpi}\varpi - \underline{\kappa}\kappa)e^{-\chi} + \varpi\kappa^3 e^{-(\chi+2\theta)} \quad . \quad . \quad . \quad . \quad (26) \end{aligned}$$

$$\begin{aligned} 2M_1M_3\sqrt{-1} = & -\varpi^3\kappa e^{\chi-2\theta} + \varpi\kappa(\varpi\underline{\varpi} - \kappa\kappa)e^{\chi} + \underline{\varpi}\underline{\kappa}^3 e^{\chi+2\theta} \\ & + \underline{\varpi}^3\kappa e^{-(\chi-2\theta)} - \varpi\kappa(\underline{\varpi}\varpi - \kappa\kappa)e^{-\chi} - \varpi\kappa^3 e^{-(\chi+2\theta)} \quad . \quad . \quad . \quad . \quad (27) \end{aligned}$$

$$\tfrac{1}{3} - M_3^2 = \tfrac{1}{3} - 2\varpi\underline{\varpi}\kappa\underline{\kappa} + \underline{\varpi}^2\kappa^2 e^{2\theta} + \varpi^2\kappa^2 e^{-2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

It may be here noted that $\varpi\varpi + \kappa\kappa = 1$, so that

$$\tfrac{1}{3} - 2\varpi\underline{\varpi}\kappa\underline{\kappa} = \tfrac{1}{3}(\varpi^2\underline{\varpi}^2 - 4\varpi\underline{\varpi}\kappa\underline{\kappa} + \kappa^2\underline{\kappa}^2)$$

These five formulas (24) to (28) are clearly equivalent to the expansion of the harmonic functions as a series of sines and cosines of angles of the form $\alpha\chi + \beta l + \gamma N$. It remains to explain the uses to be made of these expressions.

§ 4. *The disturbing function.*

In the theory of the disturbing function the differentiation with respect to the elements of the orbit of the disturbed body is an artifice to avoid the determination of the three component disturbing forces, by means of differentiation with regard to

the radius vector, longitude and latitude. In the present problem we have to determine the perturbation of a satellite under the influence of the tides raised by itself and by another satellite. Where the tides are raised by the satellite itself, the elements of that satellite's orbit of course enter in the disturbing function in expressing the state of tidal distortion of the planet, but they also enter as expressing the position of the satellite. It is clear that, in effecting the differentiations above referred to, we must only regard the elements of the orbit as entering in the disturbing function in the latter sense. Hence it follows that even although there may be only one satellite, yet in the evaluation of the disturbing function we must suppose that there are two satellites, viz.: one a tide-raising satellite and another a disturbed satellite.

In this place, where the planet is called the earth, the tide-raising satellite may be conveniently called Diana, and the satellite whose motion is disturbed may be called the moon. After the formation of the differential equations Diana may be made identical with the moon or with the sun at will, or the analysis may be made applicable to a planet with any number of satellites.

As above stated, unaccented symbols will be taken to apply to Diana, and accented symbols to the moon.

The first step, then, is to find the tidal distortion due to Diana.

Let M be the projection of Diana on the celestial sphere concentric with the earth, and P the projection of any point in the earth.

Let $\rho\xi$, $\rho\eta$, $\rho\zeta$ be the rectangular coordinates of P and $r\mathrm{M}_1$, $r\mathrm{M}_2$, $r\mathrm{M}_3$ the rectangular coordinates of Diana referred to axes A, B, C fixed in the earth.

Then since ρ, r are radii vectores, ξ, η, ζ and M_1, M_2, M_3 are direction-cosines.

The tide-generating potential V (of the second degree of harmonics, which will be alone considered) at P is given by

$$\mathrm{V}=\tfrac{3}{2}\frac{\mu m}{r^3}\rho^2(\cos^2\mathrm{PM}-\tfrac{1}{3})$$

according to the usual theory.

Now

$$\cos\mathrm{PM}=\xi\mathrm{M}_1+\eta\mathrm{M}_2+\zeta\mathrm{M}_3$$

and

$$\cos^2\mathrm{PM}-\tfrac{1}{3}=2\xi\eta\mathrm{M}_1\mathrm{M}_2+2\frac{\xi^2-\eta^2}{2}\frac{\mathrm{M}_1{}^2-\mathrm{M}_2{}^2}{2}+2\eta\zeta\mathrm{M}_2\mathrm{M}_3+2\xi\zeta\mathrm{M}_1\mathrm{M}_3$$
$$+\tfrac{3}{2}\frac{\xi^2+\eta^2-2\zeta^2}{3}\frac{\mathrm{M}_1{}^2+\mathrm{M}_2{}^2-2\mathrm{M}_3{}^2}{3}$$

Also by previous definition, $\tau=\frac{3}{2}\mu m/c^3$; so that

$$\tfrac{3}{2}\frac{\mu m}{r^3}=\frac{\tau}{(1-\mathrm{e}^2)^3}\left[\frac{c(1-\mathrm{e}^2)}{r}\right]^3$$

Now let

$$\mathrm{X}=\left[\frac{c(1-\mathrm{e}^2)}{r}\right]^{\frac{3}{2}}\mathrm{M}_1,\ \ \mathrm{Y}=\left[\frac{c(1-\mathrm{e}^2)}{r}\right]^{\frac{3}{2}}\mathrm{M}_2,\ \ \mathrm{Z}=\left[\frac{c(1-\mathrm{e}^2)}{r}\right]^{\frac{3}{2}}\mathrm{M}_3\quad . \quad . \quad . \quad . \qquad (29)$$

Then clearly

$$V \div \frac{\tau}{(1-e^2)^3}\rho^2 = 2\xi\eta XY + 2\frac{\xi^2-\eta^2}{2}\frac{X^2-Y^2}{2} + 2\eta\zeta YZ + 2\xi\zeta XZ + \tfrac{3}{2}\frac{\xi^2+\eta^2-2\zeta^2}{3}\frac{X^2+Y^2-2Z^2}{3}$$

Now assume that the five functions $2XY$, X^2-Y^2, YZ, XZ, $X^2+Y^2-2Z^2$ are each expressed as a series of simple time-harmonics; it will appear below that this may always be done. We now have V expressed as the sum of five solid harmonics $\rho^2\xi\eta$, $\rho^2(\xi^2-\eta^2)$, &c., each multiplied by a simple time-harmonic. According to any tidal theory each such term must raise a tide expressible by a surface harmonic of the same type, and multiplied by a simple time-harmonic of the same speed; moreover, each such tide must have a height which is some fraction of the corresponding equilibrium tide of a perfectly fluid spheroid, but the simple time-harmonic will in general be altered in phase.

Now if $r=a+\sigma$ be the equation to the wave-surface, corresponding to a generating potential $V=[\tau/(1-e^2)^3]\,\rho^2\ 2\xi\eta XY$, then when the spheroid is *perfectly fluid*, $\sigma/a=[\tau/\mathfrak{g}(1-e^2)^3]\ 2\xi\eta XY$, where $\mathfrak{g}=\frac{2}{5}g/a$, according to the ordinary equilibrium theory of tides. (It will now be assumed that we are dealing with bodily tides of the spheroid; if the tides were oceanic a slight modification would have to be introduced.)

In a frictional fluid, the tide σ will be reduced in height and altered in phase.

Let $\mathfrak{XY}$ represent a function of the same form as XY, save that each simple time-harmonic term of XY is multiplied by some fraction expressive of reduction of height of tide, and that the argument of each such simple harmonic term is altered in phase; the constants so introduced will be functions of the constitution of the spheroid, and of the speed of the harmonic terms. Also extend the same notation to the other functions of X, Y, Z which occur in V.

Then it is clear that, if $r=a+\sigma$ be the equation to the complete wave surface corresponding to the potential V,

$$(1-e^2)^3\frac{\mathfrak{g}}{\tau}\frac{\sigma}{a} = 2\xi\eta\ \mathfrak{XY} + 2\frac{\xi^2-\eta^2}{2}\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2} + 2\eta\zeta\ \mathfrak{YZ} + 2\xi\zeta\ \mathfrak{XZ} + \tfrac{3}{2}\frac{\xi^2+\eta^2-2\zeta^2}{3}\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3} \quad . \quad (30)$$

This expression shows that σ is a surface harmonic of the second order.

Then by (17) we have for the disturbing function for the moon, due to Diana's tides,

$$W = \tau'\sigma$$

where σ is the height of tide, at the point where the moon's radius vector pierces the wave surface.

Hence in the expression (30) for σ, we must put

$$\xi=\mathrm{M}_1', \ \eta=\mathrm{M}_2', \ \zeta=\mathrm{M}_3'$$

Then by analogy with (29), let

$$\mathrm{X}'=\left[\frac{c'(1-\mathrm{e}'^2)}{r'}\right]^{\frac{3}{2}}\mathrm{M}_1', \ \mathrm{Y}'=\left[\frac{c'(1-\mathrm{e}'^2)}{r'}\right]^{\frac{3}{2}}\mathrm{M}_2', \ \mathrm{Z}'=\left[\frac{c'(1-\mathrm{e}'^2)}{r'}\right]^{\frac{3}{2}}\mathrm{M}_3'$$

and we have

$$\mathrm{W}=\frac{\tau\tau'}{\mathfrak{g}}\frac{1}{(1-\mathrm{e}^2)^3(1-\mathrm{e}'^2)^3}\Big[2\mathrm{X}'\mathrm{Y}'\,\mathfrak{X}\mathfrak{Y}+2\frac{\mathrm{X}'^2-\mathrm{Y}'^2}{2}\,\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}+2\mathrm{Y}'\mathrm{Z}'\,\mathfrak{Y}\mathfrak{Z}+2\mathrm{X}'\mathrm{Z}'\,\mathfrak{X}\mathfrak{Z}$$
$$+\tfrac{3}{2}\,\frac{\mathrm{X}'^2+\mathrm{Y}'^2-2\mathrm{Z}'^2}{3}\,\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3}\Big] \quad . \quad . \quad . \quad . \quad (31)$$

This is the required expression for the disturbing function on the moon, due to Diana's tides.

So far the investigation is general, but we now have to develop this function so as to make it applicable to the several problems to be considered.

II.

SECULAR CHANGES IN THE INCLINATION OF THE ORBIT OF A SATELLITE.

§ 5. *The perturbed satellite moves in a circular orbit inclined to a fixed plane.—Subdivision of the problem.*

In this case $\mathrm{e}=0$, $\mathrm{e}'=0$, $r=c$, $r'=c'$, so that the functions X, Y, Z and X′, Y′, Z′ are simply the direction cosines of Diana and the moon, referred to the axes A, B, C fixed in the earth. Hence $\mathrm{X}=\mathrm{M}_1$, $\mathrm{Y}=\mathrm{M}_2$, $\mathrm{Z}=\mathrm{M}_3$, and the five formulas (24–8) give the functions $\mathrm{X}^2-\mathrm{Y}^2$, $2\mathrm{XY}$, $2\mathrm{YZ}$, $2\mathrm{ZX}$, $\frac{1}{3}-Z^2$. In order to form the functions in gothic letters we must express these functions as simple time-harmonics.

The formulas (24) to (28) are equivalent to the expression of the five functions as a series of terms of the type A cos $(\alpha\chi+\beta\theta+\gamma N+\delta)$. Now χ is the angle between a point fixed on the equator and the autumnal equinox, and therefore (neglecting alterations in the diurnal rotation and the precessional motion) increases uniformly with the time, being equal to $nt+$a constant, which constant may be treated as zero by a proper choice of axes A, B, C.

θ is the true longitude measured from the autumnal equinox, and is equal to $\Omega t+\epsilon-\psi$, since the orbit is circular; also ψ may for the present be put equal to zero, without any loss of generality.

Then if in forming the expressions for the state of tidal distortion of the earth we neglect the motion of the node, the five functions are expressed as a series of simple time-harmonics of the type A cos $(\alpha nt+\beta\Omega t+\zeta)$.

The corresponding term in the corresponding gothic-letter function will be KA cos $(\alpha nt+\beta\Omega t+\zeta-\mathrm{k})$, where K is the fraction by which the tide is reduced and k is the alteration of phase.

It appears, from the inspection of the five formulas (24–8), that there are tides of seven speeds, viz.: $2(n-\Omega)$, $2n$, $2(n+\Omega)$, $n-2\Omega$, n, $n+2\Omega$, 2Ω.

The following schedule gives the symbols to be introduced for reduction of tide and alteration of phase or lag.

	Semi-diurnal.			Diurnal.			Fortnightly.
	Slow.	Sidereal.	Fast.	Slow.	Sidereal.	Fast.	
Speed	$2(n-\Omega)$,	$2n$,	$2(n+\Omega)$,	$n-2\Omega$,	n,	$n+2\Omega$,	2Ω
Fraction of equilibrium tide .	F_1	F	F_2	G_1	G	G_2	H
Retardation of phase or lag. .	$2\mathrm{f}_1$	2f	$2\mathrm{f}_2$	g_1	g	g_2	2h

The gothic-letter functions may now at once be written down from (24–8). Thus,

$$
\begin{aligned}
2(\mathfrak{X}^2-\mathfrak{Y}^2)=\mathrm{F}_1\varpi^4e^{2(\chi-\theta)-2\mathrm{f}_1}&+\mathrm{F}2\varpi^2\underline{\kappa}^2e^{2\chi-2\mathrm{f}}+\mathrm{F}_2\kappa^4e^{2(\chi+\theta)-2\mathrm{f}_2}\\
&+\mathrm{F}_1\underline{\varpi}^4e^{-2(\chi-\theta)+2\mathrm{f}_1}+\mathrm{F}2\varpi^2\kappa^2e^{-2\chi+2\mathrm{f}}+\mathrm{F}_2\kappa^4e^{-2(\chi+\theta)+2\mathrm{f}_2}\quad .\ .\ .\quad (32)
\end{aligned}
$$

$$
-4\mathfrak{X}\mathfrak{Y}\sqrt{-1}=\text{the same, with second line of opposite sign}\ .\ .\ .\ .\quad (33)
$$

$$
\begin{aligned}
2\mathfrak{Y}\mathfrak{Z}=-\mathrm{G}_1\varpi^3\kappa e^{\chi-2\theta-\mathrm{g}_1}&+\mathrm{G}\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})e^{\chi-\mathrm{g}}+\mathrm{G}_2\varpi\underline{\kappa}^3e^{\chi+2\theta-\mathrm{g}_2}\\
-\mathrm{G}_1\underline{\varpi}^3\underline{\kappa}e^{-(\chi-2\theta)+\mathrm{g}_1}&+\mathrm{G}\varpi\kappa(\varpi\varpi-\underline{\kappa}\kappa)e^{-\chi+\mathrm{g}}+\mathrm{G}_2\varpi\kappa^3e^{-(\chi+2\theta)+\mathrm{g}_2}\ .\ .\ .\quad (34)
\end{aligned}
$$

$$
2\mathfrak{X}\mathfrak{Z}\sqrt{-1}=\text{the same, with second line of opposite sign}\ .\ .\ .\ .\ .\ .\quad (35)
$$

$$
\tfrac{1}{3}-\mathfrak{Z}^2=\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\kappa+\mathrm{H}\underline{\varpi}^2\underline{\kappa}^2e^{2\theta-2\mathrm{h}}+\mathrm{H}\varpi^2\kappa^2e^{-2\theta+2\mathrm{h}}\ .\ .\ .\ .\ .\quad (36)
$$

The fact that there is no factor of the same kind as H in the first pair of (36) results from the assumption that the tides due to the motion of the nodes of the orbit are the equilibrium tides unaltered in phase.

The formulas for $2(\mathrm{X}'^2-\mathrm{Y}'^2)$, $-4\mathrm{X}'\mathrm{Y}'\sqrt{-1}$, $2\mathrm{Y}'\mathrm{Z}'$, $2\mathrm{X}'\mathrm{Z}'\sqrt{-1}$, $\frac{1}{3}-\mathrm{Z}'^2$ are found by symmetry, by merely accenting all the symbols in the five formulas (24–8) for the M functions. In the use made of these formulas this accentuation will be deemed to be done.

At present we shall not regard χ as being accented, but in § 12 and in Part III. we shall have to regard χ as also accented.

We now have to develop the several products of the X′ functions multiplied by the $\mathfrak{X}$ functions.

Before making these multiplications, it must be considered what are the terms which are required for finding secular changes in the elements, since all others are superfluous for the problem in hand.

Such terms are clearly those in which θ and θ' are wanting, and also those where $\theta-\theta'$ occurs, for these will be wanting in θ when Diana is made identical with the moon. It follows therefore that we need only multiply together terms of the like speeds. In the following developments all superfluous terms are omitted.

Semi-diurnal terms.

These are $2X'Y'\mathfrak{X}\mathfrak{Y}+2\dfrac{X'^2-Y'^2}{2}\cdot\dfrac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}$.

If we multiply (24) (with accented symbols) by (32), and (25) (with accented symbols) by (33), and subtract the latter from the former, we see that χ disappears from the expression, and that,

$$8X'Y'\mathfrak{X}\mathfrak{Y}+2(X'^2-Y'^2)(\mathfrak{X}^2-\mathfrak{Y}^2)=\text{First line of (24)}\times\text{second of (32)}$$
$$+\text{Second of (25)}\times\text{first of (33)}$$

Then as far as we are concerned

$$2X'Y'\mathfrak{X}\mathfrak{Y}+2\frac{X'^2-Y'^2}{2}\cdot\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}$$
$$=\tfrac{1}{4}\left[F_1\varpi^4\varpi'^4e^{2(\theta-\theta)-2f_1}+4F\varpi^2\kappa^2\varpi'^2\kappa'^2e^{-2f}+F_2\kappa^4\kappa'^4e^{-2(\theta-\theta)-2f_2}\right]$$
$$+\tfrac{1}{4}\left[F_1\varpi^4\varpi'^4e^{-2(\theta'-\theta)+2f_1}+4F\varpi^2\kappa^2\varpi'^2\kappa'^2e^{2f}+F_2\kappa^4\kappa'^4e^{2(\theta'-\theta)+2f_2}\right]\quad.\quad(37)$$

If χ had been accented in the X' functions, we should have had $2(\chi-\chi')$ in all the indices of exponentials of the first line, and $-2(\chi-\chi')$ in all the indices of the second line. These three pairs of terms will be called W_{I}, W_{II}, W_{III}.

Diurnal terms.

These are $2Y'Z'\mathfrak{Y}\mathfrak{Z}+2X'Z'\mathfrak{X}\mathfrak{Z}$.

If the multiplications be performed as in the previous case, it will be found that χ disappears in the sum of the two products, and, as far as concerns terms in $\theta'-\theta$ and those independent of θ and θ', we have

$$2Y'Z'\mathfrak{Y}\mathfrak{Z}+2X'Z'\mathfrak{X}\mathfrak{Z}$$
$$=G_1\varpi^3\kappa\varpi'^3\kappa'e^{2(\theta-\theta)-g_1}+G\varpi\kappa(\varpi\varpi-\kappa\kappa)\varpi'\kappa'(\varpi'\varpi'-\kappa'\kappa')e^{-g}+G_2\varpi\kappa^3\varpi'\kappa'^3e^{-2(\theta'-\theta)-g_2}$$
$$+G_1\varpi^3\kappa\varpi'^3\kappa'e^{-2(\theta'-\theta)+g_1}+G\varpi\kappa(\varpi\varpi-\kappa\kappa)\varpi'\kappa'(\varpi'\varpi'-\kappa'\kappa')e^{g}+G_2\varpi\kappa^3\varpi'\kappa'^3e^{2(\theta-\theta)+g_2}\quad.\quad(38)$$

If χ had been accented in the X' functions we should have had $\chi-\chi'$ in all the

indices of the exponentials of the first line, and $-(\chi-\chi')$ in all the indices of the second line. These three pairs of terms will be called W_1, W_2, W_3.

Fortnightly term.

This is $\frac{3}{2}(\frac{1}{3}-Z'^2)(\frac{1}{3}-\mathfrak{Z}^2)$.

Multiplying (36) by (28) when the symbols are accented, and only retaining desired terms,

$$\frac{3}{2}(\tfrac{1}{3}-Z'^2)(\tfrac{1}{3}-\mathfrak{Z}^2)=\tfrac{3}{2}(\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa})(\tfrac{1}{3}-2\varpi'\underline{\varpi}'\kappa'\underline{\kappa}')+\tfrac{3}{2}\mathrm{H}\underline{\varpi}^2\underline{\kappa}^2\varpi'^2\kappa'^2e^{-2(\theta'-\theta)-2\mathrm{h}}$$
$$+\tfrac{3}{2}\mathrm{H}\varpi^2\kappa^2\underline{\varpi}'^2\underline{\kappa}'^2e^{2(\theta'-\theta)+2\mathrm{h}} \quad . \quad (39)$$

Even if χ had been accented in the X′ functions, neither χ or χ' would have entered in this expression. These terms will be called W_0.

Then the sum of the three expressions (37), (38), and (39), when multiplied by $\tau\tau'/\mathfrak{g}$, is equal to W, the disturbing function.

If Diana be a different body from the moon the terms in $\theta'-\theta$ are periodic, and the only part of W, from which secular changes in the moon's mean distance and inclination can arise, are the sidereal semi-diurnal and diurnal terms, viz.: those in F and G, and also the term independent of H in (39). These terms being independent of θ' are independent of ϵ', the moon's epoch. Hence it follows that, as far as concerns the influence of Diana's tides upon the moon, $d\mathrm{W}/d\epsilon'$ is zero, and we conclude that—*the tides raised by any one satellite can produce directly no secular change in the mean distance of any other satellite.**

But Diana being still distinct from the moon, the F-, G-, and part of the fortnightly term, which are independent of θ, do involve N and N'; for W contains terms of the forms $e^{\pm\alpha N}$, $e^{\pm\alpha N'}$, $e^{\pm(\alpha N+\beta N')}$, also it has terms independent of N, N'. Hence $d\mathrm{W}/dN'$ will contain terms of the form $e^{\pm\alpha N'}$, $e^{\pm(\alpha N+\beta N')}$, or their equivalent sines or cosines.

Now by hypothesis there are two disturbing bodies, and we know by lunar theory that the direct influence of Diana on the moon is such as to tend to make the nodes of the moon's orbit revolve on the ecliptic; on the other hand, there is a direct influence of the permanent oblateness of the earth on the nodes of the moon's orbit.

If the oblateness of the earth be large, the result of the joint influence of these two causes may be such as either to make the nodes of the moon's orbit rotate with a very unequal angular velocity, or perform oscillations (possibly large ones) about a mean position. If this be the case the mean value of $d\mathrm{W}/dN'$ may differ considerably from zero. This case is considered in detail in Part III. of this paper.

If on the other hand the oblateness be small the nodes of the orbit revolve with a

* If there be a rigorous relationship between the mean motions of a pair of satellites this may not be true. This appears to be (at least very nearly) the case between two pairs of satellites of the planet Saturn.

sensibly uniform angular velocity on the ecliptic. This is the case at present with the earth and moon. Here then dW/dN', as far as concerns the influence of Diana's tides on the moon, is sensibly periodic according to simple harmonic functions of the time. From this we conclude that:—

If the nodes of the satellites' orbits revolve uniformly on the plane of reference, then the tides raised by any one satellite can produce no secular change in the inclination of the orbit of any other satellite.

There are thus two cases in which the problem is simplified by our being permitted to consider only the case of identity between Diana and the moon:

1st. Where there are two or more satellites, but where the nodes of the perturbed satellite's orbit revolve with sensible uniformity on the plane of reference.

2nd. Where the planet and satellite are the only bodies in existence.

In these two cases, after differentiation of the disturbing function with respect to the accented elements, we shall be able to drop the accents.

There is also a third case in which Diana's tides *will* produce a secular effect on the inclination of the moon's orbit, and this is where the nodes of the moon's orbit either revolve irregularly or oscillate. This case is enormously more complicated than the others, and forms the subject of Part III. of this paper; I have only attempted to solve it on the supposition of the smallness both of the inclination of the orbit, and of the obliquity of the ecliptic.

The first of these three cases is that which actually represents the moon and earth, together with solar perturbation of the moon at the present time.

In tracing the configuration of the lunar orbit backwards from the present state, we shall start with the first case; this will graduate into the third, and from this it will pass to a state represented to a very close degree of approximation by the second.

We are not at present concerned to know what are the conditions under which there may be approximate uniformity in the motion of the nodes; this will be investigated below.

We will begin with the first of the three cases, and will find also the rate of change of the diurnal rotation and of the obliquity of the planet.

The second case will then be taken, and afterwards the third case will have to be discussed almost *ab initio* in Part III.

6. *Secular change of inclination of the orbit of a satellite, where there is a second disturbing body, and where the nodes revolve with sensible uniformity on the fixed plane of reference.*

By (13) the equation giving the change of inclination is

$$-\frac{\xi'}{k'}\frac{dj'}{dt}=\frac{1}{\sin j'}\frac{dW}{dN'}+\tan\tfrac{1}{2}j'\frac{dW}{d\epsilon'}$$

As shown above, however, we need here only deal with a single satellite, so that Diana and the moon may be considered as identical and the accents may be dropped to all the symbols, except in the differential coefficients of W. Also we need only maintain the distinction between Diana and the moon as regards N, N' and ϵ, ϵ'; and after the differentiations of W these distinctions must also be dropped. Hence ϖ only differs from ϖ', κ from κ', $\underline{\varpi}$ from $\underline{\varpi}'$, and $\underline{\kappa}$ from $\underline{\kappa}'$ in the accentuation of N.

Also since $\theta=\Omega t+\epsilon$, $\theta'=\Omega' t+\epsilon'$, therefore we may replace $\theta'-\theta$ in the three expressions (37–9) by $\epsilon'-\epsilon$.

If we put $\sin j=2pq$, $\tan\frac{1}{2}j=q/p$, and write $\phi(N,\epsilon)$ for the operation $\frac{1}{2pq}\frac{d}{dN'}+\frac{q}{p}\frac{d}{d\epsilon'}$, putting $N=N'$, $\epsilon=\epsilon'$ after differentiation; then from (13) we have

$$-\frac{\xi}{k}\frac{dj}{dt}=\phi(N,\epsilon)\mathrm{W}$$

Also for brevity, let $\phi(N)=\frac{1}{2pq}\frac{d}{dN'}$, $\phi(\epsilon)=\frac{q}{p}\frac{d}{d\epsilon'}$; so that $\phi(N,\epsilon)=\phi(N)+\phi(\epsilon)$.

The terms corresponding to the tides of the seven speeds will now be taken separately, the coefficients in ϖ, κ will be developed, and the terms involving $N'-N$ selected, the operation $\phi(N,\epsilon)$ performed, and then N' put equal to N, and ϵ to ϵ'. For the sake of brevity the coefficient $\tau^2/\mathfrak{g}$ will be dropped and will be added in the final result. The component parts of W taken from the equations (37–9) will be indicated as W_{I}, W_{II}, $\mathrm{W}_{\mathrm{III}}$ for the slow, sidereal, and fast semi-diurnal parts; as W_1, W_2, W_3 for the slow, sidereal, and fast diurnal parts; and as W_0 for the fortnightly part.

Slow semi-diurnal terms $(2n-2\Omega)$.

$$\mathrm{W}_{\mathrm{I}}=\tfrac{1}{4}\mathrm{F}_1[\varpi^4\underline{\varpi}'^4e^{2(\epsilon'-\epsilon)-2f_1}+\underline{\varpi}^4\varpi'^4e^{-2(\epsilon'-\epsilon)+2f_1}] \quad \ldots \ldots \quad (40)$$

Let

$$\mathrm{w}_{\mathrm{I}}=\tfrac{1}{4}\varpi^4\underline{\varpi}'^4e^{2(\epsilon'-\epsilon)-2f_1}$$

Since

$$\varpi=Pp-qQe^{N}$$

Therefore

$$\varpi^4=P^4p^4-4P^3Qp^3qe^N+6P^2Q^2p^2q^2e^{2N}-4PQ^3pq^3e^{3N}+Q^4q^4e^{4N}$$

$$\underline{\varpi}'^4=\text{ the same with } -N' \text{ in place of } N$$

Therefore

$$\mathrm{w}_{\mathrm{I}}=\tfrac{1}{4}\{P^8p^8+16P^6Q^2p^6q^2e^{N-N'}+36P^4Q^4p^4q^4e^{2(N-N')}+16P^2Q^6p^2q^6e^{3(N-N')}$$
$$+Q^8q^8e^{4(N-N')}\}e^{2(\epsilon'-\epsilon)-2f_1}$$

Therefore

$$\mathrm{w}_{\mathrm{I}}=\Sigma\mathrm{A}_nP^{8-2n}Q^{2n}p^{8-2n}q^{2n}e^{n(N-N')+2(\epsilon'-\epsilon)-2f_1}$$

where $n=0, 1, 2, 3, 4$.

Then

$$\phi(N)\mathrm{w}_{\mathrm{I}}=\Sigma\frac{n}{2\sqrt{-1}}\mathrm{A}_n P^{8-2n}Q^{2n}p^{7-2n}q^{2n-1}e^{-2\mathrm{f}_1}$$

$$\phi(\epsilon)\mathrm{w}_{\mathrm{I}}=-\Sigma\frac{4}{2\sqrt{-1}}\mathrm{A}_n P^{8-2n}Q^{2n}p^{7-2n}q^{2n+1}e^{-2\mathrm{f}_1}$$

Therefore by addition

$$\phi(N,\epsilon)\mathrm{w}_{\mathrm{I}}=\Sigma[n(p^2+q^2)-4q^2]P^{8-2n}Q^{2n}p^{7-2n}q^{2n-1}\frac{e^{-2\mathrm{f}_1}}{2\sqrt{-1}}$$

Now when

$$\begin{array}{lll} n=0, & \mathrm{A}_n=\tfrac{1}{4}, & n(p^2+q^2)-4q^2=-4q^2 \\ =1, & \phantom{\mathrm{A}_n}=4, & =p^2-3q^2 \\ =2, & \phantom{\mathrm{A}_n}=9, & =2(p^2-q^2) \\ =3, & \phantom{\mathrm{A}_n}=4, & =3p^2-q^2 \\ =4, & \phantom{\mathrm{A}_n}=\tfrac{1}{4}, & =4p^2 \end{array}$$

If we had taken the second term of W_{I} we should have had the same coefficients but multiplied by $-e^{2\mathrm{f}_1}/2\sqrt{-1}$ instead of by $e^{-2\mathrm{f}_1}/2\sqrt{-1}$. Therefore, since $(e^{2\mathrm{f}_1}-e^{-2\mathrm{f}_1})/2\sqrt{-1}=\sin 2\mathrm{f}_1$

$$\phi(N,\epsilon)\mathrm{W}_{\mathrm{I}}=-\mathrm{F}_1\sin 2\mathrm{f}_1[-P^8p^7q+4P^6Q^2p^5q(p^2-3q^2)+18P^4Q^4p^3q^3(p^2-q^2) \\ +4P^2Q^6pq^5(3p^2-q^2)+Q^8pq^7]$$

Then let

$$\mathfrak{F}_1=\tfrac{1}{4}[P^8p^6-4P^6Q^2p^4(p^2-3q^2)-18P^4Q^4p^2q^2(p^2-q^2)-4P^2Q^6q^4(3p^2-q^2)-Q^8q^6] \quad (41)$$

and remembering that $2pq=\sin j$, we have

$$\phi(N,\epsilon)\mathrm{W}_{\mathrm{I}}=2\mathfrak{F}_1\mathrm{F}_1\sin 2\mathrm{f}_1\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Sidereal semi-diurnal terms (2n).

$$\mathrm{W}_{\mathrm{II}}=\mathrm{F}[\underline{\varpi}^2\underline{\kappa}^2\varpi'^2\kappa'^2e^{-2\mathrm{f}}+\underline{\varpi}^2\kappa^2\varpi'^2\underline{\kappa}'^2e^{2\mathrm{f}}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Here the epoch is wanting, so that $\phi(N,\epsilon)=\phi(N)$.

Let

$$\mathrm{w}_{\text{II}}=(\varpi\underline{\kappa}\underline{\varpi}'\kappa')^2$$

$$\varpi=Pp-Qqe^N,\ \underline{\kappa}=Qp+Pqe^{-N}$$

$$\varpi\underline{\kappa}=PQ(p^2-q^2)+pq(P^2e^{-N}-Q^2e^N)$$

$$\underline{\varpi}'\kappa'=PQ(p^2-q^2)+pq(P^2e^{N'}-Q^2e^{-N'})$$

$$\sqrt{\mathrm{w}_{\text{II}}}=P^2Q^2(p^2-q^2)^2+PQpq(p^2-q^2)[P^2(e^{N'}+e^{-N})-Q^2(e^{-N'}+e^N)]$$
$$+p^2q^2[P^4e^{-(N-N')}+Q^4e^{(N-N')}-P^2Q^2(e^{N+N'}+e^{-(N+N')})]$$

$$\mathrm{w}_{\text{II}}=P^4Q^4[(p^2-q^2)^4-4p^2q^2(p^2-q^2)^2+4p^4q^4]+4P^6Q^2p^2q^2(p^2-q^2)^2e^{-(N-N')}$$
$$+4P^2Q^6p^2q^2(p^2-q^2)^2e^{N-N'}+p^4q^4[P^8e^{-2(N-N')}+Q^8e^{2(N-N')}]$$

$$\phi(N)\mathrm{w}_{\text{II}}=-\frac{1}{2\sqrt{-1}}[4pq(p^2-q^2)^2P^2Q^2(P^4-Q^4)+2p^3q^3(P^8-Q^8)]$$

If we had operated on the other term of W_{II} we should have got the same with the opposite sign, and $e^{2\mathrm{f}}$ in place of $e^{-2\mathrm{f}}$.

Then let

$$\mathfrak{F}=\tfrac{1}{2}(P^2-Q^2)\{2(p^2-q^2)^2P^2Q^2+p^2q^2(P^4+Q^4)\} \quad . \quad . \quad . \quad . \quad (44)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_{\text{II}}=2\mathfrak{F}\mathrm{F}\sin 2\mathrm{f}\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Fast semi-diurnal terms $(2n+2\Omega)$.

$$\mathrm{W}_{\text{III}}=\tfrac{1}{4}\mathrm{F}_2[\underline{\kappa}^4\kappa'^4e^{-2(\epsilon'-\epsilon)-2\mathrm{f}_2}+\kappa^4\underline{\kappa}'^4e^{2(\epsilon'-\epsilon)+2\mathrm{f}_2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Since κ is obtained from ϖ by writing Q for P, and $-P$ for Q, therefore by writing $-2\mathrm{f}_2$ for $2\mathrm{f}_1$, and interchanging Q's and P's we may write down the result by symmetry with the slow semi-diurnal terms. Then let

$$\mathfrak{F}_2=\tfrac{1}{4}[Q^8p^6-4P^2Q^6p^4(p^2-3q^2)-18P^4Q^4p^2q^2(p^2-q^2)-4P^6Q^2q^4(3p^2-q^2)-P^8q^6] \,. \quad (47)$$

and

$$\phi(N,\epsilon)\mathrm{W}_{\text{III}}=-2\mathfrak{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Slow diurnal terms.

$$\mathrm{W}_1=\mathrm{G}_1[\varpi^3\underline{\kappa}\underline{\varpi}'^3\kappa'e^{2(\epsilon'-\epsilon)-\mathrm{g}_1}+\underline{\varpi}^3\kappa\varpi'^3\underline{\kappa}'e^{-2(\epsilon'-\epsilon)+\mathrm{g}_1}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Let $\mathrm{w}_1=\varpi^3\underline{\kappa}\underline{\varpi}'^3\kappa'e^{2(\epsilon'-\epsilon)-\mathrm{g}_1}$.

For the moment let $\mathrm{I}=\frac{1}{2}i$, then since $\varpi=Pp-Qqe^{N}$, and since $P=\cos\mathrm{I}$, $Q=\sin\mathrm{I}$, therefore $d\varpi/d\mathrm{I}=-\kappa$, and therefore $d\varpi^4/d\mathrm{I}=-4\varpi^3\kappa$.

Hence (see slow semi-diurnal terms)

$$\varpi^3\kappa=P^3Qp^4+P^2(P^2-3Q^2)p^3qe^{N}-3PQ(P^2-Q^2)p^2q^2e^{2N}+Q^2(3P^2-Q^2)pq^3e^{3N}-PQ^3q^4e^{4N}$$

$\underline{\varpi}'^3\underline{\kappa}'=$ same with $-N'$ for N

Hence

$$\mathrm{w}_1=[P^6Q^2p^8+P^4(P^2-3Q^2)^2p^6q^2e^{N-N'}+9P^2Q^2(P^2-Q^2)^2p^4q^4e^{2(N-N')}$$
$$+Q^4(3P^2-Q^2)^2p^2q^6e^{3(N-N')}+P^2Q^6q^8e^{4(N-N')}]\,e^{2(\epsilon'-\epsilon)-\mathrm{g}_1}$$

$$\phi(N)\mathrm{w}_1=\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}\Big[\qquad P^4(P^2-3Q^2)^2p^5q+18P^2Q^2(P^2-Q^2)^2p^3q^3$$
$$+3Q^4(3P^2-Q^2)^2pq^5+4P^2Q^6\frac{q^7}{p}\Big]$$

$$\phi(\epsilon)\mathrm{w}_1=-\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}\Big[4P^6Q^2p^7q+4P^4(P^2-3Q^2)^2p^5q^3+36P^2Q^2(P^2-Q^2)^2p^3q^5$$
$$+4Q^4(3P^2-Q^2)^2pq^7+4P^2Q^6\frac{q^9}{p}\Big]$$

Adding

$$\phi(N,\epsilon)\mathrm{w}_1=-\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}[4P^6Q^2p^7q-P^4(P^2-3Q^2)^2p^5q(p^2-3q^2)$$
$$-18P^2Q^2(P^2-Q^2)^2p^3q^3(p^2-q^2)-Q^4(3P^2-Q^2)^2pq^5(3p^2-q^2)-4P^2Q^6pq^7]$$

Then let

$$\mathfrak{G}_1=\tfrac{1}{4}[4P^6Q^2p^6-P^4(P^2-3Q^2)^2p^4(p^2-3q^2)-18P^2Q^2(P^2-Q^2)^2p^2q^2(p^2-q^2)$$
$$-Q^4(3P^2-Q^2)^2q^4(3p^2-q^2)-4P^2Q^6q^6]\quad . \quad (50)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_1\doteq2\mathfrak{G}_1\mathrm{G}_1\sin\mathrm{g}_1\sin j\quad . \ . \ . \ . \ . \ . \ . \quad (51)$$

Sidereal diurnal terms (n).

$$\mathrm{W}_2=\mathrm{G}[\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})\underline{\varpi}'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{-\mathrm{g}}+\underline{\varpi}\kappa(\varpi\underline{\varpi}-\kappa\underline{\kappa})\varpi'\underline{\kappa}'(\varpi'\underline{\varpi}'-\underline{\kappa}'\kappa')e^{\mathrm{g}}]\quad . \quad (52)$$

Here the epoch is wanting, so that $\phi(N,\epsilon)=\phi(N)$.

Let

$$\mathrm{w}_2=\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})\underline{\varpi}'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')$$

$$\varpi\underline{\kappa}=PQ(p^2-q^2)+pq(P^2e^{-N}-Q^2e^{N})$$

$$\varpi\underline{\varpi}-\kappa\underline{\kappa}=(P^2-Q^2)(p^2-q^2)-2PQpq(e^{N}+e^{-N})$$

$$\begin{aligned}\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})=PQ(P^2-Q^2)[(p^2-q^2)^2-2p^2q^2]+P^2(P^2-3Q^2)pq(p^2-q^2)e^{-N}\\ -Q^2(3P^2-Q^2)pq(p^2-q^2)e^{N}-2PQp^2q^2(P^2e^{-2N}-Q^2e^{2N})\end{aligned}$$

$\underline{\varpi}'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')=$ the same with $-N'$ instead of N

$$\begin{aligned}\mathrm{w}_2=P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2-2p^2q^2]^2+P^4(P^2-3Q^2)^2p^2q^2(p^2-q^2)^2e^{-(N-N')}\\ +Q^4(3P^2-Q^2)^2p^2q^2(p^2-q^2)^2e^{N-N'}+4P^2Q^2p^4q^4(P^4e^{-2(N-N')}+Q^4e^{2(N-N')})\end{aligned}$$

$$\phi(N)\mathrm{w}_2=-\frac{1}{2\sqrt{-1}}\{pq(p^2-q^2)^2(P^4(P^2-3Q^2)^2-Q^4(3P^2-Q^2)^2)+8P^2Q^2(P^4-Q^4)p^3q^3\}$$

Now

$$P^4(P^2-3Q^2)^2-Q^4(3P^2-Q^2)^2=(P^2-Q^2)(P^4+Q^4-6P^2Q^2)$$

Put therefore

$$\mathfrak{G}=\tfrac{1}{4}(P^2-Q^2)\{(p^2-q^2)^2(P^4+Q^4-6P^2Q^2)+8P^2Q^2p^2q^2\} \quad . \quad . \quad . \quad (53)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_2=2\mathfrak{G}\mathrm{G}\sin\mathrm{g}\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Fast diurnal terms $(n+2\Omega)$.

$$\mathrm{W}_3=\mathrm{G}_2[\underline{\varpi}\underline{\kappa}^3\varpi'\kappa'^3e^{-2(\epsilon'-\epsilon)-\mathrm{g}_2}+\varpi\kappa^3\underline{\varpi}'\underline{\kappa}'^3e^{2(\epsilon'-\epsilon)+\mathrm{g}_2}] \quad . \quad . \quad . \quad . \quad . \quad (55)$$

By an analogy similar to that by which the fast semi-diurnal was derived from the slow, we have

$$\begin{aligned}\mathfrak{G}_2=\tfrac{1}{4}[4P^2Q^6p^6-Q^4(3P^2-Q^2)^2p^4(p^2-3q^2)-18P^2Q^2(P^2-Q^2)^2p^2q^2(p^2-q^2)\\ -P^4(P^2-3Q^2)^2q^4(3p^2-q^2)-4P^6Q^2q^6] \quad . \quad (56)\end{aligned}$$

and

$$\phi(N,\epsilon)\mathrm{W}_3=-2\mathfrak{G}_2\mathrm{G}_2\sin\mathrm{g}_2\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Fortnightly terms (2Ω).

$$\mathrm{W}_0=\tfrac{3}{2}[(\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa})(\tfrac{1}{3}-2\varpi'\underline{\varpi}'\kappa'\underline{\kappa}')+\mathrm{H}\underline{\varpi}^2\underline{\kappa}^2\varpi'^2\kappa'^2e^{-2(\epsilon'-\epsilon)-2\mathrm{h}}+\mathrm{H}\varpi^2\kappa^2\underline{\varpi}'^2\underline{\kappa}'^2e^{2(\epsilon'-\epsilon)+2\mathrm{h}}] \ . \ (58)$$

It will be found that $\phi(N)$ performed on the first term is zero, as it ought to be according to the general principles of energy—for the system is a conservative one as far as regards these terms.

Let

$$\mathrm{w}_0=(\varpi\kappa\underline{\varpi}'\underline{\kappa}')^2 e^{2(\epsilon'-\epsilon)+2\mathrm{h}}$$

$$\varpi\kappa=PQp^2+pq(P^2-Q^2)e^N-PQq^2e^{2N}$$

$$\varpi^2\kappa^2=P^2Q^2p^4+2PQ(P^2-Q^2)p^3qe^N+[(P^2-Q^2)^2-2P^2Q^2]p^2q^2e^{2N}$$
$$-2PQ(P^2-Q^2)pq^3e^{3N}+P^2Q^2q^4e^{4N}$$

$\underline{\varpi}'^2\underline{\kappa}'^2=$the same with $-N'$ for N

$$\mathrm{w}_0=[P^4Q^4p^8+4P^2Q^2(P^2-Q^2)^2p^6q^2e^{N-N'}+[(P^2-Q^2)^2-2P^2Q^2]p^4q^4e^{2(N-N')}$$
$$+4P^2Q^2(P^2-Q^2)^2p^2q^6e^{3(N-N')}+P^4Q^4q^8e^{4(N-N')}]e^{2(\epsilon'-\epsilon)+2\mathrm{h}}$$

$$\phi(N)\mathrm{w}_0=\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}\Big[\quad 4P^2Q^2(P^2-Q^2)^2p^5q+2[(P^2-Q^2)^2-2P^2Q^2]^2p^3q^3$$
$$+12P^2Q^2(P^2-Q^2)^2pq^5+4P^4Q^4\frac{q^7}{p}\Big]$$

$$\phi(\epsilon)\mathrm{w}_0=-\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}\Big[4P^4Q^4p^7q+16P^2Q^2(P^2-Q^2)^2p^5q^3+4[(P^2-Q^2)^2-2P^2Q^2]^2p^3q^5$$
$$+16P^2Q^2(P^2-Q^2)^2pq^7+4P^4Q^4\frac{q^9}{p}\Big]$$

Adding and arranging the terms

$$\phi(N,\epsilon)\mathrm{w}_0=-\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}pq\{4P^4Q^4(p^6-q^6)-4P^2Q^2(P^2-Q^2)^2(p^2-q^2)^3$$
$$-2p^2q^2(p^2-q^2)[(P^2-Q^2)^2-2P^2Q^2]^2\}$$

Then let

$$\mathfrak{H}=\tfrac{3}{4}\{2P^4Q^4(p^6-q^6)-2P^2Q^2(P^2-Q^2)^2(p^2-q^2)^3$$
$$-p^2q^2(p^2-q^2)[(P^2-Q^2)^2-2P^2Q^2]^2\} \quad . \quad . \quad (59)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_0=-2\mathfrak{H}\mathrm{H}\sin 2\mathrm{h}\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

This is the last of the seven sets of terms.

Then collecting results from (42–5–8, 51–4–7, 60), we have

$$\frac{1}{\sin j}\frac{dj}{dt}=-\frac{\tau^2}{\mathfrak{g}}\frac{k}{\mathfrak{k}}\{2\mathfrak{F}_1\mathrm{F}_1\sin 2\mathrm{f}_1+2\mathfrak{F}\mathrm{F}\sin 2\mathrm{f}-2\mathfrak{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2+2\mathfrak{G}_1\mathrm{G}_1\sin \mathrm{g}_1$$
$$+2\mathfrak{G}\mathrm{G}\sin \mathrm{g}-2\mathfrak{G}_2\mathrm{G}_2\sin \mathrm{g}_2-2\mathfrak{H}\mathrm{H}\sin 2\mathrm{h}\} \quad . \quad (61)$$

The seven gothic-letter functions defined by (41–4–7, 50–3–6–9) are functions of the sines and cosines of half the obliquity and of half the inclination, but they are reducible to forms which may be expressed in the following manner :—

$$\left.\begin{aligned}
\mathfrak{F}_1+\mathfrak{F}_2&=\tfrac{1}{4}\cos j\left[1-\tfrac{1}{4}\sin^2 j-2\sin^2 i(1-\tfrac{5}{8}\sin^2 j)+\tfrac{5}{8}\sin^4 i(1-\tfrac{7}{4}\sin^2 j)\right]\\
\mathfrak{F}_1-\mathfrak{F}_2&=\tfrac{1}{4}\cos i\left[1-\tfrac{3}{4}\sin^2 j-\tfrac{3}{2}\sin^2 i(1-\tfrac{5}{4}\sin^2 j)\right]\\
\mathfrak{G}_1+\mathfrak{G}_2&=-\tfrac{1}{4}\cos j\left[1-\sin^2 j-\tfrac{7}{2}\sin^2 i(1-\tfrac{10}{7}\sin^2 j)+\tfrac{5}{2}\sin^4 i(1-\tfrac{7}{4}\sin^2 j)\right]\\
\mathfrak{G}_1-\mathfrak{G}_2&=-\tfrac{1}{4}\cos i\left[1-\tfrac{3}{2}\sin^2 j-3\sin^2 i(1-\tfrac{5}{4}\sin^2 j)\right]\\
\mathfrak{F}&=\tfrac{1}{4}\cos i\left[\tfrac{1}{2}\sin^2 j+\sin^2 i-\tfrac{5}{4}\sin^2 i\sin^2 j\right]\\
\mathfrak{G}&=\tfrac{1}{4}\cos i\left[1-\sin^2 j-2\sin^2 i+\tfrac{5}{2}\sin^2 i\sin^2 j\right]\\
\mathfrak{H}&=-\tfrac{1}{4}\cos j\left[\tfrac{3}{4}\sin^2 j+\tfrac{3}{2}\sin^2 i(1-\tfrac{5}{2}\sin^2 j)-\tfrac{15}{8}\sin^4 i(1-\tfrac{7}{4}\sin^2 j)\right]
\end{aligned}\right\}\quad . \quad (62)$$

These coefficients will be applicable whatever theory of tides be used, and no approximation, as regards either the obliquity or inclination, has been used in obtaining them.

§ 7. *Application to the case where the planet is viscous.*

If the planet or earth be viscous with a coefficient of viscosity v, then according to the theory of viscous tides, when inertia is neglected, the tangent of the phase-retardation or lag of any tide is equal to $19v/2gaw$ multiplied by the speed of that tide; and the height of tide is equal to the equilibrium tide of a perfectly fluid spheroid multiplied by the cosine of the lag. If therefore we put $\frac{2gaw}{19v}=\mathfrak{p}$, we have

$$\tan 2\mathrm{f}_1=\frac{2(n-\Omega)}{\mathfrak{p}},\ \tan 2\mathrm{f}=\frac{2n}{\mathfrak{p}},\ \tan 2\mathrm{f}_2=\frac{2(n+\Omega)}{\mathfrak{p}}$$

$$\tan \mathrm{g}_1=\frac{n-2\Omega}{\mathfrak{p}},\ \tan \mathrm{g}=\frac{n}{\mathfrak{p}},\ \tan 2\mathrm{g}_2=\frac{n+2\Omega}{\mathfrak{p}},\ \tan 2\mathrm{h}=\frac{2\Omega}{\mathfrak{p}}$$

$\mathrm{F}_1=\cos 2\mathrm{f}_1$, $\mathrm{F}=\cos 2\mathrm{f}$, $\mathrm{F}_2=\cos 2\mathrm{f}_2$, $\mathrm{G}_1=\cos \mathrm{g}_1$, $\mathrm{G}=\cos \mathrm{g}$, $\mathrm{G}_2=\cos \mathrm{g}_2$ and $\mathrm{H}=\cos 2\mathrm{h}$.

Therefore

$$-\frac{\xi}{k\sin j}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\{\mathfrak{F}_1\sin 4\mathrm{f}_1+\mathfrak{F}\sin 4\mathrm{f}-\mathfrak{F}_2\sin 4\mathrm{f}_2+\mathfrak{G}_1\sin 2\mathrm{g}_1$$
$$+\mathfrak{G}\sin 2\mathrm{g}-\mathfrak{G}_2\sin 2\mathrm{g}_2-\mathfrak{H}\sin 4\mathrm{h}] \quad . \quad . \quad . \quad (63)$$

This equation involves such complex functions of i and j, that it does not present to the mind any physical meaning. It will accordingly be illustrated graphically.

For this purpose the case is taken when the planet rotates fifteen times as fast as the

satellite revolves. Then the speeds of the seven tides are proportional to the following numbers: 28, 30, 32 (semi-diurnal); 13, 15, 17 (diurnal); and 2 (fortnightly).

It would require a whole series of figures to illustrate the equation for all values of i and j, and for all viscosities. The case is therefore taken where the inclination j of the orbit to the ecliptic is so small that we may neglect squares and higher powers of $\sin j$. Then the formulas (62) become

$$\mathfrak{F}_1+\mathfrak{F}_2=\tfrac{1}{4}(1-2\sin^2 i+\tfrac{5}{8}\sin^4 i)$$
$$\mathfrak{F}_1-\mathfrak{F}_2=\tfrac{1}{4}\cos i(1-\tfrac{3}{2}\sin^2 i)$$
$$\mathfrak{G}_1+\mathfrak{G}_2=-\tfrac{1}{4}(1-\tfrac{7}{2}\sin^2 i+\tfrac{5}{2}\sin^4 i)$$
$$\mathfrak{G}_1-\mathfrak{G}_2=-\tfrac{1}{4}\cos i(1-3\sin^2 i)$$
$$\mathfrak{F}=\tfrac{1}{4}\cos i\sin^2 i,\ \mathfrak{G}=\tfrac{1}{4}\cos i(1-2\sin^2 i)$$
$$\mathfrak{H}=-\tfrac{3}{8}\sin^2 i(1-\tfrac{5}{4}\sin^2 i)$$

From these we may compute a series of values corresponding to $i=0°$, 15°, 30°, 45°, 60°, 75°, 90°. (I actually did compute them from the P, Q formulas.)

I then took as five several standards of the viscosity of the planet, such viscosities as would make the lag f_1 of the slow semi-diurnal tide (of speed $2n-2\Omega$) equal to 10°, 20°, 30°, 40°, 44°. Then it is easy to compute tables giving the five corresponding values of each of the following, viz.: $\sin 4\mathrm{f}_1$, $\sin 4\mathrm{f}$, $\sin 4\mathrm{f}_2$, $\sin 2\mathrm{g}_1$, $\sin 2\mathrm{g}$, $\sin 2\mathrm{g}_2$, $\sin 4\mathrm{h}$.

Then these numerical values were appropriately multiplied (with CRELLE's three figure table) by the sets of values before found for the $\mathfrak{F}$'s, $\mathfrak{G}$'s, &c.

From the sets of tables formed, the proper sets were selected and added up. The result was to have a series of numbers which were proportional to $dj/\sin j dt$.

Then the series corresponding to each degree of viscosity were set off in a curve, as shown in fig. 4.

The ordinates, which are generally negative, represent $dj/\sin j dt$, and the abscissæ correspond to i, the obliquity of the planet's equator to the ecliptic.

This figure shows that the inclination j of the orbit will diminish, unless the obliquity be very large.

It appears from the results of previous papers, that the satellite's distance will increase as the time increases, unless the obliquity be very large, and if the obliquity be very large the mean distance decreases more rapidly for large than for small viscosity. This statement, taken in conjunction with our present figure, shows that in general the inclination will decrease as long as the mean distance increases, and *vice versâ*. This is not, however, necessarily true for all speeds of rotation of the planet and revolution of the satellite.

The most remarkable feature in these curves is that they show that, for moderate degrees of viscosity (f_1 less than 20°), the inclination j decreases most rapidly when i

the obliquity is zero; whilst for larger viscosities (f_1 between 20° and 45°), there is a very marked maximum rate of decrease for obliquities ranging from 30° to 40°.

Fig. 4.

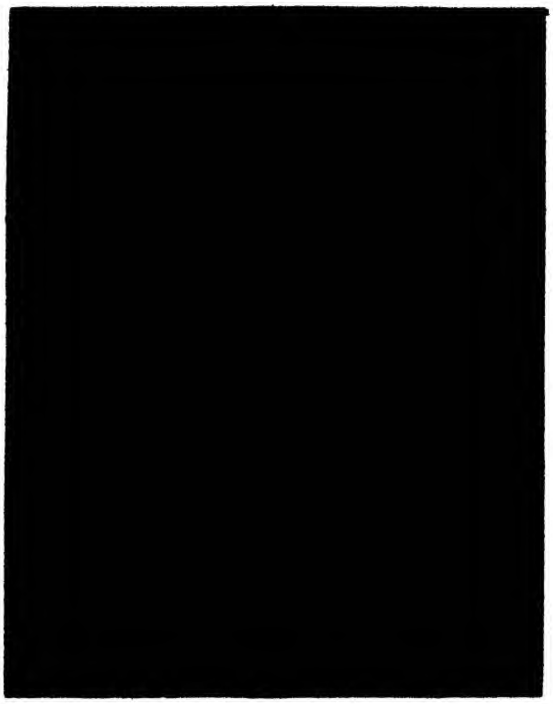


Diagram illustrating the rate of change of the inclination of a satellite's orbit to a fixed plane on which its nodes revolve, for various obliquities and viscosities of the planet $\left(\frac{1}{\sin j}\frac{dj}{dt}\right.$ when j is small$\left.\right)$.

We now return to the analytical investigation.

If the viscosity be sufficiently small to allow the phase retardations to be small, so that the lag of each tide is proportional to its speed, we may express the lags of all the tides in terms of that of the sidereal semi-diurnal tide, viz.: 2f. Then on this hypothesis we have

$$\frac{\sin 4f_1}{\sin 4f}=1-\lambda,\ \frac{\sin 4f}{\sin 4f}=1,\ \frac{\sin 4f_2}{\sin 4f}=1+\lambda,\ \frac{\sin 2g_1}{\sin 4f}=\tfrac{1}{2}-\lambda,\ \frac{\sin 2g}{\sin 4f}=\tfrac{1}{2}$$

$$\frac{\sin 2g_2}{\sin 4f}=\tfrac{1}{2}+\lambda,\ \frac{\sin 4h}{\sin 4f}=\lambda,\text{ where }\lambda=\frac{\Omega}{n}$$

And

$$-\frac{\xi}{k\sin j}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin 4f[\mathfrak{F}_1+\mathfrak{F}-\mathfrak{F}_2+\tfrac{1}{2}(\mathfrak{G}_1+\mathfrak{G}-\mathfrak{G}_2)-\lambda(\mathfrak{F}_1+\mathfrak{F}_2+\mathfrak{G}_1+\mathfrak{G}_2+\mathfrak{H})]$$

But by (62)

$$\mathfrak{F}_1-\mathfrak{F}_2+\tfrac{1}{2}(\mathfrak{G}_1-\mathfrak{G}_2)=\tfrac{1}{8}\cos i\text{ and }\mathfrak{F}+\tfrac{1}{2}\mathfrak{G}=\tfrac{1}{8}\cos i$$

and

$$\mathfrak{F}_1+\mathfrak{F}_2+\mathfrak{G}_1+\mathfrak{G}_2+\mathfrak{H}=0.$$

These results may of course be also obtained when the functions are expressed in terms of P, Q, p, q.

Whence on this hypothesis

$$-\frac{\xi}{k\sin j}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}.\tfrac{1}{4}\cos i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

§ 8. *Secular change in the mean distance of a satellite, where there is a second disturbing body, and where the nodes revolve with sensible uniformity on the fixed plane of reference.*

By (11) the equation giving the rate of change of ξ' is

$$\frac{1}{k'}\frac{d\xi'}{dt}=\frac{d\mathrm{W}}{d\epsilon'}$$

As before, we may drop the accents, except as regards ϵ'.

In § 6 we wrote $\phi(\epsilon)$ for the operation $\tan\tfrac{1}{2}j\frac{d}{d\epsilon'}$; hence $\frac{d\mathrm{W}}{d\epsilon'}=\frac{p}{q}\phi(\epsilon)\mathrm{W}$, and by reference to that section the result may be at once written down. We have

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}}\{2\Phi_1\mathrm{F}_1\sin 2\mathrm{f}_1-2\Phi_2\mathrm{F}_2\sin 2\mathrm{f}_2+2\Gamma_1\mathrm{G}_1\sin \mathrm{g}_1-2\Gamma_2\mathrm{G}_2\sin \mathrm{g}_2-2\Lambda\mathrm{H}\sin 2\mathrm{h}] \quad . \quad (65)$$

Where

$$\begin{aligned}
\Phi_1&=\tfrac{1}{2}[P^8p^8+Q^8q^8+16P^2p^2Q^2q^2(P^4p^4+Q^4q^4)+36P^4Q^4p^4q^4]\\
\Phi_2&=\text{ the same with } Q \text{ and } P \text{ interchanged}\\
\Gamma_1&=2[P^2Q^2(P^4p^8+Q^4q^8)+P^4(P^2-3Q^2)^2p^6q^2+Q^4(3P^2-Q^2)^2p^2q^6\\
&\qquad\qquad+9P^2Q^2(P^2-Q^2)^2p^4q^4] \quad . \quad (66)\\
\Gamma_2&=\text{ the same with } Q \text{ and } P \text{ interchanged}\\
\Lambda&=3[P^4Q^4(p^8+q^8)+4P^2Q^2(P^2-Q^2)^2p^2q^2(p^4+q^4)\\
&\qquad\qquad+p^4q^4[(P^2-Q^2)^2-2P^2Q^2]^2].
\end{aligned}$$

These functions are reducible to the following forms

$$\left.\begin{aligned}
2(\Phi_1+\Phi_2)&=1-\sin^2 j+\tfrac{1}{8}\sin^4 j-\sin^2 i(1-2\sin^2 j+\tfrac{5}{8}\sin^4 j)\\
&\qquad+\tfrac{1}{8}\sin^4 i(1-5\sin^2 j+\tfrac{35}{8}\sin^4 j)\\
2(\Phi_1-\Phi_2)&=\cos i\cos j[1-\tfrac{1}{2}\sin^2 j-\tfrac{1}{2}\sin^2 i(1-\tfrac{5}{2}\sin^2 j)]\\
2(\Gamma_1+\Gamma_2)&=\sin^2 j-\tfrac{1}{2}\sin^4 j+\sin^2 i(1-\tfrac{7}{2}\sin^2 j+\tfrac{5}{2}\sin^4 j)\\
&\qquad-\tfrac{1}{2}\sin^4 i(1-5\sin^2 j+\tfrac{35}{8}\sin^4 j)\\
2(\Gamma_1-\Gamma_2)&=\cos i\cos j[\sin^2 j+\sin^2 i(1-\tfrac{5}{2}\sin^2 j)]\\
2\Lambda\qquad&=\tfrac{3}{8}\sin^4 j+\sin^2 i(\tfrac{3}{2}\sin^2 j-\tfrac{15}{8}\sin^4 j)\\
&\qquad+\tfrac{3}{8}\sin^4 i(1-5\sin^2 j+\tfrac{35}{8}\sin^4 j)
\end{aligned}\right\} \quad (67)$$

§ 9. *Application to the case where the planet is viscous.*

As in § 7

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}}\{\Phi_1 \sin 4f_1-\Phi_2 \sin 4f_2+\Gamma_1 \sin 2g_1-\Gamma_2 \sin 2g_2-\Lambda \sin 4h\} \quad . \quad . \quad (68)$$

If j be put equal to zero this equation will be found to be the same as that used as the equation of tidal reaction in the previous paper on "Precession."

If the viscosity be small, with the same notation as before

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin 4f[\Phi_1-\Phi_2+\tfrac{1}{2}(\Gamma_1-\Gamma_2)-\lambda(\Phi_1+\Phi_2+\Gamma_1+\Gamma_2+\Lambda)] \quad . \quad . \quad . \quad (69)$$

Now

$$\Phi_1-\Phi_2+\tfrac{1}{2}(\Gamma_1-\Gamma_2)=\tfrac{1}{2}\cos i \cos j$$

and

$$\Phi_1+\Phi_2+\Gamma_1+\Gamma_2+\Lambda=\tfrac{1}{2}$$

Therefore

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f[\cos i \cos j-\lambda] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

We see that the rate of tidal reaction diminishes as the inclination of the orbit increases.

§ 10. *Secular change in the inclination of the orbit of a single satellite to the invariable plane, where there is no other disturbing body than the planet.*

This is the second of the two cases into which the problem subdivides itself.

If there be only two bodies, then the fixed plane of reference, which was called the ecliptic, may be taken as the invariable plane of the system. It follows from the principle of the composition of moments of momentum that the planet's axis of rotation, the normal to the satellite's orbit and the normal to the invariable plane, necessarily lie in one plane. Whence it follows that the orbit and the equator necessarily intersect in the invariable plane. From this principle it would of course be possible either to determine the motion of the node from the precession of the planet or *vice versâ*, and the change of obliquity of the planet's axis (if any) from the change in the plane of the orbit or *vice versâ*; this principle will be applied later.

We have found it convenient to measure longitudes from a line in the fixed plane, which is instantaneously coincident with the descending node of the equator on the fixed plane. Hence it follows that where there are only two bodies we shall after differentiation have to put $N=N'=0$.

Then since $\varpi' = Pp - Qqe^{N'}$ therefore $\dfrac{d\varpi'}{dN'} = \dfrac{1}{\sqrt{-1}} Qq$, and similarly

$$\frac{d\varpi'}{dN'} = -\frac{Qq}{\sqrt{-1}}, \ \frac{d\kappa'}{dN'} = -\frac{Pq}{\sqrt{-1}}, \ \frac{d\underline{\kappa}'}{dN'} = \frac{Pq}{\sqrt{-1}}, \text{ when } N' = 0.$$

Also after differentiation when $N=0$, $\varpi = \underline{\varpi} = \cos \frac{1}{2}(i+j)$, $\kappa = \underline{\kappa} = \sin \frac{1}{2}(i+j)$

In order to find dj/dt we must, as before, perform $\phi(N, \epsilon)$ on W. Then take the same notation as before for the W's and w's with suffixes.

Slow semi-diurnal term.

$$\frac{d}{dN'}(\tfrac{1}{4}\varpi^4 \underline{\varpi}'^4) = \varpi^7 \frac{d\underline{\varpi}'}{dN'} = -\frac{\varpi^7 Qq}{\sqrt{-1}}$$

and

$$\phi(N)(\tfrac{1}{4}\varpi^4 \underline{\varpi}'^4) = -\frac{1}{2\sqrt{-1}} \cdot \varpi^7 \cdot \frac{Q}{p}, \text{ also } \phi(\epsilon) e^{2(\epsilon'-\epsilon)-2f_1} = -\frac{1}{2\sqrt{-1}} \cdot \frac{4q}{p} e^{-2f_1}$$

Hence

$$\phi(N, \epsilon)\mathrm{w}_1 = \frac{e^{-2f_1}}{2\sqrt{-1}}\left[-\varpi^7 \frac{Q}{p} - \varpi^8 \cdot \frac{q}{p}\right] = -\varpi^7 \kappa \frac{e^{-2f_1}}{2\sqrt{-1}}$$

and

$$\phi(N, \epsilon)\mathrm{W}_{\mathrm{I}} = \varpi^7 \kappa \mathrm{F}_1 \sin 2\mathrm{f}_1.$$

Sidereal semi-diurnal term.

$$\frac{d\mathrm{w}_{\mathrm{II}}}{dN'} = 2\varpi^3\kappa^3\left(\varpi \frac{d\kappa'}{dN'} + \kappa \frac{d\varpi'}{dN'}\right) = -\frac{2\varpi^3\kappa^3}{\sqrt{-1}}(\varpi Pq + \kappa Qq) = -\frac{2\varpi^3\kappa^3}{\sqrt{-1}} \cdot pq$$

and since $\phi(\epsilon)\mathrm{W}_{\mathrm{II}} = 0$, therefore

$$\phi(N, \epsilon)\ \mathrm{W}_{\mathrm{II}} = 2\varpi^3\kappa^3 \mathrm{F} \sin 2\mathrm{f}$$

Fast semi-diurnal term.

By symmetry

$$\phi(N, \epsilon)\mathrm{W}_{\mathrm{III}} = \varpi\kappa^7 \mathrm{F}_2 \sin 2\mathrm{f}_2$$

Slow diurnal term.

$$\frac{d}{dN'}\underline{\varpi}'^3 \underline{\kappa}' = 3\varpi^2\kappa \frac{d\underline{\varpi}'}{dN'} + \varpi^3 \frac{d\underline{\kappa}'}{dN'} = -\frac{\varpi^2}{\sqrt{-1}}(3\kappa Qq - \varpi Pq)$$

$$\phi(N, \epsilon)\mathrm{w}_1 = -\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}} \frac{\varpi^5\kappa}{p}(3Q\kappa - P\varpi + 4q\varpi\kappa) = -\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}} \varpi^5\kappa(\varpi^2 - 3\kappa^2)$$

and

$$\phi(N, \epsilon)\mathrm{W}_1 = -\varpi^5\kappa(\varpi^2 - 3\kappa^2)\mathrm{G}_1 \sin \mathrm{g}_1$$

Sidereal diurnal term.

$$\frac{d}{dN'}\underline{\varpi}'\kappa' = -\frac{q}{\sqrt{-1}}(Q\kappa + P\varpi) = -\frac{pq}{\sqrt{-1}} \text{ and } \frac{d}{dN'}(\varpi'\underline{\varpi}' - \kappa'\underline{\kappa}') = 0$$

Therefore

$$\phi(N, \epsilon)\mathrm{w}_2 = -\frac{\epsilon^{-\mathrm{g}}}{\sqrt{-1}}\varpi\kappa(\varpi^2 - \kappa^2)^2$$

and

$$\phi(N, \epsilon)\mathrm{W}_2 = \varpi\kappa(\varpi^2 - \kappa^2)^2 \mathrm{G} \sin \mathrm{g}$$

Fast diurnal term.

By symmetry

$$\phi(N, \epsilon)\mathrm{W}_3 = \varpi\kappa^5(3\varpi^2 - \kappa^2)\mathrm{G}_2 \sin \mathrm{g}_2$$

Fortnightly term.

$$-\frac{d}{dN'}(\underline{\varpi}'\underline{\kappa}')^2 = -\frac{2\varpi\kappa}{\sqrt{-1}}q(\kappa Q - P\varpi)$$

and

$$\phi(N, \epsilon)\mathrm{w}_0 = -\frac{\epsilon^{2\mathrm{h}}}{2\sqrt{-1}}\varpi^2\kappa^2\left[2\varpi\kappa(Q\kappa - P\varpi) + 4\varpi^2\kappa^2 q\right] = \frac{\epsilon^{2\mathrm{h}}}{2\sqrt{-1}}2\varpi\kappa(\varpi^2 - \kappa^2)$$

Whence

$$\phi(N, \epsilon)\mathrm{W}_0 = 3\varpi^3\kappa^3(\varpi^2 - \kappa^2)\mathrm{H} \sin 2\mathrm{h}$$

Then collecting terms we have, on applying the result to the case of viscosity,

$$-\frac{\xi}{k}\frac{dj}{dt} = \frac{\tau^2}{\mathfrak{g}}\Big[\tfrac{1}{2}\varpi^7\kappa \sin 4\mathrm{f}_1 + \varpi^3\kappa^3 \sin 4\mathrm{f} + \tfrac{1}{2}\varpi\kappa^7 \sin 4\mathrm{f}_2 + \tfrac{3}{2}\varpi^3\kappa^3(\varpi^2 - \kappa^2) \sin 4\mathrm{h}$$
$$- \tfrac{1}{2}\varpi^5\kappa(\varpi^2 - 3\kappa^2) \sin 2\mathrm{g}_1 + \tfrac{1}{2}\varpi\kappa(\varpi^2 - \kappa^2)^2 \sin 2\mathrm{g} + \tfrac{1}{2}\varpi\kappa^5(3\varpi^2 - \kappa^2) \sin 2\mathrm{g}_2\Big] \quad . \quad (71)$$

In the particular case where the viscosity is small, this becomes

$$-\frac{\xi}{k}\frac{dj}{dt} = \frac{1}{2}\frac{\tau^2}{\mathfrak{g}} \sin 4\mathrm{f}\, \varpi\kappa = \frac{1}{4}\frac{\tau^2}{\mathfrak{g}} \sin 4\mathrm{f} \sin(i + j) \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

The right hand side is necessarily positive, and therefore the inclination of the orbit to the invariable plane will always diminish with the time.

The general equation (71) for any degree of viscosity is so complex as to present no idea to the mind, and it will accordingly be graphically illustrated.

The case taken is where $n/\Omega = 15$, which is the same relation as in the previous graphical illustration of § 7.

The general method of illustration is sufficiently explained in that section.

Fig. 5 illustrates the various values which dj/dt (the rate of increase of inclination to the invariable plane) is capable of assuming for various viscosities of the planet, and

for various inclinations of the satellite's orbit to the planet's equator. Each curve corresponds to one degree of viscosity, the viscosity being determined by the lag of the slow semi-diurnal tide of speed $2n-2\Omega$. The ordinates give dj/dt (not as before $dj/\sin j dt$) and the abscissæ give $i+j$, the inclination of the orbit to the equator.

Fig. 5.

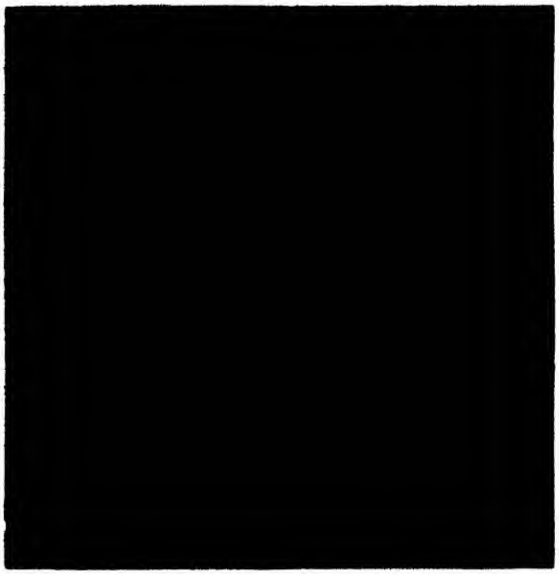


Diagram illustrating the rate of change of the inclination of a single satellite's orbit to the invariable plane, for various viscosities of the planet, and various inclinations of the orbit to the planet's equator $\left(\frac{dj}{dt}\right)$.

We see from this figure that the inclination to the invariable plane will always decrease as the time increases, and the only noticeable point is the maximum rate of decrease for large viscosities, for inclinations of the orbit and equator ranging from 60° to 70°. If n/Ω had been taken considerably smaller than 15, the inclination would have been found to increase with the time for large viscosity of the planet.

§ 11. *Secular change in the mean distance of the satellite, where there is no other disturbing body than the planet.—Comparison with result of previous paper.*

To find the variation of ξ we have to differentiate with respect to ϵ', and the following result may be at once written down

$$\frac{d\xi}{kdt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}[\varpi^8 \sin 4\mathrm{f}_1-\kappa^8 \sin 4\mathrm{f}_2+4\varpi^6\kappa^2 \sin 2\mathrm{g}_1-4\varpi^2\kappa^6 \sin 2\mathrm{g}_2-6\varpi^4\kappa^4 \sin 4\mathrm{h}]. \quad (73)$$

This agrees with the result of a previous paper (viz.: (57) or (79) of "Precession"), obtained by a different method; but in that case the inclination of the orbit was zero, so that ϖ and κ were the cosine and sine of half the obliquity, instead of the cosine and sine of $\frac{1}{2}(i+j)$.

In the case where the viscosity is small this becomes

$$\frac{d\xi}{kdt}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}[\cos(i+j)-\lambda] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

It will now be shown that the preceding result (71) for dj/dt may be obtained by means of the principle of conservation of moment of momentum, and by the use of the results of a previous paper.

It is easily shown that the moment of momentum of orbital motion of the moon and earth round their common centre of inertia is $C\xi/k$, and the moment of momentum of the earth's rotation is clearly Cn. Also j and i are the inclinations of the two axes of moment of momentum to the axis of resultant moment of momentum of the system. Hence

$$\frac{\xi}{k}\sin j=n\sin i$$

By differentiation of which

$$\frac{\xi}{k}\frac{dj}{dt}\cos j=\frac{dn}{dt}\sin i+n\cos i\frac{di}{dt}-\frac{1}{k}\frac{d\xi}{dt}\sin j$$

$$=\left[\frac{dn}{dt}\sin(i+j)+n\cos(i+j)\frac{di}{dt}\right]\cos j-\left[\frac{dn}{dt}\cos(i+j)-n\sin(i+j)\frac{di}{dt}+\frac{1}{k}\frac{d\xi}{dt}\right]\sin j$$

Now from equation (52) of the paper on "Precession," the second term on the right-hand side is zero, and therefore

$$\frac{\xi}{k}\frac{dj}{dt}=\frac{dn}{dt}\sin(i+j)+n\cos(i+j)\frac{di}{dt}$$

But by equations (21) and (16) and (29) of the paper on "Precession" (when ϖ and κ are written for the p, q of that paper)

$$\frac{dn}{dt}=-\frac{\tau^2}{\mathfrak{g}}[\tfrac{1}{2}\varpi^8\sin 4\mathrm{f}_1+2\varpi^4\kappa^4\sin 4\mathrm{f}+\tfrac{1}{2}\kappa^8\sin 4\mathrm{f}_2+\varpi^6\kappa^2\sin 2\mathrm{g}_1$$
$$+\varpi^2\kappa^2(\varpi^2-\kappa^2)^2\sin 2\mathrm{g}+\varpi^2\kappa^6\sin 2\mathrm{g}_2]$$

$$n\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}}[\tfrac{1}{2}\varpi^7\kappa\sin 4\mathrm{f}_1-\varpi^3\kappa^3(\varpi^2-\kappa^2)\sin 4\mathrm{f}-\tfrac{1}{2}\varpi\kappa^7\sin 4\mathrm{f}_2+\tfrac{1}{2}\varpi^5\kappa(\varpi^2+3\kappa^2)\sin 2\mathrm{g}_1$$
$$-\tfrac{1}{2}\varpi\kappa(\varpi^2-\kappa^2)^3\sin 2\mathrm{g}-\tfrac{1}{2}\varpi\kappa^5(3\varpi^2+\kappa^2)\sin 2\mathrm{g}_2-\tfrac{3}{2}\varpi^3\kappa^3\sin 4\mathrm{h}]$$

Then if we multiply the former of these by $\sin(i+j)$ or $2\varpi\kappa$, and the latter by $\cos(i+j)$ or $\varpi^2-\kappa^2$, and add, we get the equation (71), which has already been established by the method of the disturbing function.

It seemed well to give this method, because it confirms the accuracy of the two long analytical investigations in the paper on "Precession" and in the present one.

§ 12. *The method of the disturbing function applied to the motion of the planet.*

In the case where there are only two bodies, viz.: the planet and the satellite, the problem is already solved in the paper on "Precession," and it is only necessary to remember that the p and q of that paper are really $\cos\frac{1}{2}(i+j)$, $\sin\frac{1}{2}(i+j)$, instead of $\cos\frac{1}{2}i$, $\sin\frac{1}{2}i$. This will not be reinvestigated, but we will now consider the case of two satellites, the nodes of whose orbits revolve with uniform angular velocity on the ecliptic. The results may be easily extended to the hypothesis of any number of satellites.

In (18) we have the equations of variation of i, ψ, χ in terms of W. But as the correction to the precession has not much interest, we will only take the two equations

$$\left.\begin{aligned} n\sin i\frac{di}{dt}&=\cos i\frac{dW}{d\chi'}-\frac{dW}{d\psi'} \\ \frac{dn}{dt}&=\frac{dW}{d\chi'} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (75)$$

which give the rate of change of obliquity and the tidal friction.

In the development of W in § 5, it was assumed that ψ, ψ' were zero, and χ, χ' did not appear, because χ was left unaccented in the X′-Y′-Z′ functions.

Longitudes were there measured from the autumnal equinox, but here we must conceive the N, N' of previous developments replaced by $N-\psi$, $N'-\psi'$; also $\Omega t+\epsilon$, $\Omega' t+\epsilon'$ must be replaced by $\Omega t+\epsilon-\psi$, $\Omega' t+\epsilon'-\psi'$.

It will not be necessary to redevelop W for the following reasons.

$\Omega' t+\epsilon'-\psi'$ occurs only in the exponentials, and $N'-\psi'$ does not occur there; and $N'-\psi'$ only occurs in the functions of ϖ and κ, and $\Omega' t+\epsilon'-\psi'$ does not occur there. Hence

$$-\frac{dW}{d\psi'}=\frac{dW}{d\epsilon'}+\frac{dW}{dN'} \quad \ldots \ldots \ldots \quad (76)$$

Again, it will be seen by referring to the remarks made as to χ, χ' in the development of W in § 5, that we have the following identities:—

For semi-diurnal terms,

$$\frac{dW_{\text{I}}}{d\chi'}=-\frac{dW_{\text{I}}}{d\epsilon'},\ \frac{dW_{\text{II}}}{d\chi'}=\frac{dW_{\text{II}}}{df},\ \frac{dW_{\text{III}}}{d\chi'}=\frac{dW_{\text{III}}}{d\epsilon'}$$

For diurnal terms,

$$\frac{dW_1}{d\chi'}=-\frac{1}{2}\frac{dW_1}{d\epsilon'},\ \frac{dW_2}{d\chi'}=\frac{dW_2}{dg},\ \frac{dW_3}{d\chi'}=\frac{1}{2}\frac{dW_3}{d\epsilon'} \quad (77)$$

For the fortnightly term,

$$\frac{dW_0}{d\chi'}=0$$

Also

$$\frac{dW_{\text{II}}}{d\epsilon'}=0,\ \frac{dW_2}{d\epsilon'}=0$$

Then making use of (76) and (77), and remembering that $\cos i = P^2 - Q^2$, $\sin i = 2PQ$, we may write equations (75), thus

$$(2PQ)n\frac{di}{dt}=\frac{d}{d\epsilon'}[2Q^2\mathrm{W}_{\mathrm{I}}+2P^2\mathrm{W}_{\mathrm{III}}+\tfrac{1}{2}(P^2+3Q^2)\mathrm{W}_1+\tfrac{1}{2}(3P^2+Q^2)\mathrm{W}_3+\mathrm{W}_0]$$
$$+(P^2-Q^2)\left[\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathfrak{f}}+\frac{d\mathrm{W}_2}{d\mathfrak{g}}\right]+\frac{d}{dN'}(\Sigma\mathrm{W}) \quad . \quad (78)$$

$$\frac{dn}{dt}=-\frac{d}{d\epsilon'}[\mathrm{W}_{\mathrm{I}}-\mathrm{W}_{\mathrm{III}}+\tfrac{1}{2}\mathrm{W}_1-\tfrac{1}{2}\mathrm{W}_3]+\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathfrak{f}}+\frac{d\mathrm{W}_2}{d\mathfrak{g}} \quad . \quad . \quad . \quad . \quad . \quad (79)$$

It is clear that by using these transformations we may put $\psi=\psi'=0$, $\chi=\chi'$ before differentiation, so that ψ and χ again disappear, and we may use the old development of W.

The case where Diana and the moon are distinct bodies will be taken first, and it will now be convenient to make Diana identical with the sun.

In this case after the differentiations are made we are *not* to put $N=N'$ and $\epsilon=\epsilon'$.

The only terms, out of which secular changes in i and n can arise, are those depending on the sidereal semi-diurnal and diurnal tides, for all others are periodic with the longitudes of the two disturbing bodies. Hence the disturbing function is reduced to W_{II} and W_2. Also $d\mathrm{W}_{\mathrm{II}}/dN'$ and $d\mathrm{W}_2/dN'$ can only contribute periodic terms, because $N-N'$ is not zero, and by hypothesis the nodes revolve uniformly on the ecliptic.

Then if we consider that here p' is not equal p, nor q' to q, we see that, as far as is of present interest,

$$\mathrm{W}_{\mathrm{II}}=2\mathrm{F}\cos 2\mathfrak{f}\;P^4Q^4[(p^2-q^2)^2-2p^2q^2][(p'^2-q'^2)^2-2p'^2q'^2]$$
$$\mathrm{W}_2=2\mathrm{G}\cos\mathfrak{g}\;P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2-2p^2q^2][(p'^2-q'^2)^2-2p'^2q'^2]$$

Also the equations of variation of i and n are simply

$$(2PQ)n\frac{di}{dt}=(P^2-Q^2)\left[\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathfrak{f}}+\frac{d\mathrm{W}_2}{d\mathfrak{g}}\right]$$
$$\frac{dn}{dt}=\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathfrak{f}}+\frac{d\mathrm{W}_2}{d\mathfrak{g}}$$

Then if we put

$$\left.\begin{aligned}\phi&=2P^4Q^4[(p^2-q^2)^2-2p^2q^2][(p'^2-q'^2)^2-2p'^2q'^2]\\&\qquad=\tfrac{1}{8}\sin^4 i(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j')\\ \tfrac{1}{2}\gamma&=P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2-2p^2q^2][(p'^2-q'^2)^2-2p'^2q'^2]\\&\qquad=\tfrac{1}{4}\sin^2 i\cos^2 i(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j')\end{aligned}\right\} \quad . \quad (80)$$

We have

$$\left.\begin{aligned} -\frac{dn}{dt}&=\frac{2\tau\tau'}{\mathrm{g}}[2\phi\mathrm{F}\sin 2\mathrm{f}+\gamma\mathrm{G}\sin\mathrm{g}] \\ n\frac{di}{dt}&=-\frac{2\tau\tau'}{\mathrm{g}}[2\phi\mathrm{F}\sin 2\mathrm{f}+\gamma\mathrm{G}\sin\mathrm{g}]\cot i \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \tag{81}$$

It will be noticed that in (81) $2\tau\tau'$ has been introduced in the equations instead of $\tau\tau'$; this is because in the complete solution of the problem these terms are repeated twice, once for the attraction of the moon on the solar tides, and again for that of the sun on the lunar tides.

The case where Diana is identical with the moon must now be considered. This will enable us to find the effects of the moon's attraction on her own tides, and then by symmetry those of the sun's attraction on his tides.

We will begin with the *tidal friction*.

By comparison with (65)

$$\frac{d}{d\epsilon'}[\mathrm{W}_{\mathrm{I}}-\mathrm{W}_{\mathrm{III}}+\tfrac{1}{2}\mathrm{W}_1-\tfrac{1}{2}\mathrm{W}_3]=2\Phi_1\mathrm{F}_1\sin 2\mathrm{f}_1+2\Phi_2\mathrm{F}_2\sin 2\mathrm{f}_2+\Gamma_1\mathrm{G}_1\sin\mathrm{g}_1+\Gamma_2\mathrm{G}_2\sin\mathrm{g}_2 \tag{82}$$

Now when we put $N=N'$ (see (43) and (52))

$$\mathrm{W}_{\mathrm{II}}=2\mathrm{F}\cos 2\mathrm{f}.\mathrm{w}_{\mathrm{II}} \text{ and } \frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}}=-4\mathrm{F}\sin 2\mathrm{f}.\mathrm{w}_{\mathrm{II}}$$

Also

$$\mathrm{W}_2=2\mathrm{G}\cos\mathrm{g}.\mathrm{w}_2 \text{ and } \frac{d\mathrm{W}_2}{d\mathrm{g}}=-2\mathrm{G}\sin\mathrm{g}.\mathrm{w}_2$$

Then let

$$\begin{aligned}\Phi=2\mathrm{w}_{\mathrm{II}}&=2P^4Q^4[(p^2-q^2)^2-2p^2q^2]^2+8P^2Q^2(P^4+Q^4)p^2q^2(p^2-q^2)^2+2p^4q^4(P^8+Q^8) \\ &=2P^4Q^4(p^2-q^2)^4+8p^2q^2(p^2-q^2)^2P^2Q^2(P^4+Q^4-P^2Q^2)+2p^4q^4(P^8+4P^4Q^4+Q^8)\end{aligned} \tag{83}$$

and let

$$\begin{aligned}\tfrac{1}{2}\Gamma=\mathrm{w}_2&=P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2-2p^2q^2]^2 \\ &\qquad+[P^4(P^2-3Q^2)^2+Q^4(3P^2-Q^2)^2]p^2q^2(p^2-q^2)^2+4P^2Q^2(P^4+Q^4)p^4q^4 \\ &=P^2Q^2(P^2-Q^2)^2(p^2-q^2)^4+[(P^2-Q^2)^4-6P^2Q^2(P^2-Q^2)^2+8P^4Q^4]p^2q^2(p^2-q^2)^2 \\ &\qquad+8P^2Q^2(P^4+Q^4-P^2Q^2)p^4q^4\end{aligned} \tag{84}$$

And we have

$$\begin{aligned}-\frac{dn}{dt}=\frac{\tau^2}{\mathrm{g}}[&2\Phi_1\mathrm{F}_1\sin 2\mathrm{f}_1+2\Phi\mathrm{F}\sin 2\mathrm{f}+2\Phi_2\mathrm{F}_2\sin 2\mathrm{f}_2+\Gamma_1\mathrm{G}_1\sin\mathrm{g}_1 \\ &+\Gamma\mathrm{G}\sin\mathrm{g}+\Gamma_2\mathrm{G}_2\sin\mathrm{g}_2] \quad .\end{aligned} \tag{85}$$

This is only a partial solution, since it only refers to the action of the moon on her own tides.

If the second satellite, say the sun, be introduced, the action of the sun on the solar tides may be written down by symmetry, and the elements of the solar (or terrestrial) orbit may be indicated by the same symbols as before, but with accents.

From (85) and (81) the complete solution may be collected.

In the case of viscosity, and where the viscosity is small, it will be found that the solution becomes

$$-\frac{dn}{dt}=\frac{1}{2}\,\frac{\sin 4\mathrm{f}}{\mathfrak{g}}\Big\{(1-\tfrac{1}{2}\sin^2 i)(\tau^2+\tau'^2)-\tfrac{1}{2}(1-\tfrac{3}{2}\sin^2 i)(\tau^2\sin^2 j+\tau'^2\sin^2 j')$$
$$-\tau^2\frac{\Omega}{n}\cos i\cos j-\tau'^2\frac{\Omega'}{n}\cos i\cos j'+\tfrac{1}{2}\tau\tau'\sin^2 i(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j')\Big\} \quad . \quad (86)$$

If j and j' be put equal to zero and Ω'/n neglected, this result will be found to agree with that given in the paper on "Precession," § 17, (83).

We will next consider *the change of obliquity.*

The combined effect has already been determined in (81), but the separate effects of the two bodies remain to be found. The terms of different speeds must now be taken one by one.

Slow semi-diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{Q}{P}\frac{dW_{\mathrm{I}}}{d\epsilon'}+\frac{1}{2PQ}\frac{dW_{\mathrm{I}}}{dN'}$$

We had before

$$-\frac{\xi}{k}\frac{dj}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{q}{p}\frac{dW_{\mathrm{I}}}{d\epsilon'}+\frac{1}{2pq}\frac{dW_{\mathrm{I}}}{dN'}$$

Now W_{I} is symmetrical with regard to P and p, Q and q, and so are its differentials with regard to ϵ' and N'. The solution may be written down by symmetry with the "slow semi-diurnal" of § 6, by writing P for p and Q for q and *vice versâ.*

Let

$$\mathsf{F}_1=\tfrac{1}{4}\{P^6p^8-4P^4(P^2-3Q^2)p^6q^2-18P^2Q^2(P^2-Q^2)p^4q^4-4Q^4(3P^2-Q^2)p^2q^6-Q^6q^8\} \quad (87)$$

and

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=2\mathsf{F}_1\mathrm{F}_1\sin 2\mathrm{f}\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Sidereal semi-diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\Big[(P^2-Q^2)\frac{dW_{\mathrm{II}}}{d\mathrm{f}}+\frac{dW_{\mathrm{II}}}{dN''}\Big]$$

Now

$$\frac{dW_{II}}{df}=-2\Phi F \sin 2f \text{ and } \frac{dW_{II}}{dN'}=4p^2q^2.2\mathfrak{F}F \sin 2f$$

Therefore

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=2F \sin 2f\left[-\frac{P^2-Q^2}{2PQ}\Phi+\frac{2p^2q^2}{PQ}\mathfrak{F}\right]$$

On substitution from (44) and (83) for Φ and $\mathfrak{F}$ and simplification, we find that if

$$\mathsf{F}=\tfrac{1}{2}\{P^2Q^2(P^2-Q^2)[(p^2-q^2)^2-2p^2q^2]^2+2p^2q^2(p^2-q^2)^2(P^2-Q^2)^3-2p^4q^4(P^6-Q^6)\} \quad (89)$$

then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{F}F \sin 2f \sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Fast semi-diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{P}{Q}\frac{dW_{III}}{d\epsilon'}+\frac{1}{2PQ}\frac{dW_{III}}{dN'}$$

Since W_{III} is found from W_I by writing Q for P, and $-P$ for Q, and $-2f_2$ for $2f_1$, therefore in this case ndi/dt is found from its value in the slow semi-diurnal term by the like changes, and if

$$\mathsf{F}_2=\tfrac{1}{4}\{Q^6p^8+4Q^4(3P^2-Q^2)p^6q^2+18P^2Q^2(P^2-Q^2)p^4q^4+4P^4(P^2-3Q^2)p^2q^6-P^6q^8\} \quad (91)$$

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{F}_2F_2 \sin 2f_2 \sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Slow diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left[\frac{P^2+3Q^2}{2}\frac{dW_1}{d\epsilon'}+\frac{dW_1}{dN'}\right]$$

$$\frac{dW_1}{dN'}=-2G_1 \sin g_1[P^4(P^2-3Q^2)^2p^6q^2+18P^2Q^2(P^2-Q^2)^2p^4q^4$$
$$+3Q^4(3P^2-Q^2)^2p^2q^6+4P^2Q^6q^8]$$

$$\frac{dW_1}{d\epsilon'}= \quad 2G_1 \sin g_1.\Gamma_1$$

Substituting these values and simplifying, it will be found that if

$$\mathsf{G}_1=\tfrac{1}{4}\{P^4(P^2+3Q^2)p^8+2P^2(P^2-3Q^2)^2p^6q^2-9(P^2-Q^2)^3p^4q^4-2Q^2(3P^2-Q^2)^2p^2q^6$$
$$-Q^4(3P^2+Q^2)q^8\} \quad (93)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=2\mathsf{G}_1G_1 \sin g_1 \sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Sidereal diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left[(P^2-Q^2)\frac{dW_2}{dg}+\frac{dW_2}{dN'}\right]$$

$$\frac{dW_2}{dg}=-2G\sin g(\tfrac{1}{2}\Gamma)\quad\text{and}\quad\frac{dW_2}{dN'}=4p^2q^2 2\mathfrak{G}G\sin g$$

Therefore

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=2G\sin g\left[-\frac{P^2-Q^2}{4PQ}\Gamma+\frac{2p^2q^2}{PQ}\mathfrak{G}\right]$$

On substitution from (53) and (84) for Γ and $\mathfrak{G}$ and simplification, we find that if

$$\mathsf{G}=\tfrac{1}{4}\{(P^2-Q^2)^3[(p^2-q^2)^2-2p^2q^2]^2-2(P^2-Q^2)[(P^2-Q^2)^2-12P^2Q^2]p^2q^2(p^2-q^2)^2$$
$$-4(P^2-Q^2)(P^4+4P^2Q^2+Q^4)p^4q^4\} \quad (95)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{G}G\sin g\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

Fast diurnal term.

$$\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left[\frac{3P^2+Q^2}{2}\frac{dW_3}{d\epsilon'}+\frac{dW_3}{dN'}\right]$$

As the fast semi-diurnal is derived from the slow, so here also ; and if

$$\mathsf{G}_2=\tfrac{1}{4}\{Q^4(3P^2+Q^2)p^8+2Q^2(3P^2-Q^2)^2p^6q^2+9(P^2-Q^2)^3p^4q^4-2P^2(P^2-3Q^2)^2p^2q^6$$
$$-P^4(P^2+3Q^2)q^8\} \quad (97)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{G}_2G_2\sin g_2\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

Fortnightly term.

$$\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left(\frac{dW_0}{d\epsilon'}+\frac{dW_0}{dN'}\right)$$

If we take the term in W_0 which has 2h positive in the exponential, we have

$$\frac{dW_0}{dN'}=\tfrac{3}{2}\frac{\epsilon^{2h}}{2\sqrt{-1}}[8P^2Q^2(P^2-Q^2)^2p^6q^2+4[(P^2-Q^2)^2-2P^2Q^2]^2p^4q^4$$
$$+24P^2Q^2(P^2-Q^2)^2p^2q^6+8P^4Q^4q^8]$$

$$\frac{dW_0}{d\epsilon'}=-\tfrac{3}{2}\frac{\epsilon^{2h}}{2\sqrt{-1}}[4P^4Q^4p^8+16P^2Q^2(P^2-Q^2)^2p^6q^2+4[(P^2-Q^2)^2-2P^2Q^2]^2p^4q^4$$
$$+16P^2Q^2(P^2-Q^2)^2p^2q^6+4P^4Q^4q^8]$$

Then if these be added and simplified, it will be found that if

$$\mathsf{H}=\tfrac{3}{4}(p^4-q^4)[(p^4+q^4)P^2Q^2+2p^2q^2(P^2-Q^2)^2] \quad . \quad . \quad . \quad . \quad . \qquad (99)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{H}\mathrm{H}\sin 2\mathrm{h}\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (100)$$

Then collecting results from the seven equations (88, 90–2–4–6–8, 100),

$$n\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin i\{2\mathsf{F}_1\mathrm{F}_1\sin 2\mathrm{f}_1-2\mathsf{F}\mathrm{F}\sin 2\mathrm{f}-2\mathsf{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2+2\mathsf{G}_1\mathrm{G}_1\sin \mathrm{g}_1$$
$$-2\mathsf{G}\mathrm{G}\sin \mathrm{g}-2\mathsf{G}_2\mathrm{G}_2\sin \mathrm{g}_2-2\mathsf{H}\mathrm{H}\sin 2\mathrm{h}\} \quad . \qquad (101)$$

This is only a partial solution, and refers only to the action of the moon on her own tides; the part depending on the sun alone may be written down by symmetry.

The various functions of i and j here introduced admit of reduction to the following forms:—

$$\left.\begin{aligned}\Phi&=\tfrac{1}{4}\{\tfrac{1}{2}\sin^4 i+\tfrac{1}{2}\sin^2 j(4\sin^2 i-5\sin^4 i)+\tfrac{1}{2}\sin^4 j(1-5\sin^2 i+\tfrac{35}{8}\sin^4 i)\}\\ \tfrac{1}{2}\Gamma&=\tfrac{1}{4}\{\sin^2 i-\sin^4 i+\sin^2 j(1-\tfrac{11}{2}\sin^2 i+5\sin^4 i)\\ &\qquad -\sin^4 j(1-5\sin^2 i+\tfrac{35}{8}\sin^4 i)\}\end{aligned}\right\}\qquad (102)$$

$$\left.\begin{aligned}\mathsf{F}_1+\mathsf{F}_2&=\tfrac{1}{4}\cos j\{1-\tfrac{3}{4}\sin^2 i-\tfrac{3}{2}\sin^2 j(1-\tfrac{5}{4}\sin^2 i)\}\\ \mathsf{F}_1-\mathsf{F}_2&=\tfrac{1}{4}\cos i\{1-\tfrac{1}{4}\sin^2 i-2\sin^2 j(1-\tfrac{5}{8}\sin^2 i)+\tfrac{5}{8}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\}\\ \mathsf{G}_1+\mathsf{G}_2&=\tfrac{1}{4}\cos j\\ \mathsf{G}_1-\mathsf{G}_2&=\tfrac{1}{4}\cos i\{1+\tfrac{1}{2}\sin^2 i-\tfrac{1}{2}\sin^2 j(1+5\sin^2 i)-\tfrac{5}{4}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\}\\ \mathsf{F}&=\tfrac{1}{4}\cos i\{\tfrac{1}{2}\sin^2 i+\sin^2 j(1-\tfrac{5}{2}\sin^2 i)-\tfrac{5}{4}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\}\\ \mathsf{G}&=\tfrac{1}{4}\cos i\{1-\sin^2 i-\tfrac{7}{2}\sin^2 j(1-\tfrac{10}{7}\sin^2 i)+\tfrac{5}{2}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\}\\ \mathsf{H}&=\tfrac{1}{4}\cos j\{\tfrac{3}{4}\sin^2 i+\tfrac{3}{2}\sin^2 j(1-\tfrac{5}{4}\sin^2 i)\}\end{aligned}\right\}\qquad (103)$$

Φ_1, Φ_2, Γ_1, Γ_2 are given in equations (67), and ϕ and γ in equations (80).

The expressions for F_1 and F_2 are found by symmetry with those for $\mathfrak{F}_1$ and $\mathfrak{F}_2$, by interchanging i and j; the first of equations (62) then corresponds with the second of (103), and *vice versâ*.

From (103) it follows that

$$\mathsf{F}_1-\mathsf{F}_2+\tfrac{1}{2}(\mathsf{G}_1-\mathsf{G}_2)=\tfrac{3}{8}\cos i(1-\tfrac{3}{2}\sin^2 j)$$

and

$$\mathsf{F}+\tfrac{1}{2}\mathsf{G}=\tfrac{1}{8}\cos i(1-\tfrac{3}{2}\sin^2 j)$$

Also

$$\mathsf{F}_1+\mathsf{F}_2+\mathsf{G}_1+\mathsf{G}_2+\mathsf{H}=\tfrac{1}{2}\cos j$$

The complete solution of the problem may be collected from the equations (101) and (81).

In the case of the viscosity of the earth, and when the viscosity is small, we easily find the complete solution to be

$$n\frac{di}{dt}=\frac{\sin 4f}{\mathfrak{g}}\cdot\tfrac{1}{4}\sin i\cos i\left\{\tau^2(1-\tfrac{3}{2}\sin^2 j)+\tau'^2(1-\tfrac{3}{2}\sin^2 j')-\frac{2\Omega}{n}\tau^2\sec i\cos j\right.$$
$$\left.-\frac{2\Omega'}{n}\tau'^2\sec i\cos j'-\tau\tau'(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j')\right\} \quad . \quad . \quad (104)$$

This result agrees with that given in (83) of "Precession," when the squares of j and j' are neglected, and when Ω'/n is also neglected.

The preceding method of finding the tidal friction and change of obliquity is no doubt somewhat artificial, but as the principal object of the present paper is to discuss the secular changes in the elements of the satellite's orbit, it did not seem worth while to develop the disturbing function in such a form as would make it applicable both to the satellite and the planet; it seemed preferable to develop it for the satellite and then to adapt it for the case of the perturbation of the planet.

In long analytical investigations it is difficult to avoid mistakes; it may therefore give the reader confidence in the correctness of the results and process if I state that I have worked out the preceding values of di/dt and dn/dt independently, by means of the determination of the disturbing couples $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$. That investigation separated itself from the present one at the point where the products of the X′-Y′-Z′ functions and $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions are formed, for products of the form $Y'Z'\times\mathfrak{X}\mathfrak{Y}$ had there to be found. From this early stage the two processes are quite independent, and the identity of the results is confirmatory of both. Moreover, the investigation here presented reposes on the values found for dj/dt and $d\xi/dt$, hence the correctness of the result of the first problem here treated was also confirmed.

III.

THE PROPER PLANES OF THE SATELLITE, AND OF THE PLANET, AND THEIR SECULAR CHANGES.

§ 13. *On the motion of a satellite moving about a rigid oblate spheroidal planet, and perturbed by another satellite.*

The present problem is to determine the joint effects of the perturbing influence of the sun, and of the earth's oblateness upon the motion of the moon's nodes, and upon the inclination of the orbit to the ecliptic; and also to determine the effects on the

obliquity of the ecliptic and on the earth's precession. In the present configuration of the three bodies the problem presents but little difficulty, because the influence of oblateness on the moon's motion is very small compared with the perturbation due to the sun; on the other hand, in the case of Jupiter, the influence of oblateness is more important than that of solar perturbation. In each of these special cases there is an appropriate approximation which leads to the result. In the present problem we have, however, to obtain a solution, which shall be applicable to the preponderance of either perturbing cause, because we shall have to trace, in retrospect, the evanescence of the solar influence, and the increase of the influence of oblateness.

The lunar orbit will be taken as circular, and the earth or planet as homogeneous and of ellipticity $\mathfrak{e}$, so that the equation to its surface is

$$\rho=a\{1+\mathfrak{e}(\tfrac{1}{3}-\cos^2\theta)\}$$

The problem will be treated by the method of the disturbing function, and the method will be applied so as to give the perturbations both of the moon and earth.

First consider only the influence of oblateness.

Let ρ, θ be the coordinates of the moon, so that $\rho=c$ and $\cos\theta=\mathrm{M}_3$. Then in the formula (17) § 2, $r=c$ and $\frac{\sigma}{a}=\mathfrak{e}(\frac{1}{3}-\mathrm{M}_3^2)$, so that the disturbing function

$$\mathrm{W}=\tau\mathfrak{e}(\tfrac{1}{3}-\mathrm{M}_3^2)$$

This function, when suitably developed, will give the perturbation of the moon's motion due to oblateness, and the lunar precession and nutation of the earth.

Then by (21) we have

$$\mathrm{M}_3=\sin i\,[p^2\sin(l+N)-q^2\sin(l-N)]+\sin j\cos i\sin l.$$

Where l is the moon's longitude measured from the node, and N is the longitude of the ascending node of the lunar orbit measured from the descending node of the equator.

Then as we are only going to find secular inequalities, we may, in developing the disturbing function, drop out terms involving l; also we must write $N-\psi$ for N, because we cannot now take the autumnal equinox as fixed.

Then omitting all terms which involve l,

$$\mathrm{M}_3^2=\sin^2 i\,[\tfrac{1}{2}(p^4+q^4)-p^2q^2\cos 2(N-\psi)]+\tfrac{1}{2}\sin^2 j\cos^2 i$$
$$+\sin j\sin i\cos i\,[p^2-q^2]\cos(N-\psi)$$

Since $p=\cos\frac{1}{2}j$, $q=\sin\frac{1}{2}j$, we have

$$p^4+q^4=1-\tfrac{1}{2}\sin^2 j,\ p^2q^2=\tfrac{1}{4}\sin^2 j,\ p^2-q^2=\cos j$$

and

$$M_3^2 = \tfrac{1}{2}\sin^2 i(1-\tfrac{1}{2}\sin^2 j)+\tfrac{1}{2}\sin^2 j(1-\sin^2 i)$$
$$+\tfrac{1}{4}\sin 2i\sin 2j\cos(N-\psi)-\tfrac{1}{4}\sin^2 i\sin^2 j\cos 2(N-\psi)$$

Now

$$\tfrac{1}{2}(\sin^2 i+\sin^2 j)-\tfrac{3}{4}\sin^2 i\sin^2 j-\tfrac{1}{3}=-\tfrac{1}{3}(1-\tfrac{3}{2}\sin^2 i)(1-\tfrac{3}{2}\sin^2 j)$$

Wherefore

$$W=\tau\mathfrak{c}\{\tfrac{1}{3}(1-\tfrac{3}{2}\sin^2 i)(1-\tfrac{3}{2}\sin^2 j)-\tfrac{1}{4}\sin 2i\sin 2j\cos(N-\psi)$$
$$+\tfrac{1}{4}\sin^2 i\sin^2 j\cos 2(N-\psi)\}\quad . \quad (105)$$

This is the disturbing function.

Before applying it, we will assume that i and j are sufficiently small to permit us to neglect $\sin^2 i\sin^2 j$ compared with unity.

Then

$$\tfrac{1}{3}(1-\tfrac{3}{2}\sin^2 i)(1-\tfrac{3}{2}\sin^2 j)=\tfrac{1}{12}+\tfrac{1}{4}-\tfrac{1}{2}\sin^2 i-\tfrac{1}{2}\sin^2 j+\sin^2 i\sin^2 j-\tfrac{1}{4}\sin^2 i\sin^2 j$$
$$=\tfrac{1}{12}+\tfrac{1}{4}\cos 2i\cos 2j-\tfrac{1}{4}\sin^2 i\sin^2 j$$

Hence, when we neglect the terms in $\sin^2 i\sin^2 j$

$$W=\tfrac{1}{4}\tau\mathfrak{c}\{\tfrac{1}{3}+\cos 2i\cos 2j-\sin 2i\sin 2j\cos(N-\psi)\}\quad . \quad . \quad . \quad . \quad (106)$$

Then since this disturbing function does not involve the epoch or χ, we have by (13), (14), and (18)

$$-\frac{\xi}{k}\sin j\frac{dj}{dt}=\frac{dW}{dN},\quad \frac{\xi}{k}\sin j\frac{dN}{dt}=\frac{dW}{dj},\quad -n\sin i\frac{di}{dt}=\frac{dW}{d\psi},\quad n\sin i\frac{d\psi}{dt}=\frac{dW}{di}$$

Thus as far as concerns the influence of the oblateness on the moon, and the reaction of the moon on the earth,

$$\left.\begin{aligned}
\frac{\xi}{k}\sin j\frac{dj}{dt}&=-\tfrac{1}{4}\tau\mathfrak{c}\sin 2i\sin 2j\sin(N-\psi)\\
\frac{\xi}{k}\sin j\frac{dN}{dt}&=-\tfrac{1}{2}\tau\mathfrak{c}\{\cos 2i\sin 2j+\sin 2i\cos 2j\cos(N-\psi)\}\\
n\sin i\frac{di}{dt}&=\tfrac{1}{4}\tau\mathfrak{c}\sin 2i\sin 2j\sin(N-\psi)\\
n\sin i\frac{d\psi}{dt}&=-\tfrac{1}{2}\tau\mathfrak{c}\{\sin 2i\cos 2j+\cos 2i\sin 2j\cos(N-\psi)\}
\end{aligned}\right\}\quad . \quad . \quad . \quad (107)$$

If there be no other disturbing body, and if we refer the motion to the invariable plane of the system, we must always have $N=\psi$.

In this case the first and third of (107) become

$$\frac{dj}{dt}=\frac{di}{dt}=0$$

and the second and fourth become

$$\frac{\xi}{k}\sin j\frac{dN}{dt}=n\sin i\frac{d\psi}{dt}=-\tfrac{1}{2}\tau\mathfrak{k}\sin 2(i+j)$$

But ξ/k is proportional to the moment of momentum of the orbital motion, and n is proportional to the moment of momentum of the earth's rotation, and so by the definition of the invariable plane

$$\frac{\xi}{k}\sin j=n\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

Wherefore $\frac{dN}{dt}=\frac{d\psi}{dt}$, and it follows that the two nodes remain coincident. This result is obviously correct.

In the present case, however, there is another disturbing body, and we must now consider

The perturbing influence of the sun.

Accented symbols will here refer to the elements of the solar orbit.

We might of course form the disturbing function, but it is simpler to accept the known results of lunar theory; these are that the inclination of the lunar orbit to the ecliptic remains constant, whilst the nodes regrede with an angular velocity $\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\left[1-\frac{3}{8}\frac{\Omega'}{\Omega}\right]\Omega\cos j$.

Now $\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\Omega=\frac{1}{2}(\frac{3}{2}\Omega'^2)\times\frac{1}{\Omega}=\frac{1}{2}\frac{\tau'}{\Omega}$ in our notation. Hence I shall write $\frac{1}{2}\frac{\tau'}{\Omega}$ for $\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\left[1-\frac{3}{8}\frac{\Omega'}{\Omega}\right]\Omega$, although if necessary (in Part IV.) I shall use the more accurate formula for numerical calculation.

For the solar precession and nutation we may obtain the results from (107) by putting $j=0$, and τ' for τ.

Thus for the solar effects we have

$$\left.\begin{aligned}\frac{dj}{dt}&=0\\ \frac{dN}{dt}&=-\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j\\ \frac{di}{dt}&=0\\ n\sin i\frac{d\psi}{dt}&=-\tfrac{1}{2}\tau'\mathfrak{k}\sin 2i\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)^*$$

* The following seems worthy of remark. By the last of (109) we have $d\psi/dt=-\tau'\mathfrak{k}\cos i/n$.

In this formula $\mathfrak{k}$ is the precessional constant, because the earth is treated as homogeneous.

The full expression for the precessional constant is $(2C-A-B)/2C$, where A, B, C are the three principal moments of inertia.

Now if we regard the earth and moon as being two particles rotating with an angular velocity Ω about

Then when the system is perturbed both by the oblateness of the earth and by the sun, we have from (107) and (109),

$$\left.\begin{aligned}
\frac{\xi}{k}\sin j\,\frac{dj}{dt}&=-\tfrac{1}{4}\tau\mathfrak{c}\sin 2i\sin 2j\sin(N-\psi)\\
\frac{\xi}{k}\sin j\,\frac{dN}{dt}&=-\tfrac{1}{2}\tau\mathfrak{c}\{\cos 2i\sin 2j+\sin 2i\cos 2j\cos(N-\psi)\}-\tfrac{1}{4}\frac{\tau'}{\Omega}\frac{\xi}{k}\sin 2j\\
n\sin i\,\frac{di}{dt}&=\ \ \tfrac{1}{4}\tau\mathfrak{c}\sin 2i\sin 2j\sin(N-\psi)\\
n\sin i\,\frac{d\psi}{dt}&=-\tfrac{1}{2}\tau\mathfrak{c}\{\sin 2i\cos 2j+\cos 2i\sin 2j\cos(N-\psi)\}-\tfrac{1}{2}\tau'\mathfrak{c}\sin 2i
\end{aligned}\right\}\quad(110)$$

The second pair of equations is derivable from the first by writing i for j and j for i; N for ψ and ψ for N; n for ξ/k; n for Ω; and $\frac{1}{2}\mathfrak{c}$ for $\frac{1}{4}$ in the term in τ'.

The first pair of equations may be put into the form

$$\cos 2j\,\frac{d(2j)}{dt}=-\frac{k}{\xi}\tau\mathfrak{c}\sin 2i\cos j\cos 2j\sin(N-\psi)$$

$$\sin 2j\,\frac{dN}{dt}=-\frac{k}{\xi}\tau\mathfrak{c}\{\cos 2i\cos j\sin 2j+\sin 2i\cos i\cos 2i\cos(N-\psi)\}-\tfrac{1}{2}\frac{\tau'}{\Omega}\sin 2j\cos j$$

Now let

$$\left.\begin{aligned}
y&=\tfrac{1}{2}\sin 2j\sin N, & \eta&=\tfrac{1}{2}\sin 2i\sin\psi\\
z&=\tfrac{1}{2}\sin 2j\cos N, & \zeta&=\tfrac{1}{2}\sin 2i\cos\psi
\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ .\quad(111)$$

Therefore

$$2\frac{dz}{dt}=\cos N\cos 2j\,\frac{d(2j)}{dt}-\sin N\sin 2j\,\frac{dN}{dt}$$

$$=\frac{k}{\xi}\tau\mathfrak{c}[\cos j\cos 2j.2\eta+\cos 2i\cos j.2y]+\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j.2y$$

or

$$\frac{dz}{dt}=\left(\frac{k\tau\mathfrak{c}}{\xi}\cos 2i\cos j+\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j\right)y+\frac{k\tau\mathfrak{c}}{\xi}\cos j\cos 2j.\eta$$

Again

$$2\frac{dy}{dt}=\sin N\cos 2j\,\frac{d(2j)}{dt}+\cos N\sin 2j\,\frac{dN}{dt}$$

$$=-\frac{k\tau\mathfrak{c}}{\xi}[\cos j\cos 2j.2\zeta+\cos 2i\cos j.2z]-\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j.2z$$

their common centre of inertia, then the three principal moments of inertia of the system are $Mmc^2/(M+m)$, $Mmc^2/(M+m)$, 0, and therefore the precessional constant of the system is $\frac{1}{2}$. Thus the formula for dN/dt is precisely analogous to that for $d\psi/dt$, each of them being equal to $\tau'\times$ prec. const. $\times$ cos inclin. $\div$ rotation.

or

$$\frac{dy}{dt} = -\left(\frac{k\tau\mathfrak{k}}{\xi}\cos 2i\cos j + \tfrac{1}{2}\frac{\tau'}{\Omega}\cos j\right)z - \frac{k\tau\mathfrak{k}}{\xi}\cos j\cos 2j.\zeta$$

Now let

$$\left.\begin{array}{ll} a_1 = \dfrac{k\tau\mathfrak{k}}{\xi} & a_2 = \tfrac{1}{2}\dfrac{\tau'}{\Omega} \\ b_1 = \dfrac{\tau\mathfrak{k}}{n} & b_2 = \dfrac{\tau'\mathfrak{k}}{n} \end{array}\right\} \quad \ldots \ldots \ldots \quad (112)$$

And we have

$$\left.\begin{array}{l} \dfrac{dz}{dt} = \quad (a_1\cos 2i\cos j + a_2\cos j)y + a_1\cos j\cos 2j.\eta \\ \dfrac{dy}{dt} = -(a_1\cos 2i\cos j + a_2\cos j)z - a_1\cos j\cos 2j.\zeta \end{array}\right\} \quad \ldots \ldots \quad (113)$$

and by symmetry from the two latter of (110)

$$\left.\begin{array}{l} \dfrac{d\zeta}{dt} = \quad (b_1\cos 2j\cos i + b_2\cos i)\eta + b_1\cos i\cos 2i.y \\ \dfrac{d\eta}{dt} = -(b_1\cos 2j\cos i + b_2\cos i)\zeta - b_1\cos i\cos 2i.z \end{array}\right\} \quad \ldots \ldots \quad (114)$$

These four simultaneous differential equations have to be solved.

The a's and b's are constant, and if it were not for the cosines on the right the equations would be linear and easily soluble.

It has already been assumed that i and j are not very large, hence it would require large variations of i and j to make considerable variations in the coefficients, I shall therefore substitute for i and j, as they occur explicitly, mean values i_0 and j_0; and this procedure will be justifiable unless it be found subsequently that i and j vary largely.

Then let

$$\left.\begin{array}{ll} \alpha = a_1\cos 2i_0\cos j_0 + a_2\cos j_0 & \beta = b_1\cos 2j_0\cos i_0 + b_2\cos i_0 \\ \mathrm{a} = a_1\cos j_0\cos 2j_0 & \mathrm{b} = b_1\cos i_0\cos 2i_0 \end{array}\right\} \quad \ldots \quad (115)$$

(Hereafter i and j will be treated as small and the cosines as unity.)

Then

$$\left.\begin{array}{l} \dfrac{dz}{dt} = \quad \alpha y + \mathrm{a}\eta \\ \dfrac{dy}{dt} = -\alpha z - \mathrm{a}\zeta \\ \dfrac{d\zeta}{dt} = \quad \beta\eta + \mathrm{b}y \\ \dfrac{d\eta}{dt} = -\beta\zeta - \mathrm{b}z \end{array}\right\} \quad \ldots \ldots \ldots \quad (116)$$

These equations suggest the solutions

$$z=\Sigma L\cos(\kappa t+\mathrm{m}) \qquad \zeta=\Sigma L'\cos(\kappa t+\mathrm{m})$$
$$y=\Sigma L\sin(\kappa t+\mathrm{m}) \qquad \eta=\Sigma L'\sin(\kappa t+\mathrm{m})$$

Then substituting in (116), we must have

$$-L\kappa=\alpha L+\mathrm{a}L'\,;\ -L'\kappa=\beta L'+\mathrm{b}L$$

Wherefore

$$\frac{L'}{L}=-\frac{\kappa+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa+\beta}$$

and

$$(\kappa+\alpha)(\kappa+\beta)-\mathrm{ab}=0 \text{ or } \kappa^2+\kappa(\alpha+\beta)+\alpha\beta-\mathrm{ab}=0.$$

This quadratic equation has two real roots (κ_1 and κ_2 suppose), because $(\alpha+\beta)^2-4(\alpha\beta-\mathrm{ab})=(\alpha-\beta)^2+4\mathrm{ab}$ is essentially positive.

Then let

$$\left.\begin{aligned}\kappa_1+\kappa_2&=-(\alpha+\beta)\\ \kappa_1-\kappa_2&=-\{(\alpha-\beta)^2+4\mathrm{ab}\}^{\frac{1}{2}}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(117)$$

And the solution is

$$\left.\begin{aligned}\tfrac{1}{2}\sin 2j\cos N&=z=L_1\cos(\kappa_1t+\mathrm{m}_1)+L_2\cos(\kappa_2t+\mathrm{m}_2)\\ \tfrac{1}{2}\sin 2j\sin N&=y=L_1\sin(\kappa_1t+\mathrm{m}_1)+L_2\sin(\kappa_2t+\mathrm{m}_2)\\ \tfrac{1}{2}\sin 2i\cos\psi&=\zeta=L_1'\cos(\kappa_1t+\mathrm{m}_1)+L_2'\cos(\kappa_2t+\mathrm{m}_2)\\ \tfrac{1}{2}\sin 2i\sin\psi&=\eta=L_1'\sin(\kappa_1t+\mathrm{m}_1)+L_2'\sin(\kappa_2t+\mathrm{m}_2)\end{aligned}\right\}\quad\ldots\quad(118)$$

where

$$\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_1+\beta}\,;\ \frac{L_2'}{L_2}=-\frac{\kappa_2+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_2+\beta}$$

From these equations we have

$$\tfrac{1}{4}\sin^2 2j=L_1^2+L_2^2+2L_1L_2\cos\left[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2\right]$$
$$\tfrac{1}{4}\sin^2 2i=L_1'+L_2'+2L_1'L_2'\cos\left[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2\right]$$

From this we see that $\sin 2j$ oscillates between $2(L_1+L_2)$ and $2(L_1\backsim L_2)$, and $\sin 2i$ between $2(L_1'+L_2')$ and $2(L_1'\backsim L_2')$.

Let us change the constants introduced by integration, and write $L_1=\frac{1}{2}\sin 2j_0$, $L_2'=\frac{1}{2}\sin 2i_0$.

Then our solution is

$$
\left.
\begin{aligned}
\sin 2j \cos N &= \sin 2j_0 \cos(\kappa_1 t+\mathrm{m}_1) - \frac{\mathrm{a}}{\kappa_2+\alpha} \sin 2i_0 \cos(\kappa_2 t+\mathrm{m}_2) \\
\sin 2j \sin N &= \sin 2j_0 \sin(\kappa_1 t+\mathrm{m}_1) - \frac{\mathrm{a}}{\kappa_2+\alpha} \sin 2i_0 \sin(\kappa_2 t+\mathrm{m}_2) \\
\sin 2i \cos \psi &= -\frac{\kappa_1+\alpha}{\mathrm{a}} \sin 2j_0 \cos(\kappa_1 t+\mathrm{m}_1) + \sin 2i_0 \cos(\kappa_2 t+\mathrm{m}_2) \\
\sin 2i \sin \psi &= -\frac{\kappa_1+\alpha}{\mathrm{a}} \sin 2j_0 \sin(\kappa_1 t+\mathrm{m}_1) + \sin 2i_0 \sin(\kappa_2 t+\mathrm{m}_2)
\end{aligned}
\right\} \qquad (119)
$$

From this it follows that

$$
\begin{aligned}
\sin 2i \sin 2j \cos(N-\psi) = &-\frac{\kappa_1+\alpha}{\mathrm{a}} \sin^2 2j_0 - \frac{\mathrm{a}}{\kappa_2+\alpha} \sin^2 2i_0 \\
&+\left(1+\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right) \sin 2i_0 \sin 2j_0 \cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2]
\end{aligned}
$$

$$
\sin 2i \sin 2j \sin(N-\psi) = \left(1-\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right) \sin 2i_0 \sin 2j_0 \sin[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2]
$$

Now

$$
\begin{aligned}
(\kappa_1+\alpha)(\kappa_2+\alpha) &= -(\kappa_1+\alpha)(\kappa_1+\beta) = -\mathrm{ab} \\
\kappa_1+\kappa_2+2\alpha &= \alpha-\beta
\end{aligned}
$$

Therefore

$$
\left.
\begin{aligned}
\sin 2i \sin 2j \cos(N-\psi) = &-\frac{1}{\kappa_2+\alpha}\{\mathrm{a} \sin^2 2i_0 - \mathrm{b} \sin^2 2j_0 \\
&-(\alpha-\beta) \sin 2i_0 \sin 2j_0 \cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2]\} \\
\sin 2i \sin 2j \sin(N-\psi) = &-\frac{\kappa_1-\kappa_2}{\kappa_2+\alpha} \sin 2i_0 \sin 2j_0 \sin[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2]
\end{aligned}
\right\} \qquad (120)
$$

From (120) it is clear that the nodes of the lunar orbit will oscillate about the equinoctial line, if

$$
\mathrm{a} \sin^2 2i_0 \sim \mathrm{b} \sin^2 2j_0 \text{ be greater than } (\alpha-\beta) \sin 2i_0 \sin 2j_0,
$$

but will rotate (although not uniformly) if the former be less than the latter.

With the present configuration of the earth and moon

$$
\mathrm{a} \sin^2 2i_0 \sim \mathrm{b} \sin^2 2j_0 \text{ is very small compared with } (\alpha-\beta) \sin 2i_0 \sin 2j_0,
$$

and the nodes of the lunar orbit revolve very nearly uniformly on the ecliptic; also the inclination of the orbit varies very slightly, as the nodes revolve.

In the investigation in Part II. the secular rate of change in the inclination of the

lunar orbit has been found, on the assumption that the nodes of the lunar orbit rotate uniformly.

It is intended to trace the effects of tidal friction on the earth and moon retrospectively. In the course of the solution the importance of the solar perturbation of the moon, relatively to the influence of the earth's oblateness, will wane; the nodes will cease to revolve uniformly, and the inclination of the lunar orbit and of the equator to the ecliptic will be subject to nutation. The differential equations of Part II. will then cease to be applicable, and new ones will have to be found.

The problem is one of such complication, that I have thought it advisable only to attempt to obtain a solution on the hypothesis of the smallness both of the obliquity and of the inclination of the orbit to the plane of reference or the ecliptic. It seems best however to give the preceding investigation, although it is more accurate than the solution subsequently used.*

The first step towards this further consideration is to obtain a clear idea of the nature of the motions represented by the analytical solutions (118) or (119) of the present problem.

Assuming then i and j to be small, we have from (112) and (115)

$$\alpha=a_1+a_2,\quad \mathrm{a}=a_1,\quad \beta=b_1+b_2,\quad \mathrm{b}=b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$\left.\begin{aligned} j\cos N&=L_1\cos(\kappa_1 t+\mathrm{m}_1)+L_2\cos(\kappa_2 t+\mathrm{m}_2)\\ j\sin N&=L_1\sin(\kappa_1 t+\mathrm{m}_1)+L_2\sin(\kappa_2 t+\mathrm{m}_2)\\ i\cos\psi&=L_1'\cos(\kappa_1 t+\mathrm{m}_1)+L_2'\cos(\kappa_2 t+\mathrm{m}_2)\\ i\sin\psi&=L_1'\sin(\kappa_1 t+\mathrm{m}_1)+L_2'\sin(\kappa_2 t+\mathrm{m}_2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

Take a set of rectangular axes; let the axis of x' pass through the fixed point in the ecliptic from which longitudes are measured, let the axis of z' be drawn perpendicular to the ecliptic northwards, and let the rotation from x' to y' be positive, and therefore consentaneous with the moon's orbital motion.

Then N is the longitude of the ascending node of the lunar orbit, and therefore the direction cosines of the normal to the lunar orbit drawn northwards are,

$\sin j\cos(N-\frac{1}{2}\pi)$, $\sin j\sin(N-\frac{1}{2}\pi)$, $\cos j$; or since j is small, $j\sin N$, $-j\cos N$, 1.

And ψ is the longitude of the descending node of the equator, and therefore the direction cosines of the earth's axis, drawn northwards are,

$\sin i\cos(\psi+\frac{1}{2}\pi)$, $\sin i\sin(\psi+\frac{1}{2}\pi)$, $\cos i$; or since i is small, $-i\sin\psi$, $i\cos\psi$, 1.

Now draw a sphere of unit radius, with the origin as centre; draw a tangent plane

* See the foot-note to § 18 for a comparison of these results with those ordinarily given.

to it at the point where the axis of z' meets the sphere, and project on this plane the poles of the lunar orbit and of the earth. We here in fact map the motion of the two poles on a tangent plane to the celestial sphere. Let x', y' be a pair of axes in this plane parallel to our previous x', y'; and let x', y' be the coordinates of the pole of the lunar orbit, and ξ', η' be the coordinates of the earth's pole. Then

$$x'=j\sin N,\ y'=-j\cos N;\ \xi'=-i\sin\psi,\ \eta'=i\cos\psi\ . \quad . \quad . \quad . \qquad (123)$$

Let x, y, ξ, η be the coordinates of these same points referred to another pair of rectangular axes in this plane, inclined at an angle ϕ to the axes x', y'.

Then

$$\begin{aligned} x &= \ \ x'\cos\phi+y'\sin\phi\ ,\quad \xi= \ \ \xi'\cos\phi+\eta'\sin\phi \\ y &= -x'\sin\phi+y'\cos\phi\ ,\quad \eta=-\xi'\sin\phi+\eta'\cos\phi \end{aligned}$$

From (123) and (118) we have therefore

$$\left.\begin{aligned} x &= \ \ L_1\ \sin(\kappa_1 t+\mathrm{m}_1-\phi)+L_2\ \sin(\kappa_2 t+\mathrm{m}_2-\phi) \\ y &= -L_1\ \cos(\kappa_1 t+\mathrm{m}_1-\phi)-L_2\ \cos(\kappa_2 t+\mathrm{m}_2-\phi) \\ \xi &= -L_1'\sin(\kappa_1 t+\mathrm{m}_1-\phi)-L_2'\sin(\kappa_2 t+\mathrm{m}_2-\phi) \\ \eta &= \ \ L_1'\cos(\kappa_1 t+\mathrm{m}_1-\phi)+L_2'\cos(\kappa_2 t+\mathrm{m}_2-\phi) \end{aligned}\right\}$$

Now suppose the new axes to rotate with an angular velocity κ_2, and that $\phi=\kappa_2 t+\mathrm{m}_2$.

$$\begin{aligned} x &= L_1\sin[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2] \\ y+L_2 &= -L_1\cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m} \\ \xi &= -L_1'\sin[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m} \\ \eta-L_2' &= L_1'\cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2] \end{aligned} \quad . \quad (12$$

These four equations represent that each pole describes a circle, relatively to tl rotating axes, with a negative angular velocity (because $\kappa_1-\kappa_2$ is negative). Tl centres of the circles are on the axis of y. The ratio

$$\frac{\text{distance of centre of terrestrial circle}}{\text{distance of centre of lunar circle}}=\frac{L_2'}{-L_2}=\frac{\kappa_2+\alpha}{\mathrm{a}}=\frac{\mathrm{b}}{\kappa_2+\beta} \quad . \quad . \quad . \quad (12$$

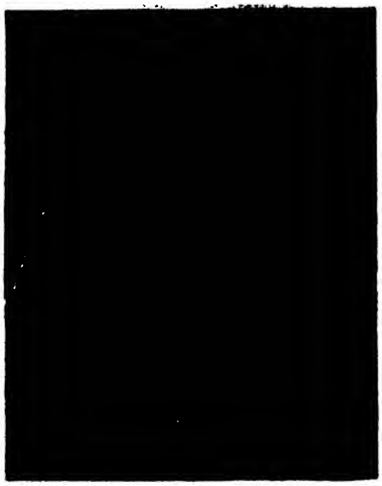
the distances being measured from the pole of the ecliptic. And the ratio

$$\frac{\text{radius of terrestrial circle}}{\text{radius of lunar circle}}=\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_1+\beta} \quad . \quad . \quad . \quad . \quad . \quad (126)$$

According to the definitions adopted in (117) of κ_1 and κ_2, $(\kappa_1+\alpha)/\mathrm{a}$ is negative and $(\kappa_2+\alpha)/\mathrm{a}$ is positive; hence L_1 has the same sign as L_1', and L_2 has the opposite sign from L_2'. When $t=-(\mathrm{m}_1-\mathrm{m}_2)/(\kappa_1-\kappa_2)$, we have

$$x=0,\ y=(-L_2)-L_1,\ \xi=0,\ \eta=L_2'+L_1'$$

Fig. 6.



In fig. 6 let Ox, Oy be the rotating axes, which revolve with a negative rotation equal to κ_2, which is negative. Let M be the centre of the lunar circle, and Q of the terrestrial circle. Then we see that L and P must be simultaneous positions of the two poles, which revolve round their respective circles with an angular velocity $\kappa_2-\kappa_1$, in the direction of the arrows.

M and Q are the poles of two planes, which may be appropriately called the proper planes of the moon and the earth. These proper planes are inclined at a constant angle to one another and to the ecliptic, and have a common node on the ecliptic, and a uniform slow negative precession relatively to the ecliptic.

The lunar orbit and the equator are inclined at constant angles to the lunar and terrestrial proper planes respectively, and the nodes of the orbit, and of the equator regrede uniformly on the respective proper planes.

In the 'Mécanique Céleste' (livre vii., chap. 2, sec. 20) LAPLACE refers to the proper plane of the lunar orbit, but the corresponding inequality of the earth is ordinarily referred to as the 19-yearly nutation. It will be proved later, that the above results are identical with those ordinarily given.

Suppose then that

I=the inclination of the earth's proper plane to the ecliptic

J=the inclination of the lunar orbit to its proper plane

$\mathrm{I}_{,}$=the inclination of the equator to the earth's proper plane

$\mathrm{J}_{,}$=the inclination of the moon's proper plane to the ecliptic

Then (127)

$$\mathrm{J}=L_1,\ \mathrm{I}=L_2',\ \mathrm{I}_{,}=L_1',\ \mathrm{J}_{,}=-L_2$$

and by (125–6)

$$\mathrm{I}_{,}=-\frac{\kappa_1+\alpha}{\mathrm{a}}\mathrm{J}=-\frac{\mathrm{b}}{\kappa_1+\beta}\mathrm{J}\ ;\ \mathrm{J}_{,}=\frac{\mathrm{a}}{\kappa_2+\alpha}\mathrm{I}=\frac{\kappa_2+\beta}{\mathrm{b}}\mathrm{I}$$

Thus I and J are the two constants introduced in the integration of the simultaneous differential equations (116).

It is interesting to examine the physical meaning of these results, and to show how the solution degrades into the two limiting cases, viz.: where the planet is spherical, and where the sun's influence is non-existent.

Let $\mathfrak{n}$ be the speed of motion of the nodes, when the ellipticity of the planet is zero.

Let $\mathfrak{l}$ be the purely lunar precession, or the precession when the solar influence is nil.

Let $\mathfrak{m}$ be the ratio of the moment of momentum of the earth's rotation to that of the orbital motion of the two bodies round their common centre of inertia.

Then

$$\mathfrak{n}=\tfrac{1}{2}\frac{\tau'}{\Omega},\quad \mathfrak{l}=\frac{\tau\mathfrak{c}}{n},\quad \mathfrak{m}=\frac{kn}{\xi}$$

Then by (121) and (115) we have

$$\alpha=\mathfrak{m}\mathfrak{l}+\mathfrak{n},\quad \mathrm{a}=\mathfrak{m}\mathfrak{l},\quad \beta=\mathfrak{l}+\frac{\tau'\mathfrak{c}}{n},\quad \mathrm{b}=\mathfrak{l}$$

First suppose that $\mathfrak{n}$ *is large compared with* $\mathfrak{l}$.

This is the case at present with the earth and moon, because the speed of motion of the moon's nodes is very great compared with the speed of the purely lunar precession.

Then a, β, b are small compared with α.

Therefore by (117)

$$\kappa_1-\kappa_2=-\alpha+\beta,\ \kappa_1+\kappa_2=-\alpha-\beta$$

and

$$\kappa_1=-\alpha,\ \kappa_2=-\beta$$

Therefore

$$\frac{\mathfrak{b}}{\kappa_1+\beta}=\frac{\mathfrak{b}}{\alpha-\beta}=\frac{\mathfrak{l}}{\mathfrak{n}-(1-\mathfrak{m})\mathfrak{l}-\frac{\tau'\mathfrak{t}}{n}}=\frac{\mathfrak{l}}{\mathfrak{n}} \text{ approximately}$$

$$\frac{\mathfrak{a}}{\kappa_2+\alpha}=\frac{\mathfrak{a}}{\alpha-\beta}=\mathfrak{m}\frac{\mathfrak{l}}{\mathfrak{n}} \text{ approximately}$$

$$\kappa_2=-\frac{\tau+\tau'}{n}\mathfrak{t},\ \kappa_2-\kappa_1=\mathfrak{n} \text{ approximately}$$

And by (127)

$$\mathrm{I}_{,}=\frac{\mathfrak{l}}{\mathfrak{n}}\mathrm{J},\quad \mathrm{J}_{,}=\mathfrak{m}\frac{\mathfrak{l}}{\mathfrak{n}}\mathrm{I}$$

Now we have shown above that $-\kappa_2$ is the common angular velocity of the pair of proper planes, and the above results show that it is in fact the luni-solar precession.

$\kappa_2-\kappa_1$ is the angular velocity of the two nodes on their proper planes, and it is nearly equal to $\mathfrak{n}$.

The ratio of the amplitude of the 19-yearly nutation to the inclination of the lunar orbit is $\mathfrak{l}/\mathfrak{n}$.

The ratio of the inclination of the lunar proper plane to the obliquity of the ecliptic is $\mathfrak{ml}/\mathfrak{n}$.

In this case, therefore, the lunar proper plane is inclined at a small angle to the ecliptic, and if the earth were spherical would be identical with the ecliptic.

Secondly, suppose that $\mathfrak{n}$ *is small compared with* $\mathfrak{l}$.

Then *à fortiori* $\frac{\tau'\mathfrak{t}}{n}$ is small compared with $\mathfrak{l}$. Hence we may put $\beta=\mathfrak{b}$.

Therefore

$$\kappa_2-\kappa_1=\sqrt{(\alpha-\beta)^2+4\mathfrak{ab}}=\mathfrak{a}+\mathfrak{b}+\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\mathfrak{n}, \text{ nearly}$$

$$=(\mathfrak{m}+1)\mathfrak{l}+\frac{\mathfrak{m}-1}{\mathfrak{m}+1}\mathfrak{n}$$

$$\kappa_2+\kappa_1=-(\mathfrak{m}+1)\mathfrak{l}-\mathfrak{n}$$

$$\kappa_2=-\frac{\mathfrak{n}}{\mathfrak{m}+1},\quad \kappa_1=-(\mathfrak{m}+1)\mathfrak{l}-\frac{\mathfrak{m}}{\mathfrak{m}+1}\mathfrak{n}$$

$$\frac{\kappa_2+\beta}{\mathfrak{b}}=1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}};\quad -\frac{\kappa_1+\alpha}{\mathfrak{a}}=\frac{1}{\mathfrak{m}}\left(1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\right)$$

Therefore

$$\mathrm{I}_{,}=\frac{1}{\mathfrak{m}}\left(1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\right)\mathrm{J},\quad \mathrm{J}_{,}=\left(1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\right)\mathrm{I}$$

From the last of these,

$$\mathrm{I}-\mathrm{J}_{,}=\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\mathrm{I}$$

$-\kappa_2$ is the precession of the system of proper planes, and the above results show that the solar precession of the planet and satellite together, considered as one system, is one $(\mathfrak{m}+1)^{\text{th}}$ of the angular velocity which the nodes of the satellite would have, if the planet were spherical.

$\kappa_2-\kappa_1$ is the lunar precession of the earth which goes on within the system, and it is approximately the same as though the sun did not exist. (Compare the second and fourth of (107) with $N=\psi$, and use (108)).

It also appears that the lunar proper plane is inclined to the planet's proper plane at a small angle the ratio of which to the inclination of the earth's proper plane to the ecliptic is equal to one $(\mathfrak{m}+1)^{\text{th}}$ part of $\mathfrak{n}/\mathfrak{l}$.

If $\mathfrak{n}$ and $\mathfrak{l}$ are of approximately equal speeds the proper plane of the moon will neither be very near the ecliptic, nor very near the earth's proper plane. The results do not then appear to be reducible to very simple forms; nor are the angular velocities κ_2 and $\kappa_2-\kappa_1$ so easily intelligible, each of them being a sort of compound precession.

If the solar influence were to wane, M and Q, the poles of the proper planes, would approach one another, and ultimately become identical. The two planes would have then become the invariable plane of the system; and the two circles would be concentric and their radii would be inversely proportional to the two moments of momentum (whose ratio is $\mathfrak{m}$).

Now in the problem which is to be here considered the solar influence will in effect wane, because the effect of tidal friction is, in retrospect, to bring the moon nearer and nearer to the earth, and to increase the ellipticity of the earth's figure; hence the relative importance of the solar influence diminishes.

We now see that the problem to be solved is to trace these proper planes, from their present condition when one is nearly identical with the ecliptic and the other is the mean equator, backwards until they are both sensibly coincident with the equator.

We also see that the present angular velocity of the moon's nodes on the ecliptic is analogous to and continuous with the purely lunar precession on the invariable plane of the moon-earth system; and that the present luni-solar precession is analogous to and continuous with a slow precessional motion of the same invariable plane.

Analytically the problem is to trace the secular changes in the constants of integration, when α, a, β, b, instead of being constant, are slowly variable under the influence of tidal friction, and when certain other small terms, also due to tides, are added to the differential equations of motion.

§ 14. *On the small terms in the equations of motion due directly to tidal friction.*

The first step is the formation of the disturbing function.

As we shall want to apply the function both to the case of the earth and to that of the moon, it will be necessary to measure longitudes from a fixed point in the ecliptic; also we must distinguish between the longitude of the equinox and the angle χ, as they enter in the two capacities (viz.: in the X′Y′ and $\mathfrak{X}\mathfrak{Y}$ functions); thus the N and N' of previous developments must become $N-\psi$, $N'-\psi'$; ϵ, ϵ' must become $\epsilon-\psi$, $\epsilon'-\psi'$; and $2(\chi-\chi')$ must be introduced in the arguments of the trigonometrical terms in the semi-diurnal terms, and $\chi-\chi'$ in the diurnal ones.

The disturbing function must be developed so that it may be applicable to the cases either where Diana, the tide-raiser, is or is not identical with the moon; but as we are only going to consider secular inequalities, all those terms which depend on the longitudes of Diana or the moon may be dropped.

In the previous development of Part II. we had terms whose arguments involved $\epsilon-\epsilon'$; in the present case this ought to be written $(\Omega t+\epsilon-\psi)-(\Omega' t+\epsilon'-\psi')$, for which it is, in fact, only an abbreviation.

Now a term involving this expression can only give rise to secular inequalities, in the case where Diana is identical with the moon; and as we shall never want to differentiate the disturbing function with regard to Ω', we may in the present development drop the Ωt and $\Omega' t$.

Having made these preliminary explanations, we shall be able to use previous results for the development of the disturbing function. The work will be much abridged by the treatment of i, j, i', j' as small.

Unaccented symbols refer to the elements of the orbit of the tide-raiser Diana, or (in the case of i, χ, ψ) to the earth as a tidally distorted body; accented symbols refer to the elements of the orbit of the perturbed satellite, or to the earth as a body whose rotation is perturbed.

Then since i, i' and j, j' are to be treated as small, (22) becomes

$$\left.\begin{array}{l} \left.\begin{array}{l}\varpi\\ \underline{\varpi}\end{array}\right\} = Pp-Qqe^{\pm(N-\psi)} = 1-\frac{1}{8}i^2-\frac{1}{8}j^2-\frac{1}{4}ije^{\pm(N-\psi)} \\ \left.\begin{array}{l}\kappa\\ \kappa\end{array}\right\} = Qp+Pqe^{\pm(N-\psi)} = \frac{1}{2}i+\frac{1}{2}je^{\pm(N-\psi)} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (128)$$

The same quantities when accented are equal to the same quantities when i, j, N, ψ are accented.

Then referring to the development in § 5 of the disturbing function, we see that, for the same reasons as before, we need only consider products of terms of the same kind in the sets of products of the type $X'Y'\times\mathfrak{X}\mathfrak{Y}$. Hence the disturbing function W is the sum of the three expressions (37–9) multiplied by $\tau\tau'/\mathfrak{g}$. Now since we only wish

to develop the expression as far as the squares of i and j, we may at once drop out all those terms in these expressions, in which κ occurs raised to a higher power than the second. This at once relieves us of the sidereal and fast semi-diurnal terms, the fast diurnal and the true fortnightly term. We are, however, left with one part of $\frac{3}{2}(\frac{1}{3}-Z'^2)(\frac{1}{3}-\mathfrak{Z}^2)$, which is independent of the moon's longitude and of the earth's rotation; this part represents the permanent increase of ellipticity of the earth, due to Diana's attraction, and to that part of the tidal action which depends on the longitude of the nodes, in which the tides are assumed to have their equilibrium value. I shall refer to it as the permanent tide.

Then as before, it will be convenient to consider the constituent parts of the disturbing function separately, and to indicate the several parts of W by suffixes as in § 5 and elsewhere; as above explained, we need only consider W_1, W_1, W_2, and W_0.

Semi-diurnal term.

From (37) we have

$$W_1 \Big/ \frac{\tau\tau'}{\mathfrak{g}} = \tfrac{1}{4}\left[F_1 \varpi^4 \varpi'^4 e^{2(\theta'-\theta)-2f_1} + F_1 \varpi^4 \varpi'^4 e^{-2(\theta'-\theta)+2f_1}\right]$$

To the indices of these exponentials we must add $\pm 2(\chi-\chi')$, and for θ write $\epsilon-\psi$, and for θ', $\epsilon'-\psi'$.

Then by (128)

$$\varpi^4 = 1-\tfrac{1}{2}i^2-\tfrac{1}{2}j^2-ije^{(N-\psi)}$$
$$\varpi'^4 = 1-\tfrac{1}{2}i'^2-\tfrac{1}{2}j'^2-i'j'e^{-(N'-\psi')}$$

Hence

$$\begin{aligned} W_1 \Big/ \frac{\tau\tau'}{\mathfrak{g}} = \tfrac{1}{2}F_1\{(1-\tfrac{1}{2}i^2-\tfrac{1}{2}j^2-\tfrac{1}{2}i'^2-\tfrac{1}{2}j'^2)\cos[2(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-2f_1] \\ -ij\cos[2(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)+(N-\psi)-2f_1] \\ -i'j'\cos[2(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-(N'-\psi')-2f_1]\} \quad . \quad . \quad (129) \end{aligned}$$

Slow diurnal term.

From (38) we have

$$W_1 \Big/ \frac{\tau\tau'}{\mathfrak{g}} = G_1\left[\varpi^3\kappa\varpi'^3\kappa' e^{2(\theta'-\theta)-g_1} + \varpi^3\kappa\varpi'^3\kappa' e^{-2(\theta'-\theta)+g_1}\right]$$

To the indices of the exponentials we must add $\pm(\chi-\chi')$; ϖ^3, ϖ'^3 may be obviously put equal to unity, and by (128)

$$\kappa\kappa' = \tfrac{1}{4}\left[ii'+i'je^{(N-\psi)}+ij'e^{-(N'-\psi')}+jj'e^{(N-N)-(\psi-\psi')}\right]$$

Hence

$$W_1\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G_1\{ii'\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-g_1]$$
$$+i'j\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)+(N-\psi)-g_1]$$
$$+ij'\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-(N'-\psi')-g_1]$$
$$+jj'\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)+(N-N')-(\psi-\psi')-g_1]\}\quad . \quad (130)$$

Sidereal diurnal term.

From (38) we have

$$W_2\Big/\frac{\tau\tau'}{\mathfrak{g}}=G[\varpi\kappa(\varpi\underline{\varpi}-\kappa\underline{\kappa})\underline{\varpi}'\underline{\kappa}'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{-g}+\underline{\varpi}\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})\varpi'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{g}]$$

To the indices of the exponentials must be added $\pm(\chi-\chi')$. ϖ, ϖ' may be treated as unity. Hence the expression becomes $G[\kappa\underline{\kappa}'e^{\chi-\chi'-g}+\underline{\kappa}\kappa'e^{-(\chi-\chi')+g}]$ and

$$W_2\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G\{ii'\cos(\chi-\chi'-g)$$
$$+i'j\cos[(\chi-\chi')-(N-\psi)-g]$$
$$+ij'\cos[(\chi-\chi')+(N'-\psi')-g]$$
$$+jj'\cos[(\chi-\chi')-(N-N')+(\psi-\psi')-g]\}\quad . \quad . \quad . \quad (131)$$

Permanent term.

From (39) we have

$$W_0\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{3}{2}(\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa})(\tfrac{1}{3}-2\varpi'\underline{\varpi}'\kappa'\underline{\kappa}')$$
$$=\tfrac{1}{6}-\kappa\underline{\kappa}-\kappa'\underline{\kappa}' \text{ to our degree of approximation.}$$

Now

$$\kappa\underline{\kappa}=\tfrac{1}{4}(i^2+j^2+ij(e^{N-\psi}+e^{-(N-\psi)}))=\tfrac{1}{4}(i^2+j^2+2ij\cos(N-\psi))$$

Hence

$$W_0\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{6}-\tfrac{1}{4}(i^2+j^2+2ij\cos(N-\psi))-\tfrac{1}{4}(i'^2+j'^2+2i'j'\cos(N'-\psi'))\quad . \quad (132)$$

W_2 and W_0 are the only terms in W which can contribute anything to the secular inequalities, unless Diana and the satellite are identical; for all the other terms involve $\epsilon-\epsilon'$, and will therefore be periodic however differentiated, unless $\epsilon=\epsilon'$.

We now have to differentiate W with respect to i', χ', ψ', j', ϵ', N'. The results will then have to be applied in the following cases.

For the moon: (i.) When the tide-raiser is the moon.
(ii.) When the tide-raiser is the sun.

For the earth: (iii.) When the tide-raiser is the moon, and the disturber the moon.
(iv.) When the tide-raiser is the sun, and the disturber the sun.
(v.) When the tide-raiser is the moon, and the disturber the sun.
(vi.) When the tide-raiser is the sun, and the disturber the moon.

The sum of the values derived from the differentiations, according to these several hypotheses, will be the complete values to be used in the differential equations (13), (14) and (18) for dj/dt, dN/dt, di/dt, $d\psi/dt$.

A little preliminary consideration will show that the labour of making these differentiations may be considerably abridged.

In the present case i and j are small, and the equations (110) which give the position of the two proper planes, and the inclinations of the orbit and equator thereto, become

$$\left.\begin{aligned}
&\frac{\xi}{k}\frac{dj}{dt}=-\tau\mathfrak{t}i\sin(N-\psi)\\
&\frac{\xi}{k}\sin j\frac{dN}{dt}=-\left(\tau\mathfrak{t}+\tfrac{1}{2}\frac{\tau'}{\Omega}\frac{\xi}{k}\right)j-\tau\mathfrak{t}i\cos(N-\psi)\\
&n\frac{di}{dt}=\tau\mathfrak{t}j\sin(N-\psi)\\
&n\sin i\frac{d\psi}{dt}=-(\tau\mathfrak{t}+\tau'\mathfrak{t})i-\tau\mathfrak{t}j\cos(N-\psi)
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (133)$$

We are now going to find certain additional terms, depending on frictional tides, to be added to these four equations. These terms will all involve τ^2, τ'^2, or $\tau\tau'$ in their coefficients, and will therefore be small compared with those in (133). If these small terms are of the same types as the terms in (133), they may be dropped; because the only effect of them would be to produce a very small and negligeable alteration in the position of the two proper planes.*

In consequence of this principle, we may entirely drop W_0 from our disturbing function, for W_0 only gives rise to a small permanent alteration of oblateness, and therefore can only slightly modify the positions of the proper planes.

Analytically the same result may be obtained, by observing that W_0 in (132) has the same form as W in (105), when i and j are treated as small.

* For example, we should find the following terms in $\frac{\xi}{k}\sin j\frac{dN}{dt}$, viz.:—

$$-\tfrac{1}{2}j\frac{\tau\tau'}{\mathfrak{g}}-\tfrac{1}{2}i\cos(N-\psi)\sin^2\mathrm{g}\frac{\tau\tau'}{\mathfrak{g}}+\tfrac{1}{2}(j+i\cos(N-\psi))\left[\sin^2 2\mathrm{f}_1-\sin^2\mathrm{g}_1-\sin^2\mathrm{g}\right]\tau^2$$

which may be all coupled up with those in the second of (133).

If the viscosity be small, so that the angles of lagging are small, it will be found that all the terms of this kind vanish in all four equations, excepting the first of those just written down, viz.: $-\frac{1}{2}j\tau\tau'/\mathfrak{g}$.

In each case, after differentiation, the transition will be made to the case of viscosity of the planet, and the proper terms will be dropped out, without further comment.

First take the perturbations of the moon.

For this purpose we have to find dW/dj' and $dW/\sin j'\, dN' + \tan \frac{1}{2}j'\, dW/d\epsilon'$ or $dW/j'dN' + \frac{1}{2}j' dW/d\epsilon'$.

By the above principle, in finding dW/dj' we may drop terms involving j and $i \cos (N-\psi)$, and in finding $dW/j'dN' + \frac{1}{2}j'\, dW/d\epsilon'$, we may drop terms involving $i \sin (N-\psi)$.

We may now suppose $\chi=\chi'$, $\psi=\psi'$.

Take the case (i.), where the tide-raiser is the moon. Then as the perturbed body is also the moon, after differentiation we may drop the accents to all the symbols.

From (129)

$$\frac{dW_1}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}} = \tfrac{1}{2}F_1\{-j \cos 2f_1 - i \cos (N-\psi+2f_1)\}$$
$$= \tfrac{1}{4}i \sin (N-\psi) \sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (134)$$

From (130)

$$\frac{dW_1}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}} = \tfrac{1}{2}G_1\{i \cos (N-\psi+g_1) + j \cos g_1\}$$
$$= -\tfrac{1}{4}i \sin (N-\psi) \sin 2g_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (135)$$

From (131) and symmetry with (135)

$$\frac{dW_2}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}} = \tfrac{1}{4}i \sin (N-\psi) \sin 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (136)$$

Adding these three (134–6) together, we have for the whole effect of the lunar tides on the moon

$$\frac{dW}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}} = \tfrac{1}{4}i \sin (N-\psi)[\sin 4f_1 - \sin 2g_1 + \sin 2g] \quad . \quad . \quad . \quad . \qquad (137)$$

Now take the case (ii.) where the tide-raiser is the sun.

Here we need only consider W_2, but although we may put $\chi=\chi'$, $\psi=\psi'$, $i=i'$, we must not put $j=j'$, $N=N'$, because the tide-raiser is distinct from the moon.

From (131)

$$\frac{dW_2}{dj'}\Big/\frac{\tau\tau'}{\mathfrak{g}} = \tfrac{1}{2}G\{i \cos (N'-\psi'-g) + j \cos (N-N'+g)\}$$

Here accented symbols refer to the moon (as perturbed), and unaccented to the sun (as tide-raiser). As we refer the motion to the ecliptic $j=0$, and the last term disappears. Also we want accented symbols to refer to the sun and unaccented to

refer to the moon, therefore make τ and τ' interchange their meanings, and drop the accents to N' and ψ'. Thus as far as important

$$\frac{dW_2}{dj'}\bigg/\frac{\tau'\tau}{\mathfrak{g}}=\tfrac{1}{4}i\sin(N-\psi)\sin 2\mathrm{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (138)$$

This gives the whole effect of the solar tides on the moon.

Then collecting results from (137–8), we have by (14)

$$\frac{\xi}{k}\sin j\frac{dN}{dt}=\tfrac{1}{4}i\sin(N-\psi)\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1-\sin 2\mathrm{g}_1+\sin 2\mathrm{g})+\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \quad . \qquad (139)$$

This gives the required additional terms due to bodily tides in the equation for dN/dt, viz.: the second of (133).

If the viscosity be small

$$\left.\begin{aligned}\sin 4\mathrm{f}_1-\sin 2\mathrm{g}_1+\sin 2\mathrm{g}&=\sin 4\mathrm{f}\\ \sin 2\mathrm{g}&=\tfrac{1}{2}\sin 4\mathrm{f}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (140)$$

Next take the secular change of inclination of the lunar orbit.

For this purpose we have to find $dW/j'dN'+\frac{1}{2}j'dW/d\epsilon'$, and may drop terms in $i\sin(N-\psi)$.

First take the case (i.), where the tide-raiser is the moon.

From (129)

$$\frac{1}{j'}\frac{dW_1}{dN'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}\mathrm{F}_1\, i\sin(N-\psi+2\mathrm{f}_1)=\tfrac{1}{4}\,i\cos(N-\psi)\sin 4\mathrm{f}_1 \quad . \quad . \quad . \qquad (141)$$

$$\tfrac{1}{2}j'\frac{dW_1}{d\epsilon'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}\mathrm{F}_1 j\sin 2\mathrm{f}_1 \qquad =\tfrac{1}{4}j\sin 4\mathrm{f}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (142$$

From (130)

$$\frac{1}{j'}\frac{dW_1}{dN'}\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\mathrm{G}_1\{i\sin(N-\psi+\mathrm{g}_1)+j\sin\mathrm{g}_1\}=-\tfrac{1}{4}(j+i\cos(N-\psi))\sin 2\mathrm{g}_1 \quad (143)$$

$$\tfrac{1}{2}j'\frac{dW_1}{d\epsilon'}\bigg/\frac{\tau^2}{\mathfrak{g}}= \quad 0 \text{ to present order of approximation} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (144)$$

From (131)

$$\frac{1}{j'}\frac{dW_2}{dN'}\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\mathrm{G}\{i\sin(N-\psi-\mathrm{g})-j\sin\mathrm{g}\}=\tfrac{1}{4}(j+i\cos(N-\psi))\sin 2\mathrm{g} \quad . \qquad (145)$$

$$\tfrac{1}{2}j'\frac{dW_2}{d\epsilon'}\bigg/\frac{\tau^2}{\mathfrak{g}}=0 \text{ absolutely} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (146)$$

Collecting results from the six equations (141–6), we have for the whole perturbation of the moon by the lunar tides

$$\left(\frac{1}{j'}\frac{dW}{dN'}+\tfrac{1}{2}j'\frac{dW}{d\epsilon'}\right)\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}(j+i\cos(N-\psi))(\sin 4f_1-\sin 2g_1+\sin 2g)\quad . \quad (147)$$

Next take the case (ii.), and suppose that the sun is the tide-raiser. Here we need only consider W_2. Then noting that $dW_2/d\epsilon'=0$ absolutely, we have from (131)

$$\left(\frac{1}{j'}\frac{dW_2}{dN'}+\tfrac{1}{2}j'\frac{dW_2}{d\epsilon'}\right)\Big/\frac{\tau\tau'}{\mathfrak{g}}=-\tfrac{1}{2}G\{i\sin(N'-\psi'-g)-j\sin(N-N'+g)\}$$

Accented symbols here refer to the moon (as perturbed), unaccented to the sun (as tide-raiser). Therefore $j=0$. Then reverting to the usual notation by shifting accents and dropping useless terms, this expression becomes

$$+\tfrac{1}{4}i\cos(N-\psi)\sin 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

Then collecting results from (147–8), we have by (13)

$$\frac{\xi}{k}\frac{dj}{dt}=-\tfrac{1}{4}(j+i\cos(N-\psi))\frac{\tau^2}{\mathfrak{g}}(\sin 4f_1-\sin 2g_1+\sin 2g)-\tfrac{1}{4}i\cos(N-\psi)\frac{\tau\tau'}{\mathfrak{g}}\sin 2g \quad (149)$$

This gives the additional terms due to bodily tides in the equation for dj/dt, viz.: the first of (133).

If the viscosity be small

$$\left.\begin{aligned}\sin 4f_1-\sin 2g_1+\sin 2g&=\sin 4f\\ \sin 2g&=\tfrac{1}{2}\sin 4f\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

Before proceeding further it may be remarked that to the present order of approximation in case (i.)

$$\frac{dW}{d\epsilon'}=\tfrac{1}{2}\sin 4f_1$$

and in case (ii.) it is zero; thus by (11)

$$\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

We now turn to the perturbations of the earth's rotation.

Here we have to find dW/di' and $\cot i\; dW/d\chi'-dW/\sin i\; d\psi'$ or $(1-\tfrac{1}{2}i^2)dW/id\chi'-dW/id\psi'$, and in the former may drop terms in i and $i\cos(N-\psi)$, and in the latter terms in $j\sin(N-\psi)$.

First take the case (iii.), where the moon is tide-raiser and disturber. Here we may take $N=N'$, $\epsilon=\epsilon'$, $j=j'$ throughout, and after differentiation may drop the accents to all the symbols.

From (129)

$$\frac{dW_1}{di'}\Big/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}F_1\{i\cos 2f_1+j\cos(N-\psi+2f_1)\}=\tfrac{1}{4}j\sin(N-\psi)\sin 4f_1 \quad . \quad (152)$$

From (130)

$$\frac{dW_1}{di'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}G_1\{i\cos g_1+j\cos(N-\psi-g_1)\}=\tfrac{1}{4}j\sin(N-\psi)\sin 2g_1 \quad . \quad (153)$$

From (131)

$$\frac{dW_2}{di'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}G\{i\cos g+j\cos(N-\psi+g)\}=-\tfrac{1}{4}j\sin(N-\psi)\sin 2g \quad . \quad (154)$$

Therefore from (152–4) we have for the whole perturbation of the earth, due to attraction of the moon on the lunar tides,

$$\frac{dW}{di'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}j\sin(N-\psi)[\sin 4f_1+\sin 2g_1-\sin 2g] \quad . \quad . \quad . \quad (155)$$

The result for case (iv.), where the sun is both tide-raiser and disturber, may be written down by symmetry; and since $j=0$ here, therefore

$$\frac{dW}{di'}\Big/\frac{\tau'^2}{\mathfrak{g}}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

Next take the cases (v.) and (vi.), where the tide-raiser and disturber are distinct. Here we need only consider W_2.

From (131)

$$\frac{dW_2}{di'}\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G\{i\cos g+j\cos(N-\psi+g)\}$$

When the moon is tide-raiser and sun disturber, this becomes

$$-\tfrac{1}{4}j\sin(N-\psi)\sin 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

When sun is tide-raiser and moon disturber it becomes zero.

Then collecting results from (155–7), we have by (18)

$$n \sin i \frac{d\psi}{dt} = \tfrac{1}{4} j \sin (N-\psi)\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4f_1 + \sin 2g_1 - \sin 2g) - \frac{\tau\tau'}{\mathfrak{g}} \sin 2g\right] \quad . \quad (158)$$

This gives the additional terms due to bodily tides in the equation for $d\psi/dt$, viz.: the last of (133).

If the viscosity be small

$$\left.\begin{aligned} \sin 4f_1 + \sin 2g_1 - \sin 2g &= \sin 4f(1-2\lambda) \\ \sin 2g &= \tfrac{1}{2} \sin 4f \\ \lambda &= \frac{\Omega}{n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (159)$$

where

Next consider the change in the obliquity of the ecliptic; for this purpose we must find $(1-\frac{1}{2}i^2)dW/id\chi' - dW/id\psi'$, and may drop terms involving $j \sin (N-\psi)$.

First take the case (iii.), where the moon is both tide-raiser and disturber.

Then from (129)

$$\frac{dW_1}{d\chi'}\bigg/\frac{\tau^2}{\mathfrak{g}} = -F_1\{(1-i^2-j^2) \sin 2f_1 + ij \sin (N-\psi-2f_1) - ij \sin (N-\psi+2f_1)\} \quad . \quad (160)$$

$$-\frac{dW_1}{d\psi'}\bigg/\frac{\tau^2}{\mathfrak{g}} = F_1\{(1-i^2-j^2) \sin 2f_1 + ij \sin (N-\psi-2f_1) - \tfrac{1}{2}ij \sin (N-\psi+2f_1)\}$$

$$-\tfrac{1}{2}i^2\frac{dW_1}{d\chi'}\bigg/\frac{\tau^2}{\mathfrak{g}} = F_1 \tfrac{1}{2}i^2 \sin 2f_1$$

Therefore

$$\begin{aligned}\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_1}{d\chi'} - \frac{1}{i}\frac{dW_1}{d\psi'}\right]\bigg/\frac{\tau^2}{\mathfrak{g}} &= \tfrac{1}{2}F_1\{i \sin 2f_1 + j \sin (N-\psi+2f_1)\} \\ &= \tfrac{1}{4}(i + j \cos (N-\psi)) \sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad (161)\end{aligned}$$

From (130)

$$\frac{dW_1}{d\chi'}\bigg/\frac{\tau^2}{\mathfrak{g}} = -\tfrac{1}{2}G_1\{i^2 \sin g_1 - ij \sin (N-\psi-g_1) + ij \sin (N-\psi+g_1) + j^2 \sin g_1\} \quad . \quad (162)$$

$$-\frac{dW_1}{d\psi'}\bigg/\frac{\tau^2}{\mathfrak{g}} = \tfrac{1}{2}G_1\{2i^2 \sin g_1 - 2ij \sin (N-\psi-g_1) + ij \sin (N-\psi+g_1) + j^2 \sin g_1\}$$

$$-\tfrac{1}{2}i^2\frac{dW_1}{d\chi'}\bigg/\frac{\tau^2}{\mathfrak{g}} = 0$$

Therefore

$$\begin{aligned}\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_1}{d\chi'} - \frac{1}{i}\frac{dW_1}{d\psi'}\right]\bigg/\frac{\tau^2}{\mathfrak{g}} &= \tfrac{1}{2}G\{i \sin g_1 - j \sin (N-\psi-g_1)\} \\ &= \tfrac{1}{4}(i + i \cos (N-\psi)) \sin 2g_1 \quad . \quad . \quad . \quad . \quad (163)\end{aligned}$$

From (131)

$$\frac{dW_2}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}G\{i^2\sin g+ij\sin(N-\psi+g)-ij\sin(N-\psi-g)+j^2\sin g\} \quad . \quad (164)$$

$$-\frac{dW_2}{d\psi'}\Big/\frac{\tau^2}{\mathfrak{g}}=\ \tfrac{1}{2}G\{\qquad\qquad -ij\sin(N-\psi-g)+j^2\sin g\}$$

$$-\tfrac{1}{2}i^2\frac{dW_2}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}}=0$$

Therefore

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_2}{d\chi'}-\frac{1}{i}\frac{dW_2}{d\psi'}\right]\Big/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}G\{i\sin g+j\sin(N-\psi+g)\}$$

$$=-\tfrac{1}{4}(i+j\cos(N-\psi))\sin 2g \quad . \quad . \quad . \quad . \quad (165)$$

Then collecting results from (161–3–5), we have for the whole perturbation of the earth due to the attraction of the moon on the lunar tides,

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW}{d\chi'}-\frac{1}{i}\frac{dW}{d\psi'}\right]\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}(i+j\cos(N-\psi))(\sin 4f_1+\sin 2g_1-\sin 2g) \quad . \quad (166)$$

The result for case (iv.), where the sun is both tide-raiser and disturber, may be written down by symmetry; and since $j=0$ here, therefore

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW}{d\chi'}-\frac{1}{i}\frac{dW}{d\psi'}\right]\Big/\frac{\tau'^2}{\mathfrak{g}}=\tfrac{1}{4}i\sin 4f \quad . \quad . \quad . \quad . \quad . \quad . \quad (167)$$

It is here assumed that the solar slow diurnal tide has the same lag as the sidereal diurnal tide, and that the solar slow semi-diurnal tide has the same lag as the sidereal semi-diurnal tide. This is very nearly true, because Ω' is small compared with n.

Next take the cases (v.) and (vi.), where the tide-raiser and disturber are distinct. Here we need only consider W_2

$$\frac{dW_2}{d\chi'}\Big/\frac{\tau\tau'}{\mathfrak{g}}=-\tfrac{1}{2}G\{i^2\sin g+ij\sin(N-\psi+g)-ij'\sin(N'-\psi'-g)$$

$$+jj'\sin(N-N'+g)\} \quad . \quad (168)$$

$$-\frac{dW_2}{d\psi'}\Big/\frac{\tau\tau'}{\mathfrak{g}}=\ \tfrac{1}{2}G\{\qquad\qquad -ij'\sin(N'-\psi'-g)+jj'\sin(N-N'+g)\}$$

$$-\tfrac{1}{2}i^2\frac{dW_2}{d\chi'}\Big/\frac{\tau\tau'}{\mathfrak{g}}=0$$

Therefore

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_2}{d\chi'}-\frac{1}{i}\frac{dW_2}{d\psi'}\right]\Big/\frac{\tau\tau'}{\mathfrak{g}}=-\tfrac{1}{2}G\{i\sin\mathrm{g}+j\sin(N-\psi+\mathrm{g})\}$$

When the moon is tide-raiser and the sun disturber, this becomes

$$-\tfrac{1}{4}(i+j\cos(N-\psi))\sin 2\mathrm{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{169}$$

When the sun is tide-raiser and the moon disturber, this becomes

$$-\tfrac{1}{4}i\sin 2\mathrm{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{170}$$

Then collecting results from (166–7–9, 170), we have by (18),

$$\left.\begin{aligned} i\frac{di}{dt}&=\tfrac{1}{4}(i+j\cos(N-\psi))\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g})-\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right]\\ &\quad+\tfrac{1}{4}i\left[\frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}-\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right]\end{aligned}\right\} \quad . \tag{171}$$

This gives the additional terms due to bodily tides in the equation for di/dt, viz.: the third of (133).

If the viscosity be small

$$\left.\begin{aligned} \sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g}&=\sin 4\mathrm{f}(1-2\lambda)\\ \sin 2\mathrm{g}&=\tfrac{1}{2}\sin 4\mathrm{f}\end{aligned}\right. \tag{172}$$

where

$$\lambda=\frac{\Omega}{n}$$

Also we have from (160–2–4–8) to the present order of approximation,

$$\frac{dW}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\sin 4\mathrm{f}_1$$

and by symmetry,

$$\frac{dW}{d\chi'}\Big/\frac{\tau'^2}{\mathfrak{g}}=-\tfrac{1}{2}\sin 4\mathrm{f}$$

Therefore by (18)

$$-\frac{dn}{dt}=\tfrac{1}{2}\left[\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1+\frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{173}$$

Now let

$$
\left.\begin{aligned}
\Gamma &= \tfrac{1}{4}\frac{k}{\xi}\,\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 - \sin 2\mathrm{g}_1 + \sin 2\mathrm{g}) \\
\mathrm{G} &= \tfrac{1}{4}\frac{k}{\xi}\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 - \sin 2\mathrm{g}_1 + \sin 2\mathrm{g}) + \frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \\
\Delta &= \frac{1}{4n}\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 + \sin 2\mathrm{g}_1 - \sin 2\mathrm{g}) + \frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f} - 2\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \\
\mathrm{D} &= \frac{1}{4n}\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 + \sin 2\mathrm{g}_1 - \sin 2\mathrm{g}) - \frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right]
\end{aligned}\right. \qquad (174)
$$

Then the four equations (139), (149), (158), and (171) may be written

$$
\left.\begin{aligned}
j\frac{dN}{dt} &= \mathrm{G}i\sin(N-\psi) \\
\frac{dj}{dt} &= -\Gamma j - \mathrm{G}i\cos(N-\psi) \\
i\frac{d\psi}{dt} &= \mathrm{D}j\sin(N-\psi) \\
\frac{di}{dt} &= \Delta i + \mathrm{D}j\cos(N-\psi)
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (175)
$$

Also from (151) and (173)

$$
\left.\begin{aligned}
\frac{1}{k}\,\frac{d\xi}{dt} &= \tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 \\
\frac{dn}{dt} &= \tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 + \tfrac{1}{2}\frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}
\end{aligned}\right. \qquad (176)
$$

These six equations (175–6) contain all the secular inequalities in the motions of the moon and earth, due to the bodily tides raised by the sun and moon, as far as is material for the present investigation. The terms which are omitted only represent a very small displacement of the proper planes and of the inclinations of the planes of motion of the two parts of the system to those proper planes.

Then reverting to the earlier notation in which

$$
\left.\begin{aligned}
y &= j\sin N,\ \eta = i\sin\psi \\
z &= j\cos N,\ \zeta = i\cos\psi
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (177)
$$

We easily find

$$\left.\begin{aligned}\frac{dz}{dt}&=-\Gamma z-\mathrm{G}\zeta\\ \frac{dy}{dt}&=-\Gamma y-\mathrm{G}\eta\\ \frac{d\zeta}{dt}&=\ \ \Delta\zeta+\mathrm{D}z\\ \frac{d\eta}{dt}&=\ \ \Delta\eta+\mathrm{D}y\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (178)$$

These equations contain the additional terms due to tides, which are to be added to the equations (116), in order to find the secular displacements of the proper planes.

The first application, which will be made hereafter, will be to the case where the viscosity is small, and it will be more convenient to make the transition to that hypothesis at present, although the greater part of what follows in this part will be equally applicable whatever may be the viscosity. In the case of small viscosity the functions Γ, Δ, G, D will be indicated by the corresponding small letters γ, δ, g, d. Then by (140), (150), (159), (172) we shall have

$$\begin{aligned}&\gamma=\tfrac{1}{4}\frac{k}{\xi}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2], \qquad\qquad \mathrm{g}=\tfrac{1}{4}\frac{k}{\xi}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2+\tfrac{1}{2}\tau\tau']\\ &\delta=\frac{1}{4n}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2(1-2\lambda)+\tau'^2-\tau\tau'],\ \mathrm{d}=\frac{1}{4n}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2(1-2\lambda)-\tfrac{1}{2}\tau\tau'] \quad . \quad (179)\\ &\qquad \text{where } \lambda=\frac{\Omega}{n}\end{aligned}$$

And in the present case where i and j are small, we have by (112) and (121)

$$\begin{aligned}&\alpha=\frac{k}{\xi}\tau\mathfrak{e}+\tfrac{1}{2}\frac{\tau'}{\Omega}, \quad \beta=\frac{\tau+\tau'}{n}\mathfrak{e}\\ &\mathrm{a}=\frac{k}{\xi}\tau\mathfrak{e}, \quad \mathrm{b}=\frac{\tau\mathfrak{e}}{n} \qquad\qquad . \quad . \quad (180)\\ &\text{where } \mathfrak{e}=\tfrac{1}{2}\frac{n^2}{\mathfrak{g}}, \text{ the permanent ellipticity of the earth}\end{aligned}$$

These equations (180) are the same whether the viscosity be supposed small or not. Then the complete equations are

$$\left.\begin{aligned}\frac{dz}{dt}&=\alpha y+\mathrm{a}\eta-(\gamma z+\mathrm{g}\zeta)\\ \frac{dy}{dt}&=-(\alpha z+\mathrm{a}\zeta)-(\gamma y+\mathrm{g}\eta)\\ \frac{d\zeta}{dt}&=\beta\eta+\mathrm{b}y+\delta\zeta+\mathrm{d}z\\ \frac{d\eta}{dt}&=-(\beta\zeta+\mathrm{b}z)+\delta\eta+\mathrm{d}y\end{aligned}\right\} \quad \ldots \ldots \quad (181)$$

If the viscosity be not small we have Γ, G, Δ, D in place of γ, g, δ, d. As it is more convenient to write small letters than capitals, in the whole of the next section the small letters will be employed, although the same investigation would be equally applicable with Γ, G, &c., in place of γ, g, &c.

The terms in γ, g, δ, d are small compared with those in α, a, β, b, and may be neglected as a first approximation. Also α, a, β, b vary slowly in consequence of tidal reaction, tidal friction, and the consequent change of ellipticity of the earth, but as a first approximation they may be treated as constant.

Then if we put

$$\left.\begin{aligned}z_1&=L_1\cos(\kappa_1 t+\mathrm{m}_1), & z_2&=L_2\cos(\kappa_2 t+\mathrm{m}_2)\\ y_1&=L_1\sin(\kappa_1 t+\mathrm{m}_1), & y_2&=L_2\sin(\kappa_2 t+\mathrm{m}_2)\\ \zeta_1&=L_1'\cos(\kappa_1 t+\mathrm{m}_1), & \zeta_2&=L_2'\cos(\kappa_2 t+\mathrm{m}_2)\\ \eta_1&=L_1'\sin(\kappa_1 t+\mathrm{m}_1), & \eta_2&=L_2'\sin(\kappa_2 t+\mathrm{m}_2)\end{aligned}\right\} \quad \ldots \quad (182)$$

By (122) or (118) the first approximation is

$$z=z_1+z_2,\ y=y_1+y_2,\ \zeta=\zeta_1+\zeta_2,\ \eta=\eta_1+\eta_2$$

where

$$\left.\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_1+\beta},\ \frac{L_2'}{L_2}=-\frac{\kappa_2+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_2+\beta}\right\} \quad \ldots \quad (183)$$

Before considering the secular changes in the constants L of integration, it will be convenient to take one other step.

The equation of tidal friction (173) may be written approximately

$$-\frac{dn}{dt}=\tfrac{1}{2}\frac{\tau^2+\tau'^2}{\mathrm{g}}\sin 4\mathrm{f}_1 \quad \ldots \ldots \quad (184)$$

because sin 4f will be nearly equal to sin $4\mathrm{f}_1$ as long as τ'^2 is not small compared with τ^2. (See however § 22, Part IV.)

Also the equation of tidal reaction (151) is

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

Dividing one by the other and putting $\tau^2=\tau_0{}^2\xi^{-12}$, we have

$$k\frac{dn}{d\xi}=1+\left(\frac{\tau'}{\tau_0}\right)^2\xi^{12}$$

and integrating,

$$\frac{n}{n_0}=1+\frac{1}{kn_0}\left[(1-\xi)+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

This is the equation of conservation of moment of momentum of the moon-earth system, as modified by solar tidal friction. From it we obtain n in terms of ξ.

§ 15. *On the secular changes of the constants of integration.*

It is often found difficult on first reading a long analytical investigation to trace the general method amidst the mass of detail, and it is only at the end that the ruling idea is perceived; in such circumstances it has often appeared to me that a preliminary sketch would be of great service to the reader. I shall act on this idea here, and consider some simple equations analogous to those to be treated.

Let the equations be

$$\frac{dz}{dt}=\alpha y, \quad \frac{dy}{dt}=-\alpha z$$

If α be constant, the solution is obviously

$$z=\text{L}\cos(\alpha t+\text{m}), \quad y=-\text{L}\sin(\alpha t+\text{m})$$

Now suppose α to be slowly varying; put therefore $\alpha+\alpha' t$ for α, and treat α, α' as constants.

Then

$$\frac{dz}{dt}=\alpha y+\alpha' t y, \qquad \frac{dy}{dt}=-\alpha z-\alpha' t z$$

Differentiating

$$\frac{d^2z}{dt^2}+\alpha^2 z=-\alpha' t\left(\alpha z-\frac{dy}{dt}\right)+\alpha' y$$

$$\frac{d^2y}{dt^2}+\alpha^2 y=-\alpha' t\left(\alpha y+\frac{dz}{dt}\right)-\alpha' z$$

The terms on the right-hand side of these equations are small, because they involve α', and therefore we may substitute in them from the first approximation.

Hence

$$\frac{d^2z}{dt^2}+\alpha^2z=-\alpha'L\sin(\alpha t+\mathrm{m})-2\alpha'\alpha tL\cos(\alpha t+\mathrm{m})$$

and a similar equation for y.

The solution of this equation is

$$z=L\cos(\alpha t+\mathrm{m})+\frac{\alpha'}{2\alpha}Lt\cos(\alpha t+\mathrm{m})-\frac{\alpha'}{2\alpha}Lt\cos(\alpha t+\mathrm{m})-\frac{\alpha'}{2}Lt^2\sin(\alpha t+\mathrm{m})$$

The terms depending on t cut one another out, and

$$z=L\cos(\alpha t+\mathrm{m})-\frac{\alpha'}{2}Lt^2\sin(\alpha t+\mathrm{m})$$

Similarly we should find

$$y=-L\sin(\alpha t+\mathrm{m})-\frac{\alpha'}{2}Lt^2\cos(\alpha t+\mathrm{m})$$

The terms in t^2 are obviously equivalent to a change in m, the phase of the oscillation; but the amplitude L is unaffected. We might have arrived at this conclusion about the amplitude if, in solving the differential equations, we had neglected in the solutions the terms depending on t^2, as will be done in considering our equations below. In those equations, however, we shall not find that the terms in t annihilate one another, and thus there will be a change of amplitude.

That this conclusion concerning amplitude is correct, may be seen from the fact that the rigorous solution of the equations

$$\frac{dz}{dt}=\alpha y,\quad \frac{dy}{dt}=-\alpha z$$

is

$$\begin{aligned} z&=L\cos\left(\int\alpha dt+\mathrm{m}_0\right), & y&=-L\sin\left(\int\alpha dt+\mathrm{m}_0\right)\\ &=L\cos\left(\alpha t+\mathrm{m}_0-\int\alpha' tdt\right), & &=-L\sin\left(\alpha t+\mathrm{m}_0-\int\alpha' tdt\right)\end{aligned}$$

Whence L is unaffected, whilst

$$\mathrm{m}=\mathrm{m}_0-\int\alpha' tdt$$

So that

$$\frac{d\mathrm{m}}{dt}=-t\frac{d\alpha}{dt}$$

Next consider the equations

$$\frac{dz}{dt}=\alpha y-\gamma z,\ \frac{dy}{dt}=-\alpha z-\gamma y$$

Where α is constant, but γ is a very small quantity compared with α, but which may vary slowly.

Treat γ as constant, and differentiate, and we have

$$\frac{d^2z}{dt^2}+\alpha^2z=-\gamma\left(\frac{dz}{dt}+\alpha y\right)$$

$$\frac{d^2y}{dt^2}+\alpha^2y=-\gamma\left(\frac{dy}{dt}-\alpha z\right)$$

Then if we neglect γ, we have the first approximation

$$z=L\cos(\alpha t+\mathrm{m}),\ y=-L\sin(\alpha t+\mathrm{m})$$

Substituting these values for z, y on the right, we have

$$\frac{d^2z}{dt^2}+\alpha^2z=\gamma 2\alpha L\sin(\alpha t+\mathrm{m})$$

And a similar equation for y.

The solutions are

$$z=\ \ L\cos(\alpha t+\mathrm{m})-\gamma Lt\cos(\alpha t+\mathrm{m})$$
$$y=-L\sin(\alpha t+\mathrm{m})+\gamma Lt\sin(\alpha t+\mathrm{m})$$

From this we see that, if we desire to retain the first approximation as the solution, we must have

$$\frac{1}{L}\frac{dL}{dt}=-\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

This will be true if γ varies slowly; hence

$$L=L_0e^{-\int\gamma dt}$$

and the solution is

$$z=\ \ L_0e^{-\int\gamma dt}\cos(\alpha t+\mathrm{m})$$
$$y=-L_0e^{-\int\gamma dt}\sin(\alpha t+\mathrm{m})$$

It is easy to verify that these are the rigorous solutions of the equations, when α is constant but γ varies.

The equation (187) gives the rate of change of amplitude of oscillation.

The cases which we have now considered, by the method of variation of parameters, are closely analogous to those to be treated below, and have been treated in the same way, so that the reader will be able to trace the process.

They are in fact more than simply analogous, for they are what our equations (181) become if the obliquity of the ecliptic be zero and $\zeta=0$, $\eta=0$. In this case $L=j$, and $dj/dt=-j\gamma$.

This shows that the secular change of figure of the earth, and the secular changes in the rate of revolution of the moon's nodes do not affect the rate of alteration of the inclination of the lunar orbit to the ecliptic, so long as the obliquity is zero. This last result contains the implicit assumption that the perturbing influence of the moon on the earth is not so large, but that the obliquity of the equator may always remain small, however the lunar nodes vary. In an exactly similar manner we may show that, if the inclination of the lunar orbit be zero, $di/dt=i\delta$.

This is the result of the previous paper "On the Precession of a Viscous Spheroid," when the obliquity is small.

According to the method which has been sketched, the equations to be integrated are given in (181), when we write $\alpha+\alpha't$ for α, $\mathrm{a}+\mathrm{a}'t$ for a, $\beta+\beta't$ for β, $\mathrm{b}+\mathrm{b}'t$ for b, and then treat α, a, &c., α', a$'$, &c., γ, g, &c., as constants.

Before proceeding to consider the equations, it will be convenient to find certain relations between the quantities α, a, &c., and the two roots κ_1 and κ_2 of the quadratic $(\kappa+\alpha)(\kappa+\beta)=\mathrm{ab}$.

We have supposed the two roots to be such that

$$\left.\begin{aligned}\kappa_1+\kappa_2&=-\alpha-\beta\\ \kappa_1-\kappa_2&=-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

Then

$$\kappa_1\kappa_2=(\alpha\beta-\mathrm{ab}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

$$\left.\begin{aligned}\kappa_1^2+\kappa_2^2&=\alpha^2+\beta^2+2\mathrm{ab}\\ \kappa_1^2\kappa_2^2&=(\alpha^2+\mathrm{ab})(\beta^2+\mathrm{ab})-\mathrm{ab}(\alpha+\beta)^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (190)$$

$$\left.\begin{aligned}\beta^2+\mathrm{ab}-\kappa_1^2&=(\kappa_1+\kappa_2)(\kappa_2+\alpha)\\ \beta^2+\mathrm{ab}-\kappa_2^2&=(\kappa_1+\kappa_2)(\kappa_1+\alpha)\\ \alpha^2+\mathrm{ab}-\kappa_1^2&=(\kappa_1+\kappa_2)(\kappa_2+\beta)\\ \alpha^2+\mathrm{ab}-\kappa_2^2&=(\kappa_1+\kappa_2)(\kappa_1+\beta)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$$

$$\left.\begin{array}{l}\kappa_1+\alpha=-(\kappa_2+\beta)\\ \kappa_2+\alpha=-(\kappa_1+\beta)\end{array}\right\} \quad \ldots \ldots \ldots \quad (192)$$

$$\mathrm{ab}(\alpha+\beta)=(\kappa_1+\alpha)(\kappa_2+\alpha)(\kappa_1+\kappa_2) \quad \ldots \ldots \quad (193)$$

Now suppose our equations (181) to be written as follows:—

$$\left.\begin{array}{l}\dfrac{dz}{dt}=\ \ \alpha y+\mathrm{a}\eta+s\\ \dfrac{dy}{dt}=-\alpha z-\mathrm{a}\zeta+u\\ \dfrac{d\zeta}{dt}=\ \ \beta\eta+\mathrm{b}y+\sigma\\ \dfrac{d\eta}{dt}=-\beta\zeta-\mathrm{b}z+v\end{array}\right\} \quad \ldots \ldots \ldots \quad (194)$$

Where s, u, σ, v comprise all the terms involving α', a$'$, &c., γ, g, &c.

Then if we write (z) as a type of z, y, ζ, η; (α) as a type of α, a, β, b; (α') as a type of α', a$'$, β', b$'$; (γ) as a type of γ, g, δ, d; and (s) as a type of s, u, σ, v; it is clear that (s) is $(z)(\alpha')t+(\gamma)(z)$.

Differentiate each of the equations (194), and substitute for $\dfrac{d(z)}{dt}$ after differentiation. Then if we write

$$\left.\begin{array}{l}\mathrm{S}=\dfrac{ds}{dt}+\alpha u+\mathrm{a}v\\ \mathrm{U}=\dfrac{du}{dt}-\alpha s-\mathrm{a}\sigma\\ \Sigma=\dfrac{d\sigma}{dt}+\beta v+\mathrm{b}u\\ \Upsilon=\dfrac{dv}{dt}-\beta\sigma-\mathrm{b}s\end{array}\right\} \quad \ldots \ldots \ldots \quad (195)$$

The result is

$$\left.\begin{array}{l}\dfrac{d^2z}{dt^2}=-(\alpha^2+\mathrm{ab})z-\mathrm{a}(\alpha+\beta)\zeta+\mathrm{S}\\ \dfrac{d^2y}{dt^2}=-(\alpha^2+\mathrm{ab})y-\mathrm{a}(\alpha+\beta)\eta+\mathrm{U}\\ \dfrac{d^2\zeta}{dt^2}=-(\beta^2+\mathrm{ab})\zeta-\mathrm{b}(\alpha+\beta)z+\Sigma\\ \dfrac{d^2\eta}{dt^2}=-(\beta^2+\mathrm{ab})\eta-\mathrm{b}(\alpha+\beta)y+\Upsilon\end{array}\right\} \quad \ldots \ldots \quad (196)$$

From the first of these

$$-(\beta^2+\mathrm{ab})\mathrm{a}(\alpha+\beta)\zeta=(\beta^2+\mathrm{ab})\frac{d^2z}{dt^2}+(\alpha^2+\mathrm{ab})(\beta^2+\mathrm{ab})z-\mathrm{S}(\beta^2+\mathrm{ab})$$

Therefore from the third

$$\mathrm{a}(\alpha+\beta)\frac{d^2\zeta}{dt^2}=(\beta^2+\mathrm{ab})\frac{d^2z}{dt^2}+\{(\alpha^2+\mathrm{ab})(\beta^2+\mathrm{ab})-\mathrm{ab}(\alpha+\beta)^2\}z-\mathrm{S}(\beta^2+\mathrm{ab})+\Sigma\mathrm{a}(\alpha+\beta)$$

and by (190)

$$\mathrm{a}(\alpha+\beta)\frac{d^2\zeta}{dt^2}=(\beta^2+\mathrm{ab})\frac{d^2z}{dt^2}+\kappa_1^2\kappa_2^2z-\mathrm{S}(\beta^2+\mathrm{ab})+\Sigma\mathrm{a}(\alpha+\beta)$$

Similarly

$$\left.\begin{aligned}\mathrm{a}(\alpha+\beta)\frac{d^2\eta}{dt^2}&=(\beta^2+\mathrm{ab})\frac{d^2y}{dt^2}+\kappa_1^2\kappa_2^2y-\mathrm{U}(\beta^2+\mathrm{ab})+\Upsilon\mathrm{a}(\alpha+\beta)\\ \mathrm{b}(\alpha+\beta)\frac{d^2z}{dt^2}&=(\alpha^2+\mathrm{ab})\frac{d^2\zeta}{dt^2}+\kappa_1^2\kappa_2^2\zeta-\Sigma(\alpha^2+\mathrm{ab})+\mathrm{Sb}(\alpha+\beta)\\ \mathrm{b}(\alpha+\beta)\frac{d^2y}{dt^2}&=(\alpha^2+\mathrm{ab})\frac{d^2\eta}{dt^2}+\kappa_1^2\kappa_2^2\eta-\Upsilon(\alpha^2+\mathrm{ab})+\mathrm{Ub}(\alpha+\beta)\end{aligned}\right\} \quad (197)$$

Differentiate the first of (196) twice, using the first of (197), and we have

$$\frac{d^4z}{dt^4}=-(\alpha^2+\mathrm{ab})\frac{d^2z}{dt^2}-(\beta^2+\mathrm{ab})\frac{d^2z}{dt^2}-\kappa_1^2\kappa_2^2z+\left(\beta^2+\mathrm{ab}+\frac{d^2}{dt^2}\right)\mathrm{S}-\Sigma\mathrm{a}(\alpha+\beta)$$

Therefore by (190)

$$\left[\frac{d^4}{dt^4}+(\kappa_1^2+\kappa_2^2)\frac{d^2}{dt^2}+\kappa_1^2\kappa_2^2\right]z=\left(\beta^2+\mathrm{ab}+\frac{d^2}{dt^2}\right)\mathrm{S}-\Sigma\mathrm{a}(\alpha+\beta)$$

Then writing (S) as a type of S, Σ, U, Υ,—

$$(\mathrm{S})\text{ is of the type }(z)(\alpha)(\alpha')t+(\alpha)(\gamma)(z)+(\alpha')(z)+\frac{d(z)}{dt}(\alpha')t+(\gamma)\frac{d(z)}{dt}$$

Hence every term of (S) contains some small term, either (α') or (γ).

Therefore on the right-hand side of the above equation we may substitute for (z) the first approximation, viz.: $(z_1)+(z_2)$ given in (182–3).

When this substitution is carried out, let (S_1), (S_2) be the parts of (S) which contain all terms of the speeds κ_1 and κ_2 respectively.

Then by (191) and (193) the right-hand side in the above equation may be written

$$(\kappa_1+\kappa_2)(\kappa_2+\alpha)\mathrm{S}_1-\frac{\Sigma_1}{\mathrm{b}}(\kappa_1+\alpha)(\kappa_2+\alpha)(\kappa_1+\kappa_2)+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1$$

+ the same with 2 and 1 interchanged.

Now let D^4 stand for the operation $\frac{d^4}{dt^4}+(\kappa_1^2+\kappa_2^2)\frac{d^2}{dt^2}+\kappa_1^2\kappa_2^2$, and we have

$$\left.\begin{aligned}
D^4 z &= (\kappa_1+\kappa_2)(\kappa_2+\alpha)\left\{S_1-\frac{\kappa_1+\alpha}{b}\Sigma_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)S_1 \\
&\qquad\qquad + \text{the same with 2 and 1 reversed} \\
D^4 y &= (\kappa_1+\kappa_2)(\kappa_2+\alpha)\left\{U_1-\frac{\kappa_1+\alpha}{b}\Upsilon_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)U_1+\&c. \\
D^4 \zeta &= (\kappa_1+\kappa_2)(\kappa_2+\beta)\left\{\Sigma_1-\frac{\kappa_1+\beta}{a}S_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1+\&c. \\
D^4 \eta &= (\kappa_1+\kappa_2)(\kappa_2+\beta)\left\{\Upsilon_1-\frac{\kappa_1+\beta}{a}U_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Upsilon_1+\&c.
\end{aligned}\right\} \quad . \quad (198)$$

The last three of these equations are to be found by a parallel process, or else by symmetry.

If the right hand sides of (198) be neglected, we clearly obtain, on integration, the first approximation (183) for z, y, ζ, η. This first approximation was originally obtained by mere inspection.

We now have to consider the effects of the small terms on the right on the constants of integration L_1, L_2, L_1', L_2' introduced in the first approximation.

The small terms on the right are, by means of the first approximation, capable of being arranged in one of the alternative forms

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\kappa_1 t + t\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\kappa_1 t + \text{the same with 2 for 1}$$

Now consider the differential equation

$$\frac{d^4x}{dt^4}+(a^2+b^2)\frac{d^2x}{dt^2}+a^2b^2x=A\cos(at+\eta)+Bt\cos(at+\eta) \quad . \quad . \quad . \quad (199)$$

First suppose that B is zero, so that the term in A exists alone.

Assume $x=Ct\sin(at+\eta)$ as the solution.

Then

$$\frac{d^2x}{dt^2}=C\{-a^2t\sin(at+\eta)+2a\cos(at+\eta)\}$$

$$\frac{d^4x}{dt^4}=C\{a^4t\sin(at+\eta)-4a^3\cos(at+\eta)\}$$

By substitution in (199), with $B=0$, we have

$$C\{-4a^3+2a(a^2+b^2)\}=A$$

Therefore the solution is

$$x=-\frac{A}{2a(a^2-b^2)}t\sin(at+\eta)$$

By writing $\eta-\frac{1}{2}\pi$ for η, we see that a term $A \sin(at+\eta)$ in the differential equation would generate $\frac{A}{2a(a^2-b^2)} t \cos(at+\eta)$ in the solution.

From this theorem it follows that the solution of the equation

$$D^4 z = F_1 y_1 + F_2 y_2$$

is

$$z = \frac{tF_1 z_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)} + \text{the same with 2 and 1 interchanged}$$

and the solution of

$$D^4 z = F_1 \eta_1 + F_2 \eta_2$$

is

$$z = \frac{tF_1 \zeta_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)} + \text{the same with 2 and 1 interchanged}$$

Also (writing the two alternatives by means of an easily intelligible notation) the solutions of

$$D^4 y = F_1 \begin{cases} z_1 \\ \zeta_1 \end{cases} + F_2 \begin{cases} z_2 \\ \zeta_2 \end{cases}$$

are

$$y = -\frac{tF_1 \begin{cases} y_1 \\ \eta_1 \end{cases}}{2\kappa_1(\kappa_1^2-\kappa_2^2)} - \text{the same with 2 and 1 interchanged}$$

The similar equations for $D^4\zeta$, $D^4\eta$ may be treated in the same way. The general rule is that

y and η in the differential equations generate in the solution tz and $t\zeta$ respectively; and z and ζ generate $-ty$ and $-t\eta$ respectively; and the terms are to be divided by $2\kappa_1(\kappa_1^2-\kappa_2^2)$ or $2\kappa_2(\kappa_2^2-\kappa_1^2)$ as the case may be.

Next suppose that $A=0$ in the equation (199), and assume as the solution

$$x = Ct^2 \sin(at+\eta) + Dt \cos(at+\eta)$$

Then

$$\begin{aligned} \frac{d^2x}{dt^2} = \quad & C\{-a^2t^2 \sin(at+\eta) + 4at \cos(at+\eta) + 2\sin(at+\eta)\} \\ & + D\{-a^2t \cos(at+\eta) - 2a \sin(at+\eta)\} \\ \frac{d^4x}{dt^4} = \quad & C\{a^4t^2 \sin(at+\eta) - 8a^3t \cos(at+\eta) - 12a^2 \sin(at+\eta)\} \\ & + D\{a^4t \cos(at+\eta) + 4a^3 \sin(at+\eta)\} \end{aligned}$$

Then substituting in (199), we must have

$$4aC(a^2+b^2)-8a^3C=B$$

and

$$2(C-aD)(a^2+b^2)-12a^2C+4a^3D=0$$

Whence

$$C=-\frac{B}{4a(a^2-b^2)},\quad D=-\frac{5a^2-b^2}{4a^2(a^2-b^2)^2}B$$

Hence the solution of (199), when $A=0$, is

$$x=-\frac{5a^2-b^2}{4a^2(a^2-b^2)^2}Bt\cos(at+\eta)-\frac{1}{4a(a^2-b^2)}Bt^2\sin(at+\eta)$$

If t be very small, the second of these terms may be neglected.

By writing $\eta-\frac{1}{2}\pi$ for η, we see that a term $Bt\sin(at+\eta)$ in the differential equation, would have given rise in the solution to

$$x=-\frac{5a^2-b^2}{4a^2(a^2-b^2)^2}Bt\sin(at+\eta)$$

t being very small.

By this theorem we see that the solutions of the two alternative differential equations

$$D^4z=tF_1\begin{Bmatrix}z_1\\ \zeta_1\end{Bmatrix}+tF_2\begin{Bmatrix}z_2\\ \zeta_2\end{Bmatrix}$$

are, when t is very small,

$$z=-\frac{5\kappa_1^2-\kappa_2^2}{4\kappa_1^2(\kappa_1^2-\kappa_2^2)^2}tF_1\begin{Bmatrix}z_1\\ \zeta_1\end{Bmatrix}-\text{the same with 2 and 1 interchanged.}$$

The similar equations for D^4y, $D^4\eta$, $D^4\zeta$ may be treated similarly. The general rule is that:—

tz and $t\zeta$ in the differential equations are reproduced, but with an opposite sign in the solution; and similarly ty and $t\eta$ are reproduced with the opposite sign; and in the solution the terms are to be multiplied by

$$\frac{5\kappa_1^2-\kappa_2^2}{4\kappa_1^2(\kappa_1^2-\kappa_2^2)^2}\text{ or }\frac{5\kappa_2^2-\kappa_1^2}{4\kappa_2^2(\kappa_2^2-\kappa_1^2)^2}$$

For the purpose of future developments it will be more convenient to write these factors in the forms

$$\frac{1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left\{\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right\}\text{ and }\frac{1}{2\kappa_2(\kappa_2^2-\kappa_1^2)}\left\{\frac{2\kappa_2}{\kappa_2^2-\kappa_1^2}+\frac{1}{2\kappa_2}\right\}$$

By means of these two rules we see that the solutions of the two alternative differential equations

$$D^4 z = A_1 \begin{cases} y_1 \\ \eta_1 \end{cases} + tB_1 \begin{cases} z_1 \\ \zeta_1 \end{cases} + \text{the same with 2 for 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

are, so long as t is very small,

$$z = z_1 + \frac{tA_1 \begin{cases} z_1 \\ \zeta_1 \end{cases}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} - \frac{tB_1 \begin{cases} z_1 \\ \zeta_1 \end{cases}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} \left[\frac{2\kappa_1}{\kappa_1^2 - \kappa_2^2} + \frac{1}{2\kappa_1} \right]$$

$$+ \text{the same with 2 and 1 interchanged} \quad (201)$$

Then putting for z_1, ζ_1, &c., their values from (182), these solutions may be written,

$$z = \cos(\kappa_1 t + m_1) \left\{ L_1 + \frac{tA_1 \begin{cases} L_1 \\ L_1' \end{cases}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} - \frac{tB_1 \begin{cases} L_1 \\ L_1' \end{cases}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} \left[\frac{2\kappa_1}{\kappa_1^2 - \kappa_2^2} + \frac{1}{2\kappa_1} \right] \right\}$$

$$+ \text{ the same with 2 for 1} \quad (202)$$

Hence we may retain the first approximation

$$z = L_1 \cos(\kappa_1 t + m_1) + L_2 \cos(\kappa_2 t + m_2)$$

as the solution, provided that L_1 and L_2 are no longer constant, but vary in such a way that

$$\left. \begin{aligned} \frac{dL_1}{dt} &= \frac{A_1 \begin{cases} L_1 \\ L_1' \end{cases}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} - \frac{B_1 \begin{cases} L_1 \\ L_1' \end{cases}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} \left[\frac{2\kappa_1}{\kappa_1^2 - \kappa_2^2} + \frac{1}{2\kappa_1} \right] \\ &\text{and a similar equation for } L_2 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (203)$$

It will be found, when we come to apply these results, that the solution of the equation for $D^4 y$ will lead to the same equations for the variation of L_1 and L_2 as are derived from the equation for $D^4 z$.

A similar treatment may be applied to the equations for $D^4\zeta$ or $D^4\eta$, and we find similar differential equations for dL_1'/dt and dL_2'/dt.

These equations will be the differential equations for the secular changes in L_1 and L_2', which are the constants of integration in the first approximation.

We will now apply these theorems to the differential equations (181); but as the analysis is rather complex, it will be more convenient to treat the variations of α, a, β, b and the terms in γ, g, δ, d independently.

We will indicate by the symbol Δ the additional terms which arise, and will write

the symbol out of which the term arises as a suffix—*e.g.*, we shall write Δz_α for the additional terms in the complete value of z, which arise from the variation of α. Also $(dL/dt)_\alpha$ will be written for the terms in dL/dt which arise from the variation of α.

Terms depending on the variation of α.

We now put for α in (181) $\alpha+\alpha' t$.

Hence in (194)

$$s=\alpha' ty,\ u=-\alpha' tz,\ \sigma=0,\ v=0$$

Therefore

$$S=\alpha'\left\{y-t\left(\alpha z-\frac{dy}{dt}\right)\right\},\ \Sigma=-\alpha'\mathrm{b}tz$$

And by substitution from (182–3)

$$S_1=\alpha'\{y_1+tz_1(\kappa_1-\alpha)\},\quad \Sigma_1=-\alpha'\mathrm{b}tz_1$$

S_2, Σ_2 have similar forms with 2 for 1.

Then

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)S_1=2\alpha'(\kappa_1-\alpha)\frac{dz_1}{dt}=-2\alpha'\kappa_1(\kappa_1-\alpha)y_1$$

$$S_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=\alpha'\{y_1+tz_1(\kappa_1-\alpha)\}+\alpha' t(\kappa_1+\alpha)z_1$$

$$=\alpha'\{y_1+2t\kappa_1 z_1\}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \quad (204)$$

Hence the equation for z is

$$\frac{1}{\alpha'}D^4 z=(\kappa_1+\kappa_2)(\kappa_2+\alpha)(y_1+2t\kappa_1 z_1)-2\kappa_1(\kappa_1-\alpha)y_1+\text{the same with 2 for 1.}$$

Hence by the rules found above for the solution of such an equation

$$\frac{1}{\alpha' t}\Delta z_\alpha=\frac{z_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left\{(\kappa_1+\kappa_2)(\kappa_2+\alpha)-2\kappa_1(\kappa_1-\alpha)-2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)\left[\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right]\right\}+\&\text{c.}$$

$$=-\frac{z_1}{\kappa_1^2-\kappa_2^2}\left[\kappa_1-\alpha+\frac{2\kappa_1(\kappa_2+\alpha)}{(\kappa_1-\kappa_2)}\right]+\&\text{c.}$$

$$=-z_1\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}-z_2\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}$$

Whence

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\alpha=-\alpha'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\alpha=-\alpha'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}\ .\ .\ .\ .\ \quad (205)$$

If we form U and Υ, and solve the equation for D^4y, we obtain the same results.

Again

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1=-2\alpha'\mathrm{b}\frac{dz_1}{dt}=2\alpha'\mathrm{b}\kappa_1y_1$$

$$\Sigma_1-\frac{\kappa_1+\beta}{\mathrm{a}}\mathrm{S}_1=\Sigma_1-\frac{\mathrm{b}}{\kappa_1+\alpha}\mathrm{S}_1=\frac{\mathrm{b}}{\kappa_2+\beta}\left\{\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1\right\}$$

$$=\alpha'\frac{\mathrm{b}}{\kappa_2+\beta}(y_1+2t\kappa_1z_1)\text{ by (204)}$$

Hence the equation for ζ is

$$\frac{1}{\alpha'\mathrm{b}}D^4\zeta=(\kappa_1+\kappa_2)(y_1+2t\kappa_1z_1)+2\kappa_1y_1+\text{the same with 2 for 1}$$

And by the rules of solution

$$\frac{1}{\alpha'\mathrm{b}t}\Delta\zeta_\alpha=\frac{z_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left[\kappa_1+\kappa_2+2\kappa_1-2\kappa_1(\kappa_1+\kappa_2)\left\{\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right\}\right]+\&\text{c.}$$

$$=\frac{z_1}{\kappa_1^2-\kappa_2^2}\left[1-\frac{2\kappa_1}{\kappa_1-\kappa_2}\right]+\&\text{c.}$$

$$\frac{\quad}{(\kappa_1-\kappa_2)^2}-\frac{\quad}{(\kappa_1-\kappa_2)^2}$$

$$=-\frac{1}{\mathrm{b}}\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}\zeta_1-\frac{1}{\mathrm{b}}\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}\zeta_2$$

Since

$$z_1=\frac{\kappa_2+\alpha}{\mathrm{b}}\zeta_1,\quad z_2=\frac{\kappa_1+\alpha}{\mathrm{b}}\zeta_2$$

Hence

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\alpha=-\alpha'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2},\quad\left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\alpha=-\alpha'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}\quad\ldots\ldots\quad(206)$$

If we form U and Υ, and solve the equation for $D^4\eta$, we obtain the same result.

Terms depending on the variation of β.

The results may be written down by symmetry.

z and y are symmetrical with ζ and η, and therefore unaccented L's are symmetrical with accented ones, and *vice-versâ*; α is symmetrical with β, and *vice-versâ*.

The suffixes 1 and 2 remain unaffected by the symmetry.

Then (by (192)) writing for $\kappa_2+\beta$, $\kappa_1+\beta$; $-(\kappa_1+\alpha)$ and $-(\kappa_2+\alpha)$ respectively, we have by symmetry with (206),

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\beta=\beta'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\beta=\beta'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2} \quad . \quad . \quad . \quad . \quad . \quad (207)$$

And by symmetry with (205),

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\beta=\beta'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\beta=\beta'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2} \quad . \quad . \quad . \quad . \quad . \quad (208)$$

Terms depending on the variation of a.

We now put for a in (181) $\mathrm{a}+\mathrm{a}'t$.

Then in (194)

$$s=\mathrm{a}'t\eta,\ u=-\mathrm{a}'t\zeta,\ \sigma=0,\ v=0$$

Therefore

$$\mathrm{S}=\mathrm{a}'\eta+\mathrm{a}'t\left(\frac{d\eta}{dt}-\alpha\zeta\right) \qquad \Sigma=-\mathrm{a}'\mathrm{b}t\zeta$$

$$\mathrm{S}_1=\mathrm{a}'\{\eta_1+t\zeta_1(\kappa_1-\alpha)\} \quad \Sigma_1=-\mathrm{a}'\mathrm{b}t\zeta_1$$

S_2, Σ_2 have similar forms with 2 for 1

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1=2\mathrm{a}'(\kappa_1-\alpha)\frac{d\zeta_1}{dt}=-2\mathrm{a}'\kappa_1(\kappa_1-\alpha)\eta_1$$

$$\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=\mathrm{a}'[\eta_1+t\zeta_1(\kappa_1-\alpha)+t\zeta(\kappa_1+\alpha)]=\mathrm{a}'[\eta_1+2\kappa_1t\zeta_1] \quad . \quad . \quad . \quad (209)$$

Hence the equation for z is

$$\frac{1}{\mathrm{a}'}\mathrm{D}^4z=-2\kappa_1(\kappa_1-\alpha)\eta_1+(\kappa_1+\kappa_2)(\kappa_2+\alpha)(\eta_1+2\kappa_1t\zeta_1)+\text{ the same with 2 for 1}$$

$$\frac{1}{\mathrm{a}'t}\Delta z_\mathrm{a}=\frac{\zeta_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\Big[-2\kappa_1(\kappa_1-\alpha)+(\kappa_1+\kappa_2)(\kappa_2+\alpha)$$
$$-2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)\left(\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right)\Big]+\&\text{c.}$$
$$=-\frac{\zeta_1}{\kappa_1^2-\kappa_2^2}\left[\kappa_1-\alpha+\frac{2\kappa_1(\kappa_2+\alpha)}{\kappa_1-\kappa_2}\right]-\&\text{c.}$$
$$=-\zeta_1\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}-\zeta_2\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}$$
$$=-z_1\frac{\mathrm{b}}{(\kappa_1-\kappa_2)^2}\,\frac{\kappa_1+\alpha}{\kappa_2+\alpha}-z_2\frac{\mathrm{b}}{(\kappa_1-\kappa_2)^2}\,\frac{\kappa_2+\alpha}{\kappa_1+\alpha},\ \text{since } \zeta_1=z_1\frac{\mathrm{b}}{\kappa_2+\alpha},\ \zeta_2=z_2\frac{\mathrm{b}}{\kappa_1+\alpha}$$

Therefore

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_{\mathrm{a}}=-\frac{\mathrm{a}'\mathrm{b}}{(\kappa_1-\kappa_2)^2}\frac{\kappa_1+\alpha}{\kappa_2+\alpha},\quad\left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_{\mathrm{a}}=-\frac{\mathrm{a}'\mathrm{b}}{(\kappa_1-\kappa_2)^2}\frac{\kappa_2+\alpha}{\kappa_1+\alpha}\quad . \quad . \quad . \quad (210)$$

Again

$$\Sigma_1-\frac{\kappa_1+\beta}{\mathrm{a}}\mathrm{S}_1=\Sigma_1-\frac{\mathrm{b}}{\kappa_1+\alpha}\mathrm{S}_1=\frac{\mathrm{b}}{\kappa_2+\beta}\left(\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1\right)$$

$$=\frac{\mathrm{a}'\mathrm{b}}{\kappa_2+\beta}(\eta_1+2\kappa_1 t\zeta_1)\text{ by (209)}$$

Also

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1=-2\mathrm{a}'\mathrm{b}\frac{d\zeta_1}{dt}=2\mathrm{a}'\mathrm{b}\kappa_1\eta_1$$

Therefore the equation for ζ is

$$\frac{1}{\mathrm{a}'\mathrm{b}}\mathrm{D}^4\zeta=(\kappa_1+\kappa_2)(\eta_1+2\kappa_1 t\zeta_1)+2\kappa_1\eta_1+\text{ the same with 2 for 1}$$

Therefore

$$\frac{1}{\mathrm{a}'\mathrm{b}t}\Delta\zeta_{\mathrm{a}}=\frac{\zeta_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left[(\kappa_1+\kappa_2)+2\kappa_1-2\kappa_1(\kappa_1+\kappa_2)\left(\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right)\right]+\&$$

$$=\frac{\zeta_1}{\kappa_1^2-\kappa_2^2}\left(1-\frac{2\kappa_1}{\kappa_1-\kappa_2}\right)+\&\text{c.}$$

$$=-\frac{\zeta_1}{(\kappa_1-\kappa_2)^2}-\frac{\zeta_2}{(\kappa_1-\kappa_2)^2}$$

Therefore

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_{\mathrm{a}}=-\frac{\mathrm{a}'\mathrm{b}}{(\kappa_1-\kappa_2)^2},\quad\left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_{\mathrm{a}}=-\frac{\mathrm{a}'\mathrm{b}}{(\kappa_1-\kappa_2)^2}\quad . \quad . \quad . \quad . \quad (211)$$

The same results might have been obtained from the equations to D^4y, $\mathrm{D}^4\eta$.

Terms depending on the variation of b.

By symmetry with (211)

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_{\mathrm{b}}=-\frac{\mathrm{b}'\mathrm{a}}{(\kappa_1-\kappa_2)^2},\quad\left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_{\mathrm{b}}=-\frac{\mathrm{b}'\mathrm{a}}{(\kappa_1-\kappa_2)^2}\quad . \quad . \quad . \quad . \quad . \quad (212)$$

By symmetry with (210), and putting $-(\kappa_2+\alpha)$ for $(\kappa_1+\beta)$ and $-(\kappa_1+\alpha)$ for $(\kappa_2+\beta)$

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_{\mathrm{b}}=-\frac{\mathrm{b}'\mathrm{a}}{(\kappa_1-\kappa_2)^2}\frac{\kappa_2+\alpha}{\kappa_1+\alpha},\quad\left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_{\mathrm{b}}=-\frac{\mathrm{b}'\mathrm{a}}{(\kappa_1-\kappa_2)^2}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}.\quad . \quad . \quad (213)$$

We now come to a different class of terms, viz. : those depending on γ, g, δ, d.

Terms depending on γ.

Here

$$s=-\gamma z, \quad u=-\gamma y, \quad \sigma=0, \quad v=0$$

$$S=-\gamma\left(\frac{dz}{dt}+\alpha y\right) \qquad \Sigma=-\mathrm{b}\gamma y$$

$$S_1=\gamma(\kappa_1-\alpha)y_1 \qquad \Sigma_1=-\gamma \mathrm{b} y_1$$

S_2, Σ_2 have similar forms with 2 for 1

Obviously

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)S_1=0$$

$$S_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=2\gamma\kappa_1 y_1 \quad \ldots \ldots \ldots \quad (214)$$

Hence the equation for z is

$$\frac{1}{\gamma}D^4 z=2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)y_1+\text{ the same with 2 for 1}$$

Therefore

$$\frac{1}{\gamma t}\Delta z_\gamma=z_1\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}+z_2\frac{\kappa_1+\alpha}{\kappa_2-\kappa_1}$$

And

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\gamma=\gamma\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}, \quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\gamma=-\gamma\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2} \quad \ldots \ldots \quad (215)$$

Again

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1=0$$

$$\Sigma_1-\frac{\kappa_1+\beta}{\mathrm{a}}S_1=\frac{\mathrm{b}}{\kappa_2+\beta}\left[S_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1\right]$$

$$=\mathrm{b}\gamma\frac{2\kappa_1}{\kappa_2+\beta}y_1$$

And the equation for ζ is

$$\frac{1}{\gamma \mathrm{b}}D^4\zeta=2\kappa_1(\kappa_1+\kappa_2)y_1+\text{ the same with 2 for 1}$$

Therefore

$$\frac{1}{\gamma \mathrm{b} t}\Delta\zeta_\gamma=\frac{z_1}{\kappa_1-\kappa_2}+\frac{z_2}{\kappa_2-\kappa_1}$$

$$=\frac{\zeta_1}{\mathrm{b}}\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}-\frac{\zeta_2}{\mathrm{b}}\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2}, \text{ since } z_1=\zeta_1\frac{\kappa_2+\alpha}{\mathrm{b}}, \ z_2=\zeta_2\frac{\kappa_1+\alpha}{\mathrm{b}}$$

Hence

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\gamma=\gamma\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\gamma=-\gamma\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2}\quad . \quad . \quad . \quad . \quad . \quad (216)$$

Terms depending on δ.

These may be written down by symmetry.

$-\delta$ is symmetrical with γ. Hence writing $-(\kappa_1+\alpha)$ for $\kappa_2+\beta$, and $-(\kappa_2+\alpha)$ for $(\kappa_1+\beta)$, we have by symmetry with (216)

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\delta=\delta\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\delta=-\delta\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}.\quad . \quad . \quad . \quad . \quad . \quad (217)$$

And by symmetry with (215)

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\delta=\delta\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\delta=-\delta\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}\quad . \quad . \quad . \quad . \quad (218)$$

Terms depending on g.

Here

$$s=-\mathrm{g}\zeta,\ u=-\mathrm{g}\eta,\ \sigma=0,\ v=0$$

$$\mathrm{S}=-\mathrm{g}\left(\frac{d\zeta}{dt}+\alpha\eta\right)\qquad \Sigma=-\mathrm{gb}\eta$$

$$\mathrm{S}_1=\ \mathrm{g}(\kappa_1-\alpha)\eta_1\qquad \Sigma_1=-\mathrm{gb}\eta_1$$

S_2, Σ_2 have similar forms with 2 for 1

Clearly

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1=0$$

$$\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=2\mathrm{g}\kappa_1\eta_1.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

Therefore the equation for z is

$$\frac{1}{\mathrm{g}}\mathrm{D}^4z=2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)\eta_1+\text{ the same with 2 for 1}$$

Thence

$$\frac{1}{\mathrm{g}t}\Delta z_\mathrm{g}=\zeta_1\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}+\zeta_2\frac{\kappa_1+\alpha}{\kappa_2-\kappa_1}$$

$$=z_1\frac{\mathrm{b}}{\kappa_1-\kappa_2}-z_2\frac{\mathrm{b}}{\kappa_1-\kappa_2},\text{ since }\zeta_1=z_1\frac{\mathrm{b}}{\kappa_2+\alpha},\ \zeta_2=z_2\frac{\mathrm{b}}{\kappa_1+\alpha}$$

Therefore

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_g = g\frac{b}{\kappa_1-\kappa_2}, \quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_g = -g\frac{b}{\kappa_1-\kappa_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

Again

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1=0$$

and

$$\Sigma_1-\frac{\kappa_1+\beta}{a}S_1=\frac{b}{\kappa_2+\beta}\left[S_1-\frac{\kappa_1+\alpha}{b}\Sigma_1\right]$$

$$=g\frac{2b\kappa_1}{\kappa_2+\beta}\eta_1 \text{ by (219)}$$

Therefore the equation for ζ is

$$\frac{1}{gb}D^4\zeta=2\kappa_1(\kappa_1+\kappa_2)\eta_1+\text{ the same with 2 for 1}$$

Hence

$$\frac{1}{gbt}\Delta\zeta_g=\frac{\zeta_1}{\kappa_1-\kappa_2}+\frac{\zeta_2}{\kappa_2-\kappa_1}$$

Therefore

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_g = g\frac{b}{\kappa_1-\kappa_2}, \quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_g = -g\frac{b}{\kappa_1-\kappa_2} \quad . \quad . \quad . \quad . \quad . \quad (221)$$

The same results may be obtained by means of the equations for D^4y, $D^4\eta$.

Terms depending on d.

These may be written down by symmetry.

—d is symmetrical with g. Therefore by symmetry with (221)

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_d = -d\frac{a}{\kappa_1-\kappa_2}, \quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_d = d\frac{a}{\kappa_1-\kappa_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

and by symmetry with (220)

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_d = -d\frac{a}{\kappa_1-\kappa_2}, \quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_d = d\frac{a}{(\kappa_1-\kappa_2)} \quad . \quad . \quad . \quad . \quad . \quad (223)$$

This completes the consideration of the effects on the constants of integration L_1, L_2, L_1', L_2' of all the small terms.

Then collecting results from (205–8, 210–13, 215–18, 220–23),

$$
\left.\begin{aligned}
\frac{1}{L_1}\frac{dL_1}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}-\mathrm{b'a}\right\}\\
&\quad+\frac{1}{(\kappa_1-\kappa_2)}\left\{\gamma(\kappa_2+\alpha)+\delta(\kappa_1+\alpha)+\mathrm{gb}-\mathrm{da}\right\}\\
\frac{1}{L_2}\frac{dL_2}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_2+\alpha)(\alpha'-\beta')-\mathrm{a'b}\frac{\kappa_2+\alpha}{\kappa_1+\alpha}-\mathrm{b'a}\right\}\\
&\quad-\frac{1}{\kappa_1-\kappa_2}\left\{\gamma(\kappa_1+\alpha)+\delta(\kappa_2+\alpha)+\mathrm{gb}-\mathrm{da}\right\}\\
\frac{1}{L_1'}\frac{dL_1'}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_2+\alpha)(\alpha'-\beta')-\mathrm{a'b}-\mathrm{b'a}\frac{\kappa_2+\alpha}{\kappa_1+\alpha}\right\}\\
&\quad+\frac{1}{\kappa_1-\kappa_2}\left\{\gamma(\kappa_2+\alpha)+\delta(\kappa_1+\alpha)+\mathrm{gb}-\mathrm{da}\right\}\\
\frac{1}{L_2'}\frac{dL_2'}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}-\mathrm{b'a}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right\}\\
&\quad\frac{1}{\kappa_1-\kappa_2}\left\{\gamma(\kappa_1+\alpha)+\delta(\kappa_2+\alpha)+\mathrm{gb}-\mathrm{da}\right\}
\end{aligned}\right\}\qquad(224)
$$

We shall now show that these four equations are equivalent to two only, and in showing this shall verify the correctness of the results.

To prove that the four equations (224) *are equivalent to two.*

In (118) we showed that

$$\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}$$

Therefore we ought to find that

$$
\begin{aligned}
\frac{1}{L_1'}\frac{dL_1'}{dt}-\frac{1}{L_1}\frac{dL_1}{dt}&=\frac{1}{\kappa_1+\alpha}\frac{d}{dt}(\kappa_1+\alpha)-\frac{\mathrm{a}'}{\mathrm{a}}\\
&=\frac{\kappa_1+\beta}{\mathrm{ab}}\frac{d}{dt}(\kappa_1+\alpha)-\frac{\mathrm{a}'}{\mathrm{a}}
\end{aligned}
$$

Now by (188)

$$2(\kappa_1+\alpha)=\alpha-\beta-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}$$

and

$$2\frac{d}{dt}(\kappa_1+\alpha)=\alpha'-\beta'+\frac{(\alpha-\beta)(\alpha'-\beta')+2(\mathrm{a'b}+\mathrm{ab'})}{\kappa_1-\kappa_2}$$

so that

$$\frac{d}{dt}(\kappa_1+\alpha)=\frac{(\alpha'-\beta')(\kappa_1+\alpha)+\mathrm{a'b}+\mathrm{ab'}}{\kappa_1-\kappa_2}.$$

And thus we ought to find that,

$$(\kappa_1-\kappa_2)\left[\frac{1}{L_1'}\frac{dL_1'}{dt}-\frac{1}{L_1}\frac{dL_1}{dt}\right]=\alpha'-\beta'-\frac{\mathrm{a}'}{\mathrm{a}}(\kappa_1+\alpha)-\frac{\mathrm{b}'}{\mathrm{b}}(\kappa_2+\alpha)$$

Now if we subtract the first of equations (224) from the third we shall find this relation to be satisfied. Hence the first and third equations are equivalent to only a single one.

Similarly it may be proved that the second and third equations are similarly related.

To prove that the four equations (224) *reduce to those of* § 6, *when the nodes revolve with uniform velocity.*

It appears from § 13 that when a and b are small compared with $\alpha-\beta$, the nodes revolve with approximate uniformity, and the nutations of the system are small.

If this be the case, we have approximately

$$\kappa_1=-\alpha,\ \kappa_2=-\beta.$$

It will appear later that $(\alpha'-\beta')/(\alpha-\beta)$, a'/a, b'/b are quantities of the same order of magnitude as γ, g, δ, d.

Now $L_1=\mathrm{J}$, the inclination of the lunar orbit to its proper plane, and $L_2'=\mathrm{I}$, the inclination of the earth's proper plane to the ecliptic.

Therefore, the first and last of equations (224) become

$$\frac{1}{\mathrm{J}}\frac{d\mathrm{J}}{dt}=-\gamma-\frac{\mathrm{gb}-\mathrm{da}}{\alpha-\beta}$$

$$\frac{1}{\mathrm{I}}\frac{d\mathrm{I}}{dt}=\delta+\frac{\mathrm{gb}-\mathrm{da}}{\alpha-\beta}$$

But since the nodes revolve uniformly, $\mathrm{b}/(\alpha-\beta)$ and $\mathrm{a}/(\alpha-\beta)$ are small, and therefore the latter terms of these equations are negligeable compared with the former.

Hence

$$\frac{1}{\mathrm{J}}\frac{d\mathrm{J}}{dt}=-\gamma,\quad \frac{1}{\mathrm{I}}\frac{d\mathrm{I}}{dt}=\delta$$

These results in no way depend on the assumption of the smallness of the viscosity of the planet, and therefore we may substitute Γ and Δ (see (174)) for γ and δ.

A comparison of the expressions for Γ and Δ, with those given in Part II. for dj/dt and in my previous paper for di/dt, will show that our present equations for $d\mathrm{J}/dt$ and $d\mathrm{I}/dt$ are what the previous ones reduce to, when i and j are small. But this comparison shows more than this, for it shows that what the equation (61) § 6 really gives is the rate of change of the inclination of the lunar orbit to its proper plane, and that the equation (66) of the paper on "Precession" really gives the rate of change of the inclination of the earth's proper plane (or mean equator) to the ecliptic.

To show how the equations (224) *reduce to those of* § 10.

We now pass to the other extreme, and suppose the solar influence infinitesimal compared with that of oblateness.

Here

$$\alpha=\mathrm{a},\quad \beta=\mathrm{b},\quad \gamma=\mathrm{g},\quad \delta=\mathrm{d}$$
$$\kappa_1=-(\mathrm{a}+\mathrm{b}),\quad \kappa_2=0$$

Then the equations (224) reduce to

$$\left.\begin{aligned}
&\frac{1}{L_1}\frac{dL_1}{dt}=-\mathrm{g}+\mathrm{d}+\frac{\mathrm{a'b}-\mathrm{b'a}}{\mathrm{a}(\mathrm{a}+\mathrm{b})}\\
&\frac{1}{L_1'}\frac{dL_1'}{dt}=-\mathrm{g}+\mathrm{d}-\frac{\mathrm{a'b}-\mathrm{b'a}}{\mathrm{b}(\mathrm{a}+\mathrm{b})}\\
&\frac{1}{L_2}\frac{dL_2}{dt}=0,\quad \frac{1}{L_2'}\frac{dL_2'}{dt}=0
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Therefore L_2 and L_2' are constant. Also from the relationship between them

$$\frac{L_2'}{L_2}=-\frac{(\kappa_2+\alpha)}{\mathrm{a}}=-1$$

Hence it follows that the two proper planes are identical with one another, and are fixed in space. They are, in fact, the invariable plane of the system, as appears as follows :—

If we use the notation of § 10, $L_1=j$, $L_1'=i$, and $L_1'/L_1=-(\kappa_1+\alpha)/\mathrm{a}=\mathrm{b}/\mathrm{a}$; so that $\mathrm{a}i=\mathrm{b}j$.

Now $\mathrm{a}=k\tau\mathfrak{C}/\xi$, $\mathrm{b}=\tau\mathfrak{C}/n$, and i and j are by hypothesis small, therefore we may write the relationship between a, b, i, j in the form

$$\frac{\xi}{k}\sin j=n\sin i.$$

This proves that the two coincident planes fixed in space are identical with the invariable plane of the system (see 108).

But the identity of equations (225) with (71) of § 10 and (29) of the paper on "Precession" remains to be proved.

If i and j be treated as small, those equations are in effect

$$\frac{dj}{dt}=-\mathrm{g}(i+j)$$
$$\frac{di}{dt}=\ \mathrm{d}(i+j)$$

(or with G and D in place of g and d if the viscosity be not small).

Hence if (225) are identical with (71) and (29) of "Precession," we must have

$$-\mathrm{g}\frac{i}{j}=\mathrm{d}+\frac{\mathrm{a'b}-\mathrm{ab'}}{\mathrm{a(a+b)}}$$

$$\mathrm{d}\frac{j}{i}=-\mathrm{g}-\frac{\mathrm{a'b}-\mathrm{ab'}}{\mathrm{b(a+b)}}$$

But $i/j=\mathrm{b/a}$; therefore the condition for the identity of (225) with (71) and (29) of "Precession" is that

$$(\mathrm{a}+\mathrm{b})(\mathrm{gb}+\mathrm{ad})+\mathrm{a'b}-\mathrm{ab'}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

Or if the viscosity be not small, a similar equation with G and D for g and d.

We cannot prove that this condition is satisfied until a′ and b′ have been evaluated, but it will be proved later in § 16.

This discussion shows that the obliquity of the earth's equator (L_1') to the invariable plane of the moon-earth system, when the solar influence is infinitesimal, degrades into the amplitude of the nineteen-yearly nutation, when the influence of oblateness is infinitesimal. The one quantity is strictly continuous with the other.

This completes the verification of the differential equations (224) in the two extreme cases.

§ 16. *Evaluation of α', a′ &c., in the case of the earth's viscosity.*

The preceding section does not involve any hypothesis as to the constitution of the earth, but it will now be supposed to be viscous, and the various functions, which occur in (224), will be evaluated.

By (184–5) we have

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

$$-\frac{dn}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1\left(1+\left(\frac{\tau'}{\tau}\right)^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

The last equation is approximate, for by writing it in this form we are neglecting $\tau'^2(\sin 4\mathrm{f}-\sin 4\mathrm{f}_1)/\tau^2\sin 4\mathrm{f}_1$ compared with unity.

This is legitimate, because when $(\sin 4\mathrm{f}-\sin 4\mathrm{f}_1)/\sin 4\mathrm{f}_1$ is not very small, τ'^2/τ^2 is very small, and *vice-versâ*; see however § 22.

Hence (228) may be written

$$-\frac{dn}{dt}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(1+\left(\frac{\tau'}{\tau}\right)^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

Let

$$\mathfrak{m}=\frac{kn}{\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

$\mathfrak{m}$ is the ratio of the moment of momentum of the earth's rotation to that of the orbital motion of moon and earth round their common centre of inertia. (The μ of my paper on "Precession" is equal to the reciprocal of $\mathfrak{m}_0$, where $\mathfrak{m}_0$ is the value of $\mathfrak{m}$ when $t=0$.)

By (121) and (112) we have,

$$\mathfrak{a}=\frac{k}{\xi}\tau\mathfrak{e}$$

Now $\mathfrak{e}=\frac{1}{2}n^2/\mathfrak{g}$, the ellipticity of the earth due to rotation; and as $\tau=\frac{3}{2}\mu m/c^3$ and $\xi=\sqrt{c/c_0}$. therefore $\tau=\tau_0/\xi^6$.

Hence

$$\mathfrak{a}=\frac{k\tau_0}{2\mathfrak{g}}\frac{n^2}{\xi^7}$$

Differentiating logarithmically

$$\frac{\mathfrak{a}'}{\mathfrak{a}}=\frac{2}{n}\frac{dn}{dt}-\frac{7}{\xi}\frac{d\xi}{dt}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left\{\frac{2}{n}\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+\frac{7k}{\xi}\right.$$

$$\frac{\mathfrak{a}'}{\mathfrak{a}}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}\left\{2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

Also since

$$\mathfrak{a}=\mathfrak{m}\frac{\tau\mathfrak{e}}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

$$\mathfrak{a}'=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\mathfrak{m}\left\{2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

Then by (121) and (112)

$$\alpha-\mathfrak{a}=\frac{1}{2}\frac{\tau'}{\Omega}$$

Since $\Omega=\Omega_0/\xi^3$, and τ' is constant (or at least varies so slowly that we may neglect its variation), we have

$$\frac{\alpha'-\mathfrak{a}'}{\alpha-\mathfrak{a}}=\frac{3}{\xi}\frac{d\xi}{dt}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}3\mathfrak{m}$$

Now $\frac{\Omega}{n}=\lambda$, hence

$$\alpha-\mathfrak{a}=\frac{\tau\mathfrak{e}}{n}\frac{\tau'}{\tau}\left(\frac{1}{2\lambda\mathfrak{e}}\right.$$

Therefore

$$\alpha'-\mathfrak{a}'=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\frac{\tau'}{\tau}\frac{3\mathfrak{m}}{2\lambda\mathfrak{e}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

From (233–4)

$$\alpha'=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\mathfrak{m}\left\{\frac{3}{2\lambda\mathfrak{e}}\frac{\tau'}{\tau}-\left[2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right]\right\} \quad . \quad . \quad . \quad . \quad (235)$$

Also

$$\alpha=\frac{\tau\mathfrak{e}}{n}\left[\frac{\tau'}{\tau}\frac{1}{2\lambda\mathfrak{e}}+\mathfrak{m}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

By (121) and (112)

$$\mathrm{b}=\frac{\tau\mathfrak{e}}{n}=\frac{\tau_0}{2\mathfrak{g}}\frac{n}{\xi^6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (237)$$

Therefore

$$\frac{\mathrm{b}'}{\mathrm{b}}=\frac{1}{n}\frac{dn}{dt}-\frac{6}{\xi}\frac{d\xi}{dt}$$

$$=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}\left\{1+\left(\frac{\tau'}{\tau}\right)^2+6\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (238)$$

And

$$\mathrm{b}'=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\left\{1+\left(\frac{\tau'}{\tau}\right)^2+6\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

From (231) and (238)

$$\frac{\mathrm{a}'}{\mathrm{a}}-\frac{\mathrm{b}'}{\mathrm{b}}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}\left\{1+\left(\frac{\tau'}{\tau}\right)^2+\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (240)$$

By (121) and (112)

$$\beta-\mathrm{b}=\frac{\tau'\mathfrak{e}}{n}=\frac{\tau'}{2\mathfrak{g}}n$$

$$\beta'-\mathrm{b}'=\frac{\tau'}{2\mathfrak{g}}\frac{dn}{dt}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\tau'}{2\mathfrak{g}}\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)$$

$$=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\left(\frac{\tau'}{\tau}\right)\left(1+\left(\frac{\tau'}{\tau}\right)^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241)$$

Therefore

$$\beta'=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\left\{1+\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2+\left(\frac{\tau'}{\tau}\right)^3+6\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad (242)$$

Lastly

$$\beta=\frac{\tau\mathfrak{e}}{n}\left(1+\frac{\tau'}{\tau}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (243)$$

By (174), (227), and (230), when the viscosity is not small, we have

$$
\left.\begin{aligned}
\Gamma &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\,\frac{(\sin 4f_1 - \sin 2g_1 + \sin 2g)}{\sin 4f_1}\\
G &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\,\frac{(\sin 4f_1 - \sin 2g_1 + \sin 2g) + \frac{\tau'}{\tau}\sin 2g}{\sin 4f_1}\\
\Delta &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\,\frac{(\sin 4f_1 + \sin 2g_1 - \sin 2g) - 2\frac{\tau'}{\tau}\sin 2g + \left(\frac{\tau'}{\tau}\right)^2 \sin 4f}{\sin 4f_1}\\
D &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\,\frac{(\sin 4f_1 + \sin 2g_1 - \sin 2g) - \frac{\tau'}{\tau}\sin 2g}{\sin 4f_1}
\end{aligned}\right\} \quad . \quad . \quad (244)
$$

If the viscosity be small we have by (179), (227), and (230)

$$
\left.\begin{aligned}
\gamma &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\,\frac{1}{1-\lambda}\\
g &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\,\frac{1+\frac{1}{2}\frac{\tau'}{\tau}}{1-\lambda}\\
\delta &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\,\frac{1-2\lambda-\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2}{1-\lambda}\\
d &= \left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\,\frac{1-2\lambda-\frac{1}{2}\frac{\tau'}{\tau}}{1-\lambda}
\end{aligned}\right\} \quad (245)
$$

I think no confusion will arise between the distinct uses made of the symbol g in (244) and (245); in the first it always must occur with a sine, in the second it never can do so.

[If τ' be zero

$$bG+aD=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\mathfrak{m}}{2n}\right)\left(\frac{\tau\mathfrak{c}}{n}\right)(2)$$

$$a+b=\frac{\tau\mathfrak{c}}{n}(1+\mathfrak{m})$$

and by (232), (237), (240)

$$ab'-ba'=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{n}\left(\frac{\tau\mathfrak{c}}{n}\right)^2(1+\mathfrak{m})$$

Therefore we have

$$(bG+aD)(a+b)+a'b-b'a=0$$

This was shown in (226) to be the criterion that the differential equations (224) should reduce to those of (71) and of (29) of "Precession," when the solar influence is evanescent, and the above is the promised proof thereof.]

From (244), (237), and (232) we have

$$\mathrm{bG}-\mathrm{aD}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\mathfrak{m}}{2n}\right)\left(\frac{\tau\mathfrak{t}}{n}\right)\frac{2\left(1+\frac{\tau'}{\tau}\right)\sin 2\mathrm{g}-2\sin 2\mathrm{g}_1}{\sin 4\mathrm{f}_1} \quad . \quad . \quad . \quad . \quad . \quad (246)$$

and similarly

$$\mathrm{bg}-\mathrm{ad}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\mathfrak{m}}{2n}\right)\left(\frac{\tau\mathfrak{t}}{n}\right)\frac{2\lambda+\frac{\tau'}{\tau}}{1-\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (247)$$

§ 17. *Change of independent variable, and formation of equations for integration.*

In the equations (224) the time t is the independent variable, but in order to integrate we shall require ξ to be the variable. It has been shown above that these equations are equivalent to only two of them; henceforth therefore we shall only consider the first and last of them. It will also serve to keep before us the physical meaning of the L's, if the notation be changed; the following notation (which has been already used in (127)) will be adopted :—

$\mathrm{J}=\ L_1=$ the inclination of the lunar orbit to the lunar proper plane.
$\mathrm{I}=\ L_2'=$ the inclination of the earth's proper plane to the ecliptic.
$\mathrm{I}_{,}=\ L_1'=$ the inclination of the equator to the earth's proper plane.
$\mathrm{J}_{,}=-L_2=$ the inclination of the lunar proper plane to the ecliptic.

Then since J, I, &c., are small, we may write

$$\frac{dL_1}{L_1}=d.\log\tan\tfrac{1}{2}\mathrm{J},\quad \frac{dL_2'}{L_2'}=d.\log\tan\tfrac{1}{2}\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (248)$$

This particular transformation is chosen because in Part II., where j and i were not small, $dj/\sin j$ seemed to arise naturally.

Also since

$$\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}},\quad \frac{L_2'}{L_2}=-\frac{\kappa_2+\alpha}{\mathrm{a}}$$

we have

$$\left.\begin{aligned}\sin\mathrm{I}_{,}&=-\frac{\kappa_1+\alpha}{\mathrm{a}}\sin\mathrm{J}\\ \sin\mathrm{J}_{,}&=\frac{\mathrm{a}}{\kappa_2+\alpha}\sin\mathrm{I}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

These equations will give $\mathrm{I}_{,}$ and $\mathrm{J}_{,}$, when J and I are found.

Now suppose we divide the first and last of (224) by $d\xi/nkdt$, then their left-hand sides may be written

$$nk\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{J} \quad \text{and} \quad nk\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{I}$$

In the last section we have determined the functions α, α', &c., and have them in such a form that Γ, G, Δ, D (or γ, g, δ, d) have all a common factor $d\xi/nkdt$.

But this is the expression by which we have to divide the equations in order to change the variable.

Therefore in computing Γ, G, &c. (or γ, g, &c.), we may drop this common factor.

Again α, a, β, b were so written as all to have a common factor $\tau\mathfrak{t}/n$; therefore κ_1 and κ_2 also have the same common factor.

Also α' a$'$, β', b$'$ all have a common factor $(d\xi/kdt)(\tau\mathfrak{t}/n^2)$.

From this it follows that when the variable is changed, we may drop the factor $\tau\mathfrak{t}/n$ from α, a, β, b, κ_1, κ_2 and the factor $(d\xi/kdt)(\tau\mathfrak{t}/n^2)$ from α', a$'$, β', b$'$.

Hence the differential equations with the new variable become

$$
\begin{aligned}
kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{J} &= \frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}-\mathrm{b'a}\right\} \\
&\quad+\frac{1}{(\kappa_1-\kappa_2)}\{\gamma(\kappa_2+\alpha)+\delta(\kappa_1+\alpha)+\mathrm{gb}-\mathrm{da}\} \\
kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{I} &= \frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}-\mathrm{b'a}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right\} \\
&\quad-\frac{1}{(\kappa_1-\kappa_2)}\{\gamma(\kappa_1+\alpha)+\delta(\kappa_2+\alpha)+\mathrm{gb}-\mathrm{da}\}
\end{aligned}
\tag{250}
$$

or similar equations with Γ, G, Δ, D in place of γ, g, δ, d if the viscosity be not small.

But we now have by (232–3–5–6–7–9, 242–3–4–5–6–7)

$$
\begin{aligned}
&\alpha=\mathfrak{m}+\frac{\tau'}{\tau}\frac{1}{2\lambda\mathfrak{t}}, \qquad \mathrm{a}=\mathfrak{m}, \qquad \beta=1+\frac{\tau'}{\tau}, \qquad \mathrm{b}=1 \\
&\alpha'=\mathfrak{m}\left\{\frac{\tau'}{\tau}\frac{3}{2\lambda\mathfrak{t}}-\left[2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right]\right\}, \ \mathrm{a}'=-\mathfrak{m}\left\{2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right\} \\
&\beta'=-\left\{1+\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2+\left(\frac{\tau'}{\tau}\right)^3+6\mathfrak{m}\right\}, \quad \mathrm{b}'=-\Big\{1+\left(\frac{\tau'}{\tau}\right)^2+6\mathfrak{m} \\
&\Gamma=\tfrac{1}{2}\mathfrak{m}\frac{\sin 4\mathrm{f}_1-\sin 2\mathrm{g}_1+\sin 2\mathrm{g}}{\sin 4\mathrm{f}_1}, \\
&\Delta=\frac{(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g})-2\frac{\tau'}{\tau}\sin 2\mathrm{g}+\left(\frac{\tau'}{\tau}\right)^2\sin 4\mathrm{f}}{2\sin 4\mathrm{f}_1} \\
&\gamma=\frac{\mathfrak{m}}{2(1-\lambda)}, \qquad \delta=\frac{1-2\lambda-\frac{\tau'}{\tau}+\Big(\frac{\tau'}{\tau}}{2(1-\lambda)} \\
&\mathrm{bG}-\mathrm{aD}=\tfrac{1}{2}\mathfrak{m}\frac{2\left(1+\frac{\;}{\tau}\right)\sin 2\mathrm{g}-2\sin 2\mathrm{g}_1}{\sin 4\mathrm{f}_1} \\
&\mathrm{bg}-\mathrm{ad}=\frac{\mathfrak{m}\left(2\lambda+\frac{\tau'}{\tau}\right)}{2(1-\lambda)}
\end{aligned}
\tag{251}
$$

In these equations we have, recapitulating the notation

$$\mathfrak{m}=\frac{kn}{\xi},\ \lambda=\frac{\Omega}{n},\ \mathfrak{c}=\frac{1}{2}\frac{n^2}{\mathfrak{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

Also

$$\left.\begin{aligned} \kappa_1+\kappa_2&=-\alpha-\beta \\ \kappa_1-\kappa_2&=-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}} \end{aligned}\right\} \quad (253)$$

Lastly we have by (186)

$$\frac{n}{n_0}=1+\frac{1}{kn_0}\left\{(1-\xi)+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

which gives parallel values of n and ξ.

These equations will be solved by quadratures for the case of the moon and earth in Part IV.

If τ'/τ be so small as to be negligeable, and $\tau'/2\lambda\mathfrak{c}\tau$ small compared with unity, then the equations (250) admit of reduction to a simple form.

With this hypothesis it is easy to find approximate values of κ_1 and κ_2, and then by some easy, but rather tedious analysis, it may be shown that (250) reduce to the following—

$$\left.\begin{aligned} kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{J}&=-\frac{\mathfrak{m}+1}{\mathfrak{m}}\mathrm{G}+\frac{\tau'}{\tau}\frac{1}{2\lambda\mathfrak{c}}\frac{1+11\mathfrak{m}}{(1+\mathfrak{m})^2} \\ kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{I}&=\frac{\tau'}{\tau}\frac{1}{2\lambda\mathfrak{c}}\frac{1+11\mathfrak{m}}{(1+\mathfrak{m})^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (255)$$

These equations would give the secular changes of J and I, when the solar influence is very small compared with that of the moon. Of course if G be replaced by g, they are applicable to the case of small viscosity.

It is remarkable that the changes of I are independent of the viscosity; they depend in fact solely on the secular change in the permanent ellipticity of the earth.

IV.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS FOR CHANGES IN THE INCLINATION OF THE ORBIT AND THE OBLIQUITY OF THE ECLIPTIC.

§ 18. *Integration in the case of small viscosity, where the nodes revolve uniformly.*

It is not, even at the present time, rigorously true that the nodes of the lunar orbit revolve uniformly on the ecliptic and that the inclination of the orbit is constant; but it is very nearly true, and the integration may be carried backwards in time for a long way without an important departure from accuracy.

The integrations will be carried out by the method of quadratures, and the process will be divided into a series of "periods of integration," as explained in § 15 and § 17 of the paper on "Precession." These periods will be the same as those in that paper, and the previous numerical work will be used as far as possible. It will be found, however, that it is not sufficiently accurate to assume the uniform revolution of the nodes beyond the first two periods of integration. For these first two periods the equations of § 7, Part II., will be used; but for the further retrospect we shall have to make the transition to the methods of Part III. It is important to defer the transition as long as possible, because Part III. assumes the smallness of i and j, whilst Part II. does not do so.

By (104) and (86) of Part II. we have, when $j'=0$, and Ω'/n is neglected,

$$\frac{di}{dt}=\frac{\sin 4\mathrm{f}}{n\mathfrak{g}}\tfrac{1}{4}\sin i\cos i\left\{\tau^2(1-\tfrac{3}{2}\sin^2 j)+\tau'^2-\frac{2\Omega}{n}\tau^2\sec i\cos j-\tau\tau'(1-\tfrac{3}{2}\sin^2 j)\right\}$$

$$=\frac{\sin 4\mathrm{f}}{n\mathfrak{g}}\left\{(1-\tfrac{1}{2}\sin^2 i)(\tau^2+\tau'^2)-\tfrac{1}{2}(1-\tfrac{3}{2}\sin^2 i)\tau^2\sin^2 j\right.$$
$$\left.-\tau^2\frac{\Omega}{n}\cos i\cos j+\tfrac{1}{2}\tau\tau'\sin^2 i(1-\tfrac{3}{2}\sin^2 j)\right\}$$

If we put $1-\frac{1}{2}\sin^2 i=\cos i$, $1-\frac{3}{2}\sin^2 j=\cos^3 j$, and neglect $\sin^2 i\sin^2 j$, these may be written

$$\left.\begin{aligned}\frac{di}{dt}&=\frac{\sin 4\mathrm{f}}{n\mathfrak{g}}\tfrac{1}{4}\sin i\cos i\cos^3 j\left\{\tau^2+\tau'^2\sec^3 j-\tau\tau'-\frac{2\Omega}{n}\tau^2\sec i\sec^2 j\right\}\\-\frac{dn}{dt}&=\frac{\sin 4\mathrm{f}}{n\mathfrak{g}}\cos i\cos j\left\{\tau^2+\tau'^2\sec j-\tau^2\frac{\Omega}{n}+\tfrac{1}{2}\tau\tau'\sin i\tan i\cos^2 j\right\}\end{aligned}\right\}\qquad(256)$$

If we treat $\sec j$ and $\cos j$ as unity in the small terms in τ'^2, $\tau\tau'$, and Ω/n, (256) only differ from (83) of "Precession" in that di/dt has a factor $\cos^3 j$ and dn/dt has a factor $\cos j$.

Again by (64) and (70)

$$\left.\begin{aligned}-\frac{1}{k}\frac{dj}{dt}&=\frac{1}{\xi}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}\tau^2\tfrac{1}{4}\cos i\sin j\\\frac{1}{k}\frac{d\xi}{dt}&=\frac{\sin 4\mathrm{f}}{\mathfrak{g}}\tau^2\tfrac{1}{2}\cos i\cos j\left(1-\frac{\Omega}{n}\sec i\sec j\right)\end{aligned}\right\}\quad .\ .\ .\ .\ .\ (257)$$

If we divide the second of (256) by the second of (257) we get an equation for $dn/d\xi$, which only differs from (84) of "Precession" in the presence of $\sec j$ in place of unity in certain of the small terms. Now j is small for the lunar orbit; hence the equation (88) of "Precession" for the conservation of moment of momentum is very nearly true. The equation is, with present notation,

$$\frac{n}{n_0}=1+\frac{1}{kn_0}\left[1-\xi+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})+\tfrac{1}{14}\frac{\tau'}{\tau_0}\sin i\tan i\,(1-\xi^7)\right]$$

$$+\tfrac{1}{4}\sin^2 i\frac{\Omega_0}{n_0}\frac{1}{kn_0+1}\left(\frac{1}{\xi}-1\right)\left(\frac{1}{\xi}+\frac{kn_0+3}{kn_0+1}\right)+\tfrac{1}{2}\sin^2 i\frac{\Omega_0}{n_0}\frac{1}{(kn_0+1)^3}\log_e\left(\frac{kn_0+1-\xi}{kn_0\xi}\right) \quad (258)$$

In this equation we attribute to i, as it occurs on the right-hand side, an average value.

By means of this equation, I had already computed a series of values of n corresponding to equidistant values of ξ.

On dividing the first of (256) by the second of (257) we get an expression which differs from the $d\log\tan^2\frac{1}{2}i/d\xi$ of (84) of "Precession" by the presence of a common factor $\cos^2 j$, and by $\sec j$ occurring in some of the small terms. Hence we may, without much error, accept the results of the integration for i in § 17 of "Precession."

Lastly, dividing the first of (257) by the second, we have

$$\frac{d}{d\xi}\log\sin j=\frac{-1}{2\xi\left(1-\frac{\Omega}{n}\sec i\sec j\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (259)$$

This equation has now to be integrated by quadratures.

All the numerical values were already computed for § 17 of "Precession," and only required to be combined.

The present mean inclination of the lunar orbit is 5° 9′, so that $j_0=5°\ 9'$. I then conjecture 5° 12′ as a proper mean value to be assigned to j, as it occurs on the right-hand side of (259) for the first period of integration, which extends from $\xi=1$ to ·88.

First period of integration.

From $\xi=1$ to ·88, four equidistant values were computed.

From the computation for § 17 of "Precession" I extract the following.

ξ	$=1$	·96	·92	·88
$\log\frac{\Omega}{n}\sec i+10=$	8·59979	8·57309	8·56411	8·56746

Then introducing $j=5°\ 12'$, I find

ξ	$=1$	·96	·92	·88
$\left[2\xi\left(1-\frac{\Omega}{n}\sec i\sec j\right)\right]^{-1}=$	·5208	·5412	·5643	·5901

Combining these four values by the rules of the calculus of finite differences, we have

$$\int_{\cdot 88}^{1} \frac{d\xi}{2\xi\left(1-\frac{\Omega}{n}\sec i \sec j\right)} = \cdot 06641$$

This is equal to $\log_e \sin j - \log_e \sin j_0$. Taking $j_0 = 5° \; 9'$, I find $j = 5° \; 30'$.

Second period of integration.

From $\xi = 1$ to $\cdot 76$, four equidistant values were computed.

From the computation for § 17 "Precession," I extract the following :—

ξ	$= 1$	$\cdot 92$	$\cdot 84$	$\cdot 76$
$\log \frac{\Omega}{n} \sec i + 10$	$= 8\cdot 56746$	$8\cdot 59743$	$8\cdot 65002$	$8\cdot 72318$

Then assuming $5° \; 55'$ as an average value for j, I find

ξ	$= 1$	$\cdot 92$	$\cdot 84$	$\cdot 76$
$\left[2\xi\left(1-\frac{\Omega}{n}\sec i \sec j\right)\right]^{-1}$	$= \cdot 5193$	$\cdot 5660$	$\cdot 6232$	$\cdot 6948$

Combining these, we have

$$\int_{\cdot 76} \frac{d\xi}{2\xi\left(1-\frac{\Omega}{n}\sec i \sec j\right)} = \cdot 14345$$

This is equal to $\log_e \sin j - \log_e \sin j_0$. Taking $j_0 = 5° \; 30'$ from the first period, we find $j = 6° \; 21'$.

This completes the integration, as far as it is safe to employ the methods of Part II. In Part III. it was proved that, in the case where the nodes revolve uniformly, equations (224) reduce to those of Part II. But it was also shown that what the equations of Part II. really give is the change of the inclination of the lunar orbit to the lunar proper plane ; also that the equations of "Precession" really give the change of the inclination of the mean equator (that is of the earth's proper plane) to the ecliptic.

The results of the present integration are embodied in the following table, of which the first three columns are taken from the table in § 17 of "Precession."

TABLE I.

	Sidereal day in m.s. hours and minutes.		Moon's sidereal period in m.s. days.	Inclination of mean equator to ecliptic.		Inclination of lunar orbit to lunar proper plane.	
	h.	m.	Days.	°	′	°	′
Initial	23	56	27·32	23	28	5	9
	15	28	18·62	20	28	5	30
Final	9	55	8·17	17	4	6	21

We will now consider what amount of oscillation the equator and the plane of the lunar orbit undergo, as the nodes revolve, in the initial and final conditions represented in the above table.

It appears from (119) that $\sin 2j$ oscillates between $\sin 2j_0 \pm \mathrm{a} \sin 2i_0/(\kappa_2+\alpha)$, and that $\sin 2i$ oscillates between $\sin 2i_0 \pm (\kappa_1+\alpha) \sin 2j_0/\mathrm{a}$, where i_0 and j_0 are the mean values of i and j.

With the numerical values corresponding to the initial condition (that is to say in the present configurations of earth, moon, and sun), it will be found on substituting in (115) and (112), with $a_2=\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\left(1-\frac{3}{8}\frac{\Omega'}{\Omega}\right)\Omega$ instead of simply $\frac{1}{2}\frac{\tau'}{\Omega}$, that

$$\alpha=\cdot 341251,\ \beta=\cdot 000318,\ \mathrm{a}=\cdot 000059,\ \mathrm{b}=\cdot 000150,$$

when the present tropical year is the unit of time.

Since 4ab is very small compared with $(\alpha-\beta)^2$, it follows that we have to a close degree of approximation

$$\kappa_1=-\alpha,\ \kappa_2=-\beta$$

Then since $(\kappa_1+\alpha)/\mathrm{a}=\mathrm{b}/(\kappa_1+\beta)$, it follows that $\sin 2j$ oscillates between $\sin 2j_0 \pm \mathrm{a} \sin 2i_0/(\alpha-\beta)$, and $\sin 2i$ between $\sin 2i_0 \mp \mathrm{b} \sin 2j_0/(\alpha-\beta)$.

Let δj and δi be the oscillations of j and i on each side of the mean, then $\delta \sin 2j=\mathrm{a} \sin 2i/(\alpha-\beta)$ and $\delta \sin 2i=\mathrm{b} \sin 2j/(\alpha-\beta)$.

Hence in seconds of arc

$$\left.\begin{aligned}\delta j&=\frac{648000}{\pi}\,\frac{1}{2}\,\frac{\mathrm{a}}{\alpha-\beta}\,\frac{\sin 2i}{\cos 2j}\\ \delta i&=\frac{648000}{\pi}\,\frac{1}{2}\,\frac{\mathrm{b}}{\alpha-\beta}\,\frac{\sin 2j}{\cos 2i}\end{aligned}\right\} \qquad (260)$$

Reducing these to numbers with $j=5° \ 9'$, $i=23° \ 28'$. we have $\delta j=13''\cdot13$, $\delta i=11''\cdot86$.*

Hence, if the earth were homogeneous, at the present time we should have δj as the inclination of the proper plane of the lunar orbit to the ecliptic, and δi as the amplitude of the 19-yearly nutation. These are very small angles, and therefore initially the method of Part II. was applicable.

* The formulas here used for the amplitude of the 19-yearly nutation and for the inclination of the lunar proper plane to the ecliptic differ so much from those given by other writers that it will be well to prove their identity.

LAPLACE ('Méc. Cél.,' liv. vii., chap. 2) gives as the inclination of the proper plane to the ecliptic

$$\frac{\alpha\rho-\frac{1}{2}\alpha\phi}{g-1}\frac{D^2}{a^2}\sin\lambda\cos\lambda$$

Here $\alpha\rho$ is the earth's ellipticity, and is my ϵ; $\alpha\phi$ is the ratio of equatorial centrifugal force to gravity, and is my n^2a/g, it is therefore $\frac{4}{5}\epsilon$ when the earth is homogeneous.

Thus his $\alpha\rho-\frac{1}{2}\alpha\phi=$ my $\frac{3}{5}\epsilon$. His $g-1$ is the ratio of the angular velocity of the nodes to that of the moon, and is therefore my $(\alpha-\beta)/\Omega$. His D is the earth's mean radius, and is my a. His a is the moon's mean distance, and is my c. His λ is the obliquity, and is my i. Thus his formula is $\frac{3}{5}\frac{\epsilon\Omega}{\alpha-\beta}\frac{a^2}{c^2}\sin i\cos i$ in my notation.

Now my $\tau=3\mu m/2c^3$, and $\frac{2}{5}a^2=C/M$.

Therefore the formula becomes

$$\frac{1}{2}\frac{\tau\epsilon}{\alpha-\beta}(\Omega c)\frac{C}{\mu Mm}\sin 2i$$

But by (5) $C\Omega c/\mu Mm=k$.

Therefore it becomes

$$\frac{1}{2}\frac{k\tau\epsilon}{\alpha-\beta}\sin 2i$$

Now by (115) and (112), when $\xi=1$, $\mathrm{a}=k\tau\epsilon\cos j\cos 2j$.

Therefore in my notation LAPLACE's result for the inclination of the lunar proper plane to the ecliptic is

$$\frac{1}{2}\frac{\mathrm{a}}{\alpha-\beta}\frac{\sin 2i}{\cos 2j}\sec j$$

This agrees with the result (260) in the text, from which the amount of oscillation of the lunar orbit was computed, save as to the sec j. Since j is small the discrepancy is slight, and I believe my form to be the more accurate.

LAPLACE states that the inclination is $20''\cdot023$ (centesimal) if the earth be heterogeneous, and $41''\cdot470$ (centesimal) if homogeneous. Since $41''\cdot470$ (centes.)$=13''\cdot44$, this result agrees very closely with mine. The difference of LAPLACE's data explains the discrepancy.

If it be desired to apply my formula to the heterogeneous earth we must take $\frac{5}{8}$ of my k, because the $\frac{2}{5}$ of the formula (6) for s will be replaced by $\frac{1}{3}$ nearly. Also ϵ, which is $\frac{1}{232}$, must be replaced by the precessional constant, which is $\cdot003272$. Hence my previous result in the text must be multiplied by $\frac{5}{8}$ of $232\times\cdot003272$ or $\cdot6326$. This factor reduces the $13''\cdot13$ of the text to $8''\cdot31$. LAPLACE's result ($20''\cdot023$ centes.) is $6''\cdot49$. Hence there is a small discrepancy in the results; but it must be remembered that LAPLACE's value of the actual ellipticity (1/334 instead of 1/295) of the earth was considerably in error. The more correct result is I think $8''\cdot31$. The amount of this inequality was found by BURG and

Now consider the final condition.

Since the integrations of the two periods have extended from $\xi=1$ to $\cdot88$, and again from $\xi=1$ to $\cdot76$,

$$\tau=\tau_0(\cdot88\times\cdot76)^{-6},\ \Omega=\Omega_0(\cdot88\times\cdot76)^{-3},\ k=k_0(\cdot88\times\cdot76)^{-1},$$

also the value of n which gives the day of 9 hrs. 55 m. is given by $\log n=3\cdot74451$, and $\log \mathfrak{g}+10=1\cdot21217$, when the year is the unit of time.

We now have $i=17°\ 4'$, $j=6°\ 21'$.

Using these values in (115) and (112), I find

$$\alpha=\cdot10872,\ \beta=\cdot00627,\ \mathrm{a}=\cdot00563,\ \mathrm{b}=\cdot00510.$$

ab is still small compared with $(\alpha-\beta)$, but not negligeable.

Then by (117)

$$\kappa_1-\kappa_2=-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}=-(\alpha-\beta)-\frac{2\mathrm{ab}}{\alpha-\beta},\ \text{also}\ \kappa_1+\kappa_2=-(\alpha+\beta)$$

Now $2\mathrm{ab}/(\alpha-\beta)=\cdot00056$.

Hence we have

$$\left.\begin{aligned}\kappa_1+\kappa_2&=-\cdot11499\\ \kappa_1-\kappa_2&=-\cdot10301\end{aligned}\right\}\ \text{whence}\ \begin{aligned}\kappa_1&=-\cdot10900\\ \kappa_2&=-\cdot00599\end{aligned}$$

BURCKHARDT from the combined observations of BRADLEY and MASKELYNE to be 8″ (GRANT'S 'Hist. Phys. Astr.,' 1852, p. 65).

For the amplitude of the 19-yearly nutation, AIRY gives ('Math. Tracts,' 1858, article "On Precession and Nutation," p. 214)

$$\frac{6\pi^2\mathrm{B}}{\mathrm{T}'^2\omega(n+1)}\frac{\tau}{4\pi}\cos \mathrm{I}\sin 2i$$

B is the precess. const. = my $\mathfrak{e}$; his T′ = my $2\pi/\Omega$; his n = my ν; his ω = my n; his I = my i; his i = my j; and his τ is the period of revolution of the nodes, and therefore = my $2\pi/(\alpha-\beta)$.

Then since my $\tau=3\Omega^2/2(1+\nu)$, the above in my notation is

$$\tfrac{1}{2}\frac{\tau\mathfrak{e}}{n}\frac{1}{\alpha-\beta}\cos i\sin 2j$$

Now by (115) and (112) $\mathrm{b}=\frac{\tau\mathfrak{e}}{n}\cos i\cos 2i$, when $\xi=1$.

Therefore his result in my notation is

$$\tfrac{1}{2}\frac{\mathrm{b}}{\alpha-\beta}\frac{\sin 2j}{\cos 2i}$$

This is the result used above (in 260) for computing the nutations of the earth.

If my formula is to be used for the heterogeneous earth, $\mathfrak{e}$ must be replaced by the precessional constant, and therefore the result in the text must be multiplied by $232\times\cdot003272$ or $\cdot759$. Hence for the heterogeneous earth the 11″·86 must be reduced to 9″·01. AIRY computes it as 10″·33, but says the observed amount is 9″·6, but he takes the precessional constant as ·00317, and the moon's mass as 1–70th of that of the earth. I believe that ·00327 and 1–82nd are more in accordance with the now accepted views of astronomers.

κ_1 and κ_2 have now come to differ a little from $-\alpha$ and $-\beta$, but still not much. With these values I find

$$\log \frac{\mathrm{a}}{\kappa_2+\alpha}+10=8\cdot76472, \quad \log \frac{-\mathrm{b}}{\kappa_1+\beta}+10=8\cdot69606$$

Substituting in the formulas

$$\delta j=\tfrac{1}{2}\frac{\mathrm{a}}{\kappa_2+\alpha}\frac{\sin 2i}{\cos 2j}, \quad \delta i=\tfrac{1}{2}\frac{\mathrm{b}}{\kappa_1+\beta}\frac{\sin 2j}{\cos 2i}$$

I find

$$\delta j=57'\ 31'', \quad \delta i=22'\ 42''$$

Thus the oscillation of the lunar orbit has increased from 13″ to nearly a degree, and that of the equator from 12″ to 23′.

It is clear therefore that we have carried out the integration by the method of Part II., as far back in retrospect as is proper, even for a speculative investigation like the present one.

We shall here then make the transition to the method of Part III.

Henceforth the formulas used regard the inclination and obliquity as small angles; the obliquity is still however so large that this is not very satisfactory.

§ 19. *Secular changes in the proper planes of the earth and moon where the viscosity is small.*

We now take up the integration, at the point where it stops in the last section, by the method of Part III. The viscosity is still supposed to be small, so that γ, δ, g, d (as defined in (251)) must be taken in place of Γ, Δ, G, D, which refer to any viscosity. The equations are ready for the application of the method of quadratures in (250), and the symbols are defined in (251–4).

The method pursued is to assume a series of equidistant values of ξ, and then to compute all the functions (251–4), substitute them in (250), and combine the equidistant values of the functions to be integrated by the rules of the calculus of finite differences.

The preceding integration terminates where the day is 9 hrs. 55 m., and the moon's sidereal period is 8·17 m.s. days. If the present tropical year be the unit of time, we have, at the beginning of the present integration $\log n_0=3\cdot74451$, $\log \Omega_0=2\cdot44836$, and $\log k+10=6\cdot20990$, k being $s\Omega_0{}^{t}$ of (7).

The first step is to compute a series of values of n/n_0, by means of (254). As a fact, I had already computed n/n_0 corresponding to $\xi=1$, ·92, ·84, ·76 for the paper on "Precession," by means of a formula, which took account of the obliquity of the ecliptic; and accordingly I computed n/n_0, by the same formula, for the values of $\xi=$·96, ·88, ·80, instead of doing the whole operation by means of (254). The difference between my results here used and those from (254) would be very small.

The following table exhibits some of the stages of the computation. The results are given just as they were found, but it is probable that the last place of decimals, and perhaps the last but one, are of no value. As however we really only require a solution in round numbers, this is of no importance.

TABLE II.

$=$	1·	·96	·92	·88	·84	·80	·76
$n/n_0=$	1·00000	1·04467	1·08931	1·13392	1·17852	1·22308	1·26763
$\iota+10=$	8 40016	8·43812	8·47446	8·50932	8·54284	8·57507	8·60614
$/\tau+10=$	8·61867	8·51230	8·40140	8·28557	8·16435	8·03721	7·90356
$\lambda+10=$	8·70384	8·73805	8·77533	8·81581	8·85966	8·90712	8·95841
$\tau'/2\lambda\iota\tau=$	16·3546	10·8418	7·0889	4·5647	2·8895	1·7947	1·0914
$\mathfrak{m}=\mathfrak{a}=$	·90035	·97976	1·06603	1·16014	1·26320	1·37648	1·50172
$\gamma+10=$	9·67591	9·71452	9·75343	9·79287	9·83307	9·87430	9·91693
$\delta+10=$	9·65551	9·65745	9·65824	9·65788	9·65631	9·65341	9·64900
$)+10=$	8·83030	8·86665	8·91307	8·96946	9·03549	9·11080	9·19510
$\alpha'=$	36·696	23·186	12·583	4·144	− 2·747	− 8·605	−13·873
$\mathfrak{a}'=$	− 7·4782	− 8·6811	−10·0883	−11·7426	−13·6966	−16·0163	−18·7899
$\beta'=$	− 6·4455	− 6·9122	− 7·4220	− 7·9805	− 8·5940	− 9·2699	−10 0184
$\mathfrak{b}'=$	− 6·4038	− 6·8796	− 7·3968	− 7·9612	− 8·5794	− 9·2590	−10·0104
$\alpha)+10=$	8·74306	8·95453	9·16587	9·37077	9·55751	9·71146	9·82404
$(\kappa_2+\alpha)$	1·21135	1·03659	·86190	·69374	·54396	·42731	·35255
$\kappa_2-\kappa_1)$	1·21283	1·04017	·87056	·71393	·58660	·50372	·46520

The further stage in the computation, when these values are used to compute the several terms of the expressions to be integrated, are given in the following table.

TABLE III.

	1·	·96	·92	·88	·84	·80	·76
$-(\alpha'-\beta')(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)^2=$	·00995	·02395	·05424	·10413	·13350	·03053	− ·26438
$\mathfrak{a}'\mathfrak{b}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2=$	·00011	·00064	·00376	·02041	·08937	·27505	·57228
$\mathfrak{a}'\mathfrak{b}/kn(\kappa_2-\kappa_1)^2=$	−·03117	−·07671	−·18670	−·42945	−·86628	−1·42975	−1·93250
$\mathfrak{b}'\mathfrak{a}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2=$	·00008	·00049	·00294	·01606	·07072	·21887	·45786
$\mathfrak{b}'\mathfrak{a}/kn(\kappa_2-\kappa_1)^2=$	−·02403	−·05970	−·14593	−·33778	−·68546	−1·13770	−1·54610
$\gamma(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·00179	−·00452	−·01141	−·02759	−·06001	− ·10969	− ·16534
$\gamma(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·52483	·54645	·56651	·58035	·58167	·57020	·55832
$\delta(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·00170	−·00397	−·00916	−·02022	−·03995	− ·06596	− ·08922
$\delta(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·50075	·47916	·45501	·42530	·38719	·34288	·30127
$(\mathfrak{b}\mathfrak{g}-\mathfrak{a}\mathfrak{d})/kn(\kappa_2-\kappa_1)=$	·00460	·00713	·01124	·01764	·02649	·03675	·04704

The method pursued in the integration of the preceding section proceeds virtually on the assumption that the term $\gamma(\kappa_2+\alpha)/kn(\kappa_1-\kappa_2)$ is the only important one in the expression for $d \log \tan \frac{1}{2}\text{J}/d\xi$, and that the term $\delta(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)$ is the only important one in the expression for $d \log \tan \frac{1}{2}\text{I}/d\xi$.

Now when $\xi=1$, at the beginning of the present integration, we see from Table III. that the said term in γ is about 22 times as large as any other occurring in $d \log \tan \frac{1}{2}\text{J}$, and that the said term in δ is about 16 times as large as any other which occurs in $d \log \tan \frac{1}{2}\text{I}$. Hence the preceding integration must have given fairly satisfactory results. But after the first column these terms in γ and δ fail to maintain their relative importance, so that when $\xi=\cdot 76$, they have both become considerably less important than other terms—notably $\text{b}'\text{a}/kn(\kappa_2-\kappa_1)^2$ and $\text{a}'\text{b}/kn(\kappa_2-\kappa_1)^2$. This is exactly what is to be expected, because the equations are tending towards the form which they would take if the solar influence were nil, and an inspection of (225) shows that these terms would then be prominent.

Now if we combine these values of the several terms together according to (250), we obtain the seven equidistant values of $d \log \tan \frac{1}{2}\text{J}/d\xi$ and $d \log \tan \frac{1}{2}\text{I}/d\xi$ exhibited in the following table:—

TABLE IV.

ξ =	1·	·96	·92	·88	·84	·80	·76
$d \log \tan \frac{1}{2}\text{J}/d\xi=$	−·49386	−·46660	−·37218	−·15627	+ ·16138	+ ·35219	+ ·19330
$d \log \tan \frac{1}{2}\text{I}/d\xi=$	+·54460	+·58194	+·69284	+·93287	+1·28273	+1·51135	+1·39323

By interpolation it appears that $d\text{J}/d\xi$ vanishes when $\xi=\cdot 8603$. This value of ξ corresponds with 8 hrs. 36 m. for the period of the earth's rotation, and 5·20 m. s. days for the period of the moon's revolution.

Since $d\xi$ is negative in our integration, we see from these values that I, the inclination of the earth's proper plane to the ecliptic, will continue diminishing, and with increasing rapidity. On the other hand, the inclination J of the lunar orbit to its proper plane will increase at first, but at a diminishing rate, and will finally diminish. This is a point of the greatest importance in explaining the present inclination of the lunar orbit to the ecliptic, and we shall recur to it later on.

Now combine the first four values by the rule of finite differences, viz.:

$$[u_0+u_3+3(u_1+u_2)]\tfrac{3}{8}h$$

and all seven by WEDDLE's rule, viz.:

$$[u_0+u_2+u_3+u_4+u_6+5(u_1+u_3+u_5)]\tfrac{3}{10}h$$

where h is our $d\xi$, and the u's are the several numbers given in the above Table IV.; then we have, on integration from 1 to ·88,

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 + \cdot 04750$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 07953$$

and on integration from 1 to ·76

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 + \cdot 02425$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 23972$$

Then if we take $J_0 = 6°$, $I_0 = 17°$, which are in round numbers the final values of J and I derived from the first method of integration, we easily find,

$$\text{when } \xi = \cdot 88,\ J = 6°\ 17',\ I = 15°\ 43'$$
$$\text{and when } \xi = \cdot 76,\ J = 6°\ 9',\ \ I = 13°\ 25'$$

Then we have by (249)

$$\sin I_{,} = -\frac{(\kappa_1 + \alpha)}{a} \sin J = \frac{b}{\kappa_2 + \alpha} \sin J$$
$$\sin J_{,} = \frac{a}{\kappa_2 + \alpha} \sin I = -\frac{\kappa_1 + \alpha}{b} \sin I$$

Now b is always unity, and the logarithms of $(\kappa_2 + \alpha)$ and $-(\kappa_1 + \alpha)$ are given in Table II.; from this we find

$$\text{when } \xi = \cdot 88,\ I_{,} = 1°\ 16',\ J_{,} = 3°\ 39'$$
$$\text{when } \xi = \cdot 76,\ I_{,} = 2°\ 43',\ J_{,} = 8°\ 54'$$

By the same formula, when $\xi = 1$ initially, we have $I_{,} = 22'$, $J_{,} = 56'$. These two results ought to be identical with the results from (260) of the last section; and they are so very nearly, for at the end of the integration we had $\delta i = 22'\ 43''$, $\delta j = 57'\ 31''$. The small discrepancy which exists is partly due to the assumed smallness of i and j in the present investigation, and also to our having taken the values 6° and 17° for J_0 and I_0 instead of 6° 21′, 17° 4′.

The value $\xi = \cdot 88$ gives the length of day as 8 hrs. 45 m., and the moon's sidereal period as 5·57 m. s. days.

The value $\xi = \cdot 76$ gives the day as 7 hrs. 49 m., and the moon's sidereal period as 3·59 m. s. days. This value of ξ brings us to the point specified as the end of the third period of integration in § 17 of the paper on "Precession."

There is one other point which it will be interesting to determine,—it is to find the rate of the precessional motion of the node of the two proper planes on the ecliptic, and the rate of the motion of the nodes of the equator and orbit upon their respective proper planes. By means of the preceding numerical values, it will be easy to find these quantities at the epochs specified by $\xi = 1$, ·88, ·76.

The period of the precession of the two proper planes is $-2\pi/\kappa_2$, and that of the precession of the two nodes on their proper planes is $2\pi/(\kappa_2-\kappa_1)$.

In the preceding computations we omitted a common factor $\tau\mathfrak{t}/n$ from α, β, a, b, κ_1, κ_2; this factor must now be reintroduced. τ' is a constant and $\log\tau'=1\cdot77242$, then by means of the numerical values given in the first table I find

ξ =	1·	·88	·76
$\log\tau\mathfrak{t}/n+10=$	7·80940	8·19708	8·62750

Also

$\log-\kappa_2+10=$	9·99401	9·89462	9·53295

$\log(\kappa_2-\kappa_1)$ is given before in Table II. Then introducing the omitted factor $\tau\mathfrak{t}/n$, I find

ξ =	1·	·88	·76
$-2\pi/\kappa_2=$	988 yrs.	509 yrs.	434 yrs.
$2\pi/(\kappa_2-\kappa_1)=$	60 yrs.	77 yrs.	51 yrs.

Thus both precessional movements on the whole increase in rapidity (because of the increasing value of $\tau\mathfrak{t}/n$), but the rate of the precession of the pair of proper planes increases all through, whilst that of the precession on the proper planes diminishes and then increases. It was pointed out towards the end of § 13 that κ_2 is, so to speak, the ancestor of the luni-solar precession, and $\kappa_2-\kappa_1$ the ancestor of the revolution of the moon's nodes. Hence the 988 years has bred (to continue the metaphor) the present 26,000 years of the precessional period, and the 60 years has bred the present $18\frac{1}{2}$ years of the revolution of the moon's nodes.

We see that the $\kappa_2-\kappa_1$ precession attains a minimum at a certain period being more rapid, both earlier and later.

All the above results will be collected and arranged in a tabular form, after further results have been obtained by means of an integration, carrying out the investigation into the more remote past.

The tidal and precessional effects of the sun's influence have now become exceedingly small, and the only way in which the sun continues to exert a sensible effect is in its tendency to make the nodes of the lunar orbit revolve on the ecliptic. In the analysis therefore we may now treat τ' as zero everywhere, except where it occurs in the form $\tau'/\lambda\mathfrak{t}\tau$. Since λ and $\mathfrak{t}$ are both pretty small, these terms in τ'/τ rise in importance.

The equation of conservation of moment of momentum now becomes

$$\frac{n}{n_0}=1+\frac{1}{kn_0}(1-\xi)$$

Here kn_0 is equal to the value of $\mathfrak{m}$ in the preceding integration when $\xi=\cdot76$; and hence $1/kn_0=\cdot665903$.

Then we now have $\beta=$b, $\gamma=$g, $\delta=$d, $\beta'=$b$'$, $\gamma'=$g$'$, $\delta'=$d$'$, but α and α' are not equal to a and a$'$.

It is proposed to carry the new integration over the field defined by $\xi=1$ to ·88, and to compute four equidistant values.

The following tables give the results of the computation, as in the previous case.

TABLE V.

ξ =	1·	·96	·92	·88
$n/n_0=$	1·00000	1·02664	1·05327	1·07991
$\log \mathfrak{e}+10=$	8·60614	8·62898	8·65122	8·67292
$\log \tau'/\tau+10=$	7·90356	7·79718	7·68628	7·57045
$\log \lambda+10=$	8·95841	9·00018	9·04451	9·09157
$\log \tau'/2\lambda\mathfrak{e}\tau+10=$	10·03798	9·86699	9·68952	9·50493
$\mathfrak{m}=\mathrm{a}=$	1·5017	1·6060	1·7193	1·8429
$\log \mathrm{g}+10=$	9·91693	9·95049	9·98531	10·02170
$\log \mathrm{d}+10=$	9·65322	9·64780	9·64118	9·63303
$\alpha'=$	−13·873	−17·719	−21·607	−25·692
$\mathrm{a}'=$	−18·790	−21·266	−24·130	−27·460
$\beta'=\mathrm{b}'=$	−10·010	−10·636	−11·316	−12·057
$\log-(\kappa_1+\alpha)+10=$	9·82285	9·88247	9·92401	9·95203
$\log(\kappa_2+\alpha)+10=$	·35374	·32327	·31133	·31347
$\log(\kappa_2-\kappa_1)+10=$	·46586	·45758	·46052	·47035

TABLE VI.

ξ =	1·	·96	·92	·88
$-(\alpha'-\beta')(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)^2=$	− ·1998	− ·4261	− ·6551	− ·8630
$\mathrm{a}'\mathrm{b}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2=$	·4312	·6077	·7500	·8445
$\mathrm{a}'\mathrm{b}/kn(\kappa_2-\kappa_1)^2=$	−1·4643	−1·6770	−1·8297	−1·9410
$\mathrm{b}'\mathrm{a}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2=$	·3450	·4882	·6047	·6833
$\mathrm{b}'\mathrm{a}/kn(\kappa_2-\kappa_1)^2=$	−1·1714	−1·3469	−1·4752	−1·5706
$\mathrm{g}(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	− ·1251	− ·1540	− ·1777	− ·1965
$\mathrm{g}(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·4248	·4248	·4335	·4517
$\mathrm{d}(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	− ·0682	− ·0767	− ·0805	− ·0803
$\mathrm{d}(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·2315	·2116	·1963	·1846
$(\mathrm{bg}-\mathrm{ad})/kn(\kappa_2-\kappa_1)=$	·0342	·0404	·0469	·0542

Combining these terms according to the formulas (250), we have

TABLE VII.

ξ =	1·	·96	·92	·88
$d \log \tan \frac{1}{2}J/d\xi=$	+ ·1496	−·0754	−·3298	−·5625
$d \log \tan \frac{1}{2}I/d\xi=$	+1·0601	+·8607	+·6354	+·4370

By interpolation it appears that $dJ/d\xi$ vanishes when $\xi=\cdot 9679$. This value of ξ corresponds with 7 hrs. 47 m. for the period of the earth's rotation, and 3·25 m. s. days for the period of the moon's revolution.

By the rules of the calculus of finite differences, integrating from $\xi=1$ to ·88,

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 + \cdot 0244$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 0898$$

Then with $J_0=6^\circ\ 9'$, $I_0=13^\circ\ 25'$ from the previous integration, we have $J=6^\circ\ 18'$, $I=12^\circ\ 16'$.

When $\xi=\cdot 88$, the length of the day is 7 hrs. 15 m., and the moon's sidereal period is 2·45 m. s. days. Also $I_{,}=3^\circ\ 3'$, $J_{,}=10^\circ\ 58'$.

Thus we have traced the changes back until the inclination of the proper planes to one another is only $12^\circ\ 16'-10^\circ\ 58'$ or $1^\circ\ 18'$.

In the same way as before it may be shown that, when $\xi=\cdot 88$, the period of the precession of the proper planes is 609 years, and the period of the revolution of the two nodes on their moving proper planes is 22 years. The former of the two precessions is therefore at this stage getting slower, whilst the latter goes on increasing in speed.

The physical results of the whole integration of the present section is embodied in the following table.

TABLE VIII.—Results of integration in the case of small viscosity.

Day in m. s. hours and min.		Moon's sidereal period in m. s. days.	Inclination of earth's proper plane to ecliptic.		Inclination of equator to earth's proper plane.		Inclination of moon's proper plane to ecliptic.		Inclination of lunar orbit to moon's proper plane.		Precessional period of the proper planes.	Period of revolution of the two nodes on their moving proper planes.
h.	m.	Days.	°	′	°	′	°	′	°	′	Years.	Years.
9	55	8·17	17	0	0	22	0	57	6	0	988	60
8	45	5·57	15	43	1	16	3	39	6	17	509	77
7	49	3·59	13	25	2	43	8	54	6	9	434	51
7	15	2·45	12	16	3	3	10	58	6	18	609	22

If the integration is to be carried still further back, the solar action may henceforth be neglected, and the motion may be referred to the invariable plane of the system. This plane undergoes a precessional motion due to the sun, which will not interfere with the treatment of it as though fixed. It is inclined to the ecliptic at about $11° 45'$, because, at the time when we suppose the solar action to cease, the moment of momentum of the earth's rotation is larger than that of orbital motion, and therefore the earth's proper plane represents the invariable plane of the system more nearly than does the moon's proper plane.

The inclination i of the equator to the invariable plane must be taken as about $3°$, and j that of the lunar orbit as something like $5° 30'$. The ratio of the two angles $5° 30'$ and $3°$ must be equal to $1·84$, which is $\mathfrak{m}$, the ratio of the moment of momentum of the earth's rotation to that of orbital motion, at the point where the preceding integration ceases.

Then in the more remote past the angle i will continue to diminish, until the point is reached where the moon's period is about 12 hours and that of the earth's rotation about 6 hours. The angle j will continue increasing at an accelerating rate.

This may be shown as follows :—

The equations of motion are now those of Part II., which may be written

$$kn\frac{dj}{d\xi}=-\mathrm{g}(i+j)$$

$$kn\frac{di}{d\xi}=\ \mathrm{d}(i+j)$$

But since $i/j=\xi/kn=1/\mathfrak{m}$, they become

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\frac{1+\mathfrak{m}}{\mathfrak{m}}\mathrm{g}$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\ (1+\mathfrak{m})\mathrm{d}$$

(Compare with the first of equations (255) given in Part III., when $\tau'=0$.)

These equations are not independent, because of the relationship which must always subsist between i and j.

Then substituting from (251) we have for the case of small viscosity

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\frac{1+\mathfrak{m}}{2(1-\lambda)}$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\ \frac{(1+\mathfrak{m})(1-2\lambda)}{2(1-\lambda)}$$

From this we see that j will always decrease as ξ increases at a rate which tends to become infinite when $\lambda=1$; and i increases as ξ increases so long as λ is less than ·5, but decreases for values of λ between ·5 and unity at a rate which tends to become infinite when $\lambda=1$. If we consider the subject retrospectively, ξ decreases, j increases, and i decreases, except for values of λ between ·5 and unity.

This continued increase (in retrospect) of the inclination of the lunar orbit to the invariable plane is certainly not in accordance with what was to be expected, if the moon once formed a part of the earth. For if we continued to trace the changes backwards to the initial condition in which (as shown in "Precession") the two bodies move round one another as parts of a rigid body, we should find the lunar orbit inclined at a considerable angle to the equator; and it is hard to see how a portion detached from the primeval planet could ever have revolved in such an orbit.

These considerations led me to consider whether some other hypothesis than that of infinitely small viscosity of the earth might not modify the above results. I therefore determined to go over the same solution again, but with the hypothesis of very large instead of very small viscosity of the planet.

This investigation is given in the next section, but I shall not retraverse the ground covered by the integration of the first method, but shall merely take up the problem at the point where it was commenced in the present section.

§ 20. *Secular changes in the proper planes of the earth and moon when the viscosity is large.*

Let $\mathfrak{p}=2\,gaw/19v$, where v is the coefficient of viscosity of the earth.

Then by the theory of viscous tides

$$\tan 2\mathrm{f}_1=\frac{2(n-\Omega)}{\mathfrak{p}},\quad \tan 2\mathrm{f}=\frac{2n}{\mathfrak{p}},\quad \tan \mathrm{g}_1=\frac{n-2\Omega}{\mathfrak{p}},\quad \tan \mathrm{g}=\frac{n}{\mathfrak{p}} \quad . \quad . \quad . \quad (261)$$

If the viscosity be very large $\mathfrak{p}$ is very small, and the angles $\frac{1}{2}\pi-2\mathrm{f}_1$, $\frac{1}{2}\pi-2\mathrm{f}$, $\frac{1}{2}\pi-\mathrm{g}_1$, $\frac{1}{2}\pi-\mathrm{g}$ are small, so that their cosines are approximately unity and their sines approximately equal to their tangents. Hence

$$\sin 4\mathrm{f}_1=\frac{\mathfrak{p}}{n-\Omega},\quad \sin 4\mathrm{f}=\frac{\mathfrak{p}}{n},\quad \sin 2\mathrm{g}_1=\frac{2\mathfrak{p}}{n-2\Omega},\quad \sin 2\mathrm{g}=\frac{2\mathfrak{p}}{n}$$

Then introducing $\lambda=\Omega/n$, we have

$$\frac{\sin 4\mathrm{f}}{\sin 4\mathrm{f}_1}=1-\lambda,\quad \frac{\sin 2\mathrm{g}_1}{\sin 4\mathrm{f}_1}=\frac{2(1-\lambda)}{1-2\lambda},\quad \frac{\sin 2\mathrm{g}}{\sin 4\mathrm{f}_1}=2(1-\lambda) \quad . \quad . \quad . \quad . \quad (262)$$

Introducing the transformations (262) into (251), we have

$$\left.\begin{array}{l}\Gamma=\tfrac{1}{2}\mathfrak{m}\left[1-\dfrac{4\lambda(1-\lambda)}{1-2\lambda}\right],\quad \Delta=\tfrac{1}{2}\left[1+\dfrac{4\lambda(1-\lambda)}{1-2\lambda}-4(1-\lambda)\dfrac{\tau'}{\tau}+(1-\lambda)\left(\dfrac{\tau'}{\tau}\right)^2\right]\\ bG-aD=-\mathfrak{m}\left[\dfrac{4\lambda(1-\lambda)}{1-2\lambda}-2(1-\lambda)\dfrac{\tau'}{\tau}\right]\end{array}\right\} \quad (263)$$

All the other expressions in (251) remain as they were.

Then the terms in Γ, Δ, G, D in (250) are the only ones which have to be recomputed. And all the other arithmetical work of the last section will be applicable here. Also all the materials for calculating these new terms are ready to hand.

The results of the computation are embodied in the following tables.

TABLE IX.

=	1·	·96	·92	·88	·84	·80	·76
g Γ+10=	9·54901	9·57529	9·59914	9·61994	9·63663	9·64791	9·65092
log Δ=	·52876	·55517	·58023	·60484	·63005	·65708	·68739
G)+10=	9·08381	9·22356	9·34416	9·45433	9·55931	9·66259	9·76574

TABLE X.

ξ =	1·	·96	·92	·88	·84	·80	·76
$\Gamma(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·00133	−·00328	−·00800	−·01853	−·03818	−·06513	−·08961
$\Gamma(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·39185	·39657	·39712	·38973	·37003	·33856	·30260
$\Delta(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·00199	−·00485	−·01168	−·02688	−·05553	−·09627	−·13761
$\Delta(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·58529	·58541	·57994	·56554	·53826	·50044	·46468
$(bG-aD)/kn(\kappa_2-\kappa_1)=$	−·00825	−·01622	−·03034	−·05388	−·08850	−·13092	−·17504

Then combining these terms with those given in Table III., according to the formulas (250), (with Γ, &c., in place of γ, &c.), we have the following equidistant values.

TABLE XI.

ξ =	1·	·96	·92	·88	·84	·80	·76
$\log\tan\frac{1}{2}J/d\xi=$	−·3477	−·2925	−·1587	+ ·1125	+ ·5036	+ ·7818	+ ·7195
$\log\tan\frac{1}{2}I/d\xi=$	+·6168	+·6661	+·7796	+1·0107	+1·3406	+1·5458	+1·4103

By interpolation it appears that $dJ/d\xi$ vanishes when $\xi=$·8966. This value of ξ corresponds with a period of 8 hrs. 54 m. for the earth's rotation, and 5·89 m. s. days for the moon's revolution.

Integrating as in the last section, from $\xi=1$ to $\cdot 88$, we have

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 + \cdot 0238$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 0895$$

Taking $I_0=6°$, $J_0=17°$, we have $I=15°\ 34'$, $J=6°\ 9'$.
These values correspond to $I_{,}=1°\ 15'$, $J_{,}=3°\ 37'$.
Again integrating from $\xi=1$ to $\cdot 76$, we have

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 - \cdot 0461$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 2552$$

These give $J=5°\ 44'$, $I=13°\ 13'$, which correspond to $I_{,}=2°\ 33'$, $J_{,}=8°\ 46'$.

The integration will now be continued over another period, as in the last section. The following are the results of the computations.

TABLE XII.

=	1·	·96	·92	·88
$\log(\Gamma=G)+10=$	9·65092	9·64491	9·62783	9·59299
$\log(\Delta=D)+10=$	9·84629	9·86040	9·87686	9·89622

TABLE XIII.

ξ =	1·	·96	·92	·88
$G(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	$-\cdot 06781$	$-\cdot 07617$	$-\cdot 07802$	$-\cdot 07323$
$G(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·23026	·21018	·19033	·16832
$D(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	$-\cdot 10634$	$-\cdot 12511$	$-\cdot 13843$	$-\cdot 14720$
$D(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·36106	·34521	·33771	·33835
$(bG-aD)/kn(\kappa_2-\kappa_1)=$	$-\cdot 13815$	$-\cdot 16352$	$-\cdot 19057$	$-\cdot 35054$

Substituting these values in the differential equations (250), we have the following equidistant values :—

TABLE XIV.

$\xi =$	$1\cdot$	$\cdot 96$	$\cdot 92$	$\cdot 88$
$d \log \tan \frac{1}{2}J/d\xi =$	$+\cdot 5547$	$+\cdot 3915$	$+\cdot 2088$	$+\cdot 1925$
$d \log \tan \frac{1}{2}I/d\xi =$	$+1\cdot 0746$	$+\cdot 8682$	$+\cdot 6391$	$+\cdot 3093$

Then integrating from $\xi=1$ to $\cdot 88$ we have

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 - \cdot 0382$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 0886$$

Then putting $I_0=13° \ 13'$ and $J_0=5° \ 44'$, from the previous integration, we have $J=5° \ 30'$, $I=12° \ 6'$.

These values of J and I give $J_{,}=10° \ 49'$, $I_{,}=2° \ 40'$.

The physical meaning of the results of the whole integration is embodied in the following table.

TABLE XV.—Results of integration in the case of large viscosity.

Day in m. s. hours and minutes.		Moon's sidereal period in m. s. days.	Inclination of earth's proper plane to ecliptic.		Inclination of equator to earth's proper plane.		Inclination of moon's proper plane to ecliptic.		Inclination of lunar orbit to moon's proper plane.	
h.	m.	Days.	°	′	°	′	°	′	°	′
9	55	8·17	17	0	0	22	0	57	6	0
8	45	5·57	15	34	1	15	3	37	6	9
7	49	3·59	13	13	2	33	8	46	5	44
7	15	2·45	12	6	2	40	10	49	5	30

If we compare these results with those in Table VIII. for the case of small viscosity, we see that the inclinations of the two proper planes to one another and to the ecliptic are almost the same as before, but there is here this important distinction, viz.: that the inclinations of the two moving systems to their respective proper planes is less (compare 5° 30′ with 6° 18′, and 2° 40′ with 3° 3′).

And besides, if we had carried the integration, in the case of small viscosity, further back we should have found the inclination of the lunar orbit increasing.

It will now be shown that, in the present case of large viscosity, the inclinations of

the equator and the orbit to their proper planes will continue to diminish, as the square root of the moon's distance diminishes, and at an increasing rate.

Suppose that, in continuing the integration, the solar influence be entirely neglected, and the motion referred to the invariable plane of the system. This plane will be in some position intermediate between the two proper planes, but a little nearer to the earth's plane, and will therefore be inclined to the ecliptic at about 11° 45′.

The equations of motion are now those of § 10, Part II., which may be written

$$kn\frac{dj}{d\xi}=-\mathrm{G}(i+j)$$

$$kn\frac{di}{d\xi}=\quad \mathrm{D}(i+j)$$

But since $i/j=\xi/kn=1/\mathfrak{m}$, they become

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\frac{1+\mathfrak{m}}{\mathfrak{m}}\mathrm{G}$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\quad (1+\mathfrak{m})\mathrm{D}$$

(compare with the first of equations (255) given in Part III., when $\tau'=0$).

These equations are not independent of one another, because of the relationship which must always subsist between i and j.

Then substituting from (263) (in which τ' is put zero, and G, D written for Γ, Δ) we have for the case of large viscosity

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\tfrac{1}{2}(1+\mathfrak{m})\left[1-\frac{4\lambda(1-\lambda)}{1-2\lambda}\right]$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\quad \tfrac{1}{2}(1+\mathfrak{m})\left[1+\frac{4\lambda(1-\lambda)}{1-2\lambda}\right]$$

When $\lambda=\frac{1}{2}$, $4\lambda(1-\lambda)/(1-2\lambda)$ is infinite, and therefore both $dj/d\xi$ and $di/d\xi$ are infinite. This result is physically absurd.

The absurdity enters by supposing that an infinitely slow tide (viz.: that of speed $n-2\Omega$) can lag in such a way as to have its angle of lagging nearly equal to 90°. The correct physical hypothesis, for values of λ nearly equal to $\frac{1}{2}$, is to suppose the lag small for the tide $n-2\Omega$, but large for the other tides. Hence when λ is nearly $=\frac{1}{2}$, we ought to put

$$\sin 4\mathrm{f}_1=\frac{\mathfrak{p}}{n-\Omega},\ \sin 2\mathrm{g}=\frac{2\mathfrak{p}}{n},\ \text{but}\ \sin 2\mathrm{g}_1=\frac{2(n-2\Omega)}{\mathfrak{p}}$$

Then we should have

$$G=\tfrac{1}{2}\mathfrak{m}\left[1+2(1-\lambda)-\frac{2n^2}{\mathfrak{p}^2}(1-\lambda)(1-2\lambda)\right]$$

$$D=\ \tfrac{1}{2}\ \left[1-2(1-\lambda)+\frac{2n^2}{\mathfrak{p}^2}(1-\lambda)(1-2\lambda)\right]$$

The last term in each of these expressions involves a small factor both in numerator and denominator, viz.: $1-2\lambda$ because $\lambda=\frac{1}{2}$ nearly, and $\mathfrak{p}$, because the viscosity is large. The evaluation of these terms depends on the actual degree of viscosity, but all that we are now concerned with is the fact that when $\lambda=\frac{1}{2}$ the true physical result is that D changes sign by passing through zero and not infinity, and that G does the same for some value of λ not far removed from $\frac{1}{2}$.

Now consider the function $\frac{4\lambda(1-\lambda)}{1-2\lambda}-1$. The following results are not stated retrospectively, and when it is said that i or j increase or decrease, it is meant increase or decrease as t or ξ increases.

(i.) From $\lambda=1$ to $\lambda=\cdot5$ the function is negative.

Hence for these values of λ the inclination j decreases, or zero inclination is dynamically stable.

When $\lambda=\cdot5$ it is infinite; but we have already remarked on this case.

(ii.) From $\lambda=\cdot5$ to $\lambda=\cdot191$ it is positive.

Therefore for these values of λ the inclination j increases, or zero inclination is dynamically unstable. It vanishes when $\lambda=\cdot191$.

(iii.) From $\lambda=\cdot191$ to $\lambda=0$ it is negative.

Therefore for these values of λ the inclination j decreases, or zero inclination is dynamically stable.

Next consider the function $1+\frac{4\lambda(1-\lambda)}{1-2\lambda}$.

(iv.) From $\lambda=1$ to $\lambda=\cdot809$ it is positive.

Therefore for these values of λ the obliquity i increases, or zero obliquity is dynamically unstable. It vanishes when $\lambda=\cdot809$.

(v.) From $\lambda=\cdot809$ to $\lambda=\cdot5$ it is negative.

Therefore for these values of λ the obliquity i decreases, or zero obliquity is dynamically stable.

When $\lambda=\cdot5$ it is infinite; but we have already remarked on this case.

(vi.) From $\lambda=\cdot5$ to $\lambda=0$ it is positive.

Therefore for these values of λ the obliquity i increases, or zero obliquity is dynamically unstable.

Therefore from $\lambda=1$ to $\cdot 809$ the inclination j decreases and the obliquity i increases.

From $\lambda=\cdot 809$ to $\cdot 5$ both inclination and obliquity decrease.

From $\lambda=\cdot 5$ to $\cdot 191$ both inclination and obliquity increase.

From $\lambda=\cdot 191$ to 0 the inclination decreases and the obliquity increases.

Now at the point where the above retrospective integration stopped, the moon's period was 2·45 days or 59 hours, and the day was 7·25 hours; hence at this point $\lambda=\cdot 123$, which falls between ·191 and ·5. Hence both inclination and obliquity decrease retrospectively at a rate which tends to become infinite when we approach $\lambda=\cdot 5$, if the viscosity be infinitely great. For large, but not infinite, viscosity the rates become large and then rapidly decrease in the neighbourhood of $\lambda=\cdot 5$.

From this it follows that by supposing the viscosity large enough, the obliquity and inclination may be made as small as we please, when we arrive at the point where $\lambda=\cdot 5$.

It was shown in § 17 of "Precession" that $\lambda=\cdot 5$ corresponds to a month of 12 hours and a day of 6 hours.

Between the values $\lambda=\cdot 5$ and ·809 the solutions for both the cases of small and of large viscosity concur in showing zero obliquity and inclination as dynamically stable. But between $\lambda=\cdot 809$ and 1 the obliquity is dynamically unstable for infinitely large, stable for infinitely small viscosity; for these values of λ zero inclination is dynamically stable both for large and small viscosity.

From this it seems probable that for some large but finite viscosity, both zero inclination and zero obliquity would be dynamically stable for values of λ between ·809 and unity.

It appears to me therefore that we have only to accept the hypothesis that the viscosity of the earth has always been pretty large, as it certainly is at present, to obtain a satisfactory explanation of the obliquity of the ecliptic and of the inclination of the lunar orbit. This subject will be again discussed in the summary of Part VII.

§ 21. *Graphical illustration of the preceding integrations.*

A graphical illustration will much facilitate the comprehension of the numerical results of the last two sections.

The integrations which have been carried out by quadratures are of course equivalent to finding the areas of certain curves, and these curves will afford a convenient illustration of the nature of those integrations.

In §§ 19, 20 two separate points of departure were taken, the first proceeding from $\xi=1$ to ·76, and the second from $\xi=1$ to ·88. It is obvious that ξ was referred to different initial values c_0 in the two integrations.

In order therefore to illustrate the rates of increase of log tan $\frac{1}{2}$J and log tan $\frac{1}{2}$I from the preceding numerical results, we must either refer the second sets of ξ's to the same initial value c_0 as the first set, or (which will be simpler) we may take $\sqrt{c}$ as the independent variable.

Then for the values between $\xi = 1$ and ·76, the ordinates of our curves will be the numerical values given in Tables IV. and XI., each divided by $\sqrt{c_0}$. By the choice of a proper scale of length, c_0 may be taken as unity.

For the values in the second integration from $\xi = 1$ to ·88, the $\sqrt{c_0}$ is the final value of $\sqrt{c}$ in the first integration. Hence in order to draw the ordinates in the second part of the curve to the same scale as those of the first, the numbers in Tables VII. and XIV. must be divided by ·76.

Also the second set of ordinates are not spaced out at the same intervals as the first set, for the $d\sqrt{c}$ of the second integration is ·76 of the $d\sqrt{c}$ of the first integration.

Hence the ordinates given in the four Tables, IV., VII., XI., and XIV., are to be drawn corresponding to the abscissæ

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 6{\cdot}76,\ 7{\cdot}52,\ 8{\cdot}28.$$

In fig. 7 these abscissæ are marked off on the horizontal axis.

The first integration corresponds to the part OO′, and the marked points correspond to the seven values of ξ from 1 to ·76 inclusive. The second integration corresponds to the part O′O″, and the values computed in Tables VII. and XIV. were divided by ·76 to give the ordinates.

The value for $\xi = {\cdot}76$ of the first integration is identical with that for $\xi = 1$ of the second.

The integrations, which have been carried out, correspond to the determination of the areas lying between these curves and the horizontal axis, areas below being esteemed negative.

The two curves for $d \log \tan \frac{1}{2}\mathrm{I}/d\sqrt{c}$ lie very close together, and we thus see that the motion of the earth's proper plane is almost independent of the degree of viscosity.

On the other hand, the two curves for $d \log \tan \frac{1}{2}\mathrm{J}/d\sqrt{c}$ differ considerably. For large viscosity the positive area is much larger than the negative, whilst for small viscosity the positive area is a little smaller than the negative.

If the figure were extended further to the right, the two curves for the variation of I would become identical, and the ordinates would become very small. The two curves for the variation of J would separate widely. That for large viscosity would go upwards in the positive direction, so that its ordinates would be infinite at the point corresponding to $\lambda = \frac{1}{2}$; the curve for small viscosity would go downwards in the negative direction, and the ordinates would be infinite at the point where $\lambda = 1$.

In this figure OO′ is 6 centimeters, OO″ is 8·28 centimeters, and the point

corresponding to $\lambda = \frac{1}{2}$ would be 15·2 centimeters from O, and the point corresponding to $\lambda = 1$ would be 17·4 centimeters from O.

We thus see that the degree of viscosity makes an enormous difference in the results.

In the figure, portions of these further parts of the two curves for the variation of J are continued conjecturally by a line of dashes.

The whole figure is to be read from left to right for a retrospective solution, and from right to left if we advance with the time.

Fig. 7.

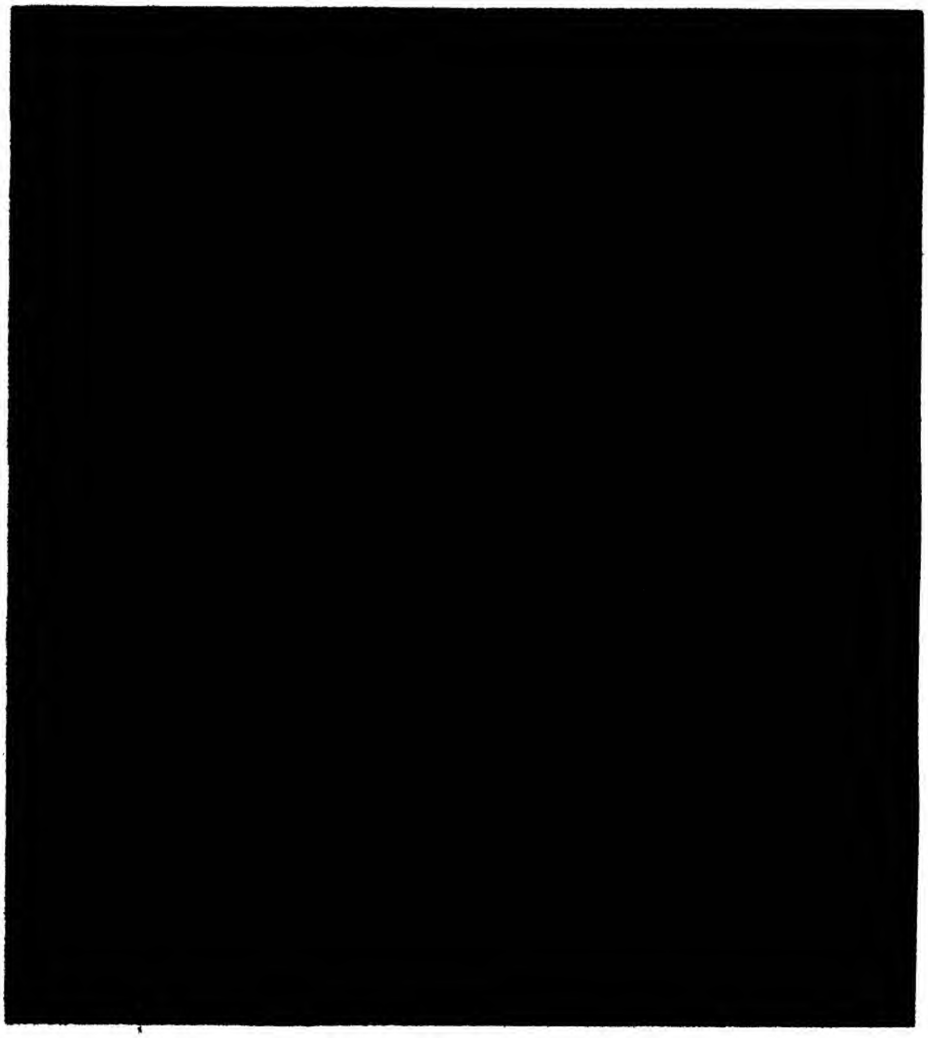


Diagram to illustrate the motion of the proper planes of the moon and earth.

§ 22. *The effects of solar tidal friction on the primitive condition of the earth and moon.*

In the paper on "Precession," § 16, I found, by the solution of a biquadratic equation, the primitive condition in which the earth and moon moved round together as a rigid body.

Since writing that paper certain additional considerations have occurred to me, which seem to be important in regard to the origin of the moon.

It was there remarked that, as we approach that critical condition of dynamical instability, the effects of solar tidal friction must have become sensible, because of the slow relative motion of the moon and earth. I did not at that time perceive the full significance of this, and I will now consider it further.

Suppose the moon to be moving orbitally nearly as fast as the earth rotates. Then the tidal reaction, which depends on the lunar tides alone, must be very small, and therefore the moon's orbital motion increases retrospectively very slowly. On the other hand, the relative motion of the earth and sun is great, and therefore if we approach the critical condition close enough, the solar tidal friction must have been greater than the lunar, however great the viscosity of the planet. The manner in which this will affect the solution of the previous paper may be shown analytically as follows.

If we neglect the obliquity, and divide the equation of tidal friction by that of tidal reaction, and suppose the viscosity small, we have from (176)

$$-k\frac{dn}{d\xi}=1+\left(\frac{\tau'}{\tau}\right)^2\frac{n}{n-\Omega}=1+\left(\frac{\tau'}{\tau_0}\right)^2\xi^{12}+\left(\frac{\tau'}{\tau}\right)^2\frac{\Omega}{n-\Omega}$$

Then integrating we have

$$n=n_0+\frac{1}{k}\left[(1-\xi)+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})\right]+\frac{1}{k}\int_\xi^1\left(\frac{\tau'}{\tau}\right)^2\frac{\Omega}{n-\Omega}d\xi$$

If we do not carry the integration to near the critical phase, where n is equal to Ω, the last integral is small, but it tends to become large as n becomes nearly equal to Ω; it has always been neglected in our integration. When however we wish to apply this equation to find the values for which n is equal to Ω, it cannot be neglected.

Suppose the integral to be equal to K. Then in the first part of the above expression we may put $n=\Omega=x^3$ and we may neglect $\frac{1}{13}(\tau'/\tau_0)^2(1-\xi^{13})$. Hence the equation for finding the angular velocity of the two bodies at the critical phase, when $n=\Omega$, is

$$x^3=n_0+\frac{1}{k}-\frac{1}{sx}+\mathrm{K}$$

$$x^4-\left(n_0+\frac{1}{k}+\mathrm{K}\right)x+\frac{1}{s}=0$$

The root of this equation, which gives the required phase, is nearly equal to the cube-root of the second coefficient, hence

$$x^3=n=\Omega=\left(n_0+\frac{1}{k}+\mathrm{K}\right)\text{ nearly.}$$

Now in the paper on "Precession" we found the initial condition, on the hypothesis that K was zero. Hence the effect of solar tidal friction is to increase the angular velocity of the two bodies when their relative motion is zero. Since K may be large, it follows that the disturbance of the solution of § 16 of "Precession" may be considerable.

This therefore shows that it is probable that an accurate solution of our problem would differ considerably from that found in "Precession," and that the common angular velocity of the two bodies might have been very great.

If KEPLER'S law holds good, then the periodic time of the moon about the earth, when their centres are 6,000 miles apart, is 2 hrs. 36 m., and when 5,000 miles apart is 1 hr. 57 m.; hence when the two spheroids are just in contact, the time of revolution of the moon would be between 2 hrs. and 2½ hrs.

Now it is a remarkable fact that the most rapid rate of revolution of a mass of fluid, of the same mean density as the earth, which is consistent with an ellipsoidal form of equilibrium, is 2 hrs. 24 m. Is this a mere coincidence, or does it not rather point to the break-up of the primæval planet into two masses in consequence of a too rapid rotation?

*It is not possible to make an adequate consideration of the subject of this section without a treatment of the theory of the tidal friction of a planet attended by a pair of satellites.

It was shown above that if the moon were to move orbitally nearly as fast as the earth rotates, the solar tidal friction would be more important than the lunar, however near the moon might be to the earth. I now (September, 1880) find that the consequence of this is that the earth's rotation continues to increase retrospectively, and the moon's orbital motion does the same; but the difference of the rotation and orbital

* From this point to the end has been added, and the section otherwise abridged since the paper was presented.—September, 1880.

motion gets continually less and less. Meanwhile, the earth's orbital motion round the sun is continually increasing, and the distance from the sun decreasing retrospectively. Theoretically this would go on until the sun and moon (treated as particles) revolve as though rigidly connected with the earth and with one another. This is the configuration of maximum energy of the system.

The solution is physically absurd, because the distance of the two bodies from the earth would then be very much less than the earth's radius, and *à fortiori* than the sun's radius.

It must be observed, however, that in the retrospect the relative motion of the moon and earth would already have become almost insensible, before the earth's distance from the sun could be sensibly affected.

V.

SECULAR CHANGES IN THE ECCENTRICITY OF THE ORBIT.

§ 23. *Formation of the disturbing function.*

We will now consider the rate of change in the eccentricity and mean distance of the orbit of a satellite, moving in an elliptic orbit, but always remaining in a fixed plane, namely, the ecliptic; and the rate of change of the obliquity of the planet's equator when perturbed by such a satellite will also be found.

Up to the end of Part I. the investigation for the formation of the disturbing function was quite general, and we therefore resume the thread at that point.

In the present problem the inclination of the satellite's orbit to the ecliptic is zero, and we have

$$\varpi=\underline{\varpi}=P=\cos\tfrac{1}{2}i,\quad \kappa=\underline{\kappa}=Q=\sin\tfrac{1}{2}i$$

We thus get rid of the ϖ and κ functions, and henceforth ϖ will indicate the longitude of the perigee.

Then by equations (24–8),

$$M_1{}^2-M_2{}^2=P^4\cos 2(\chi-\theta)+2P^2Q^2\cos 2\chi+Q^4\cos 2(\chi+\theta)$$

$$-2M_1M_2=\text{ The same with sines for cosines}$$

$$M_2M_3=-P^3Q\cos(\chi-2\theta)+PQ(P^2-Q^2)\cos\chi+PQ^3\cos(\chi+2\theta)$$

$$M_1M_3=\text{ The same with sines for cosines}$$

$$\tfrac{1}{3}-M_3{}^2=\tfrac{1}{3}(P^4-4P^2Q^2+Q^4)+2P^2Q^2\cos 2\theta$$

By the definitions (29)

$$X=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}M_1,\ Y=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}M_2,\ Z=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}M_3$$

Now let

$$\Phi(\alpha)=\left[\frac{c(1-e^2)}{r}\right]^3\cos(2\theta+\alpha),\ \Psi(\alpha)=\left[\frac{c(1-e^2)}{r}\right]^3\cos\alpha,\ R=\left[\frac{c(1-e^2)}{r}\right]^3 \quad . \quad (264)$$

Then

$$\begin{aligned}
X^2-Y^2&=P^4\Phi(-2\chi)+2P^2Q^2\Psi(2\chi)+Q^4\Phi(2\chi)\\
2XY&=\text{The same when } \chi+\tfrac{1}{4}\pi \text{ is substituted for } \chi\\
YZ&=-P^3Q\Phi(-\chi)+PQ(P^2-Q^2)\Psi(\chi)+PQ^3\Phi(\chi) \quad . \quad . \quad (265)\\
XZ&=\text{The same when } \chi-\tfrac{1}{2}\pi \text{ is substituted for } \chi\\
\tfrac{1}{3}(X^2+Y^2-2Z^2)&=\tfrac{1}{3}(P^4-P^2Q^2+Q^4)R+2P^2Q^2\Phi(0)
\end{aligned}$$

Hence all the terms of the five X-Y-Z functions belong to one of the three types Φ, Ψ, or R.

The equation to the ellipse described by the satellite Diana is

$$\frac{c(1-e^2)}{r}=1+e\cos(\theta-\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (266)$$

Hence

$$\left.\begin{aligned}
R&=1+\tfrac{3}{2}e^2+3e(1+\tfrac{1}{4}e^2)\cos(\theta-\varpi)+\tfrac{3}{2}e^2\cos 2(\theta-\varpi)+\tfrac{1}{4}e^3\cos 3(\theta-\varpi)\\
\Phi(\alpha)&=R\cos(2\theta+\alpha)=(1+\tfrac{3}{2}e^2)\cos(2\theta+\alpha)\\
&\quad+\tfrac{3}{2}e(1+\tfrac{1}{4}e^2)[\cos(3\theta+\alpha-\varpi)+\cos(\theta+\alpha+\varpi)]\\
&\quad+\tfrac{3}{4}e^2[\cos(4\theta+\alpha-2\varpi)+\cos(\alpha+2\varpi)]\\
&\quad+\tfrac{1}{8}e^3[\cos(5\theta+\alpha-3\varpi)+\cos(\theta-\alpha-3\varpi)]
\end{aligned}\right\} (267)$$

and $\Psi(\alpha)=R\cos\alpha$.

Now by the theory of elliptic motion, θ the true longitude may be expressed in terms of $\Omega t+\epsilon$ and ϖ, in a series of ascending powers of e the eccentricity. Hence $\Phi(\alpha)$, R, and $\Psi(\alpha)$ may be expressed as the sum of a number of cosines of angles of the form $l(\Omega t+\epsilon)+m\varpi+n\alpha$, and in using these functions we shall require to make α either a multiple of χ or zero, or to differ from a multiple of χ by a constant. Therefore the X-Y-Z functions are expressible as the sums of a number of sines or cosines of angles of the form $l(\Omega t+\epsilon)+m\varpi+n\chi$.

Now χ increases uniformly with the time (being equal to nt+a constant); hence, if

we regard the elements of the elliptic orbit as constant, the X-Y-Z functions are expressible as a number of simple time-harmonics. But in § 4, where the state of tidal distortion due to Diana was found, they were assumed to be so expressible; therefore that assumption was justifiable, and the remainder of that section concerning the formation of the disturbing function is applicable.

The problem may now be simplified by the following considerations:—The equation (12) for the rate of variation of the ellipticity of the orbit involves only differentials of the disturbing function with regard to epoch and perigee. It is obvious that in the disturbing function the epoch and perigee will only occur in the argument of trigonometrical functions, therefore after the required differentiations they only occur in the like forms. Now the epoch never occurs except in conjunction with the mean longitude, and the longitude of the perigee increases uniformly with the time (or nearly so), either from the action of other disturbing bodies or from the disturbing action of the permanent oblateness of the planet, which causes a progression of the apses. Hence it follows that the only way in which these differentials of the disturbing function can be non-periodic is when the tide-raiser Diana is identical with the moon. Whence we conclude that—

The tides raised by any one satellite can produce no secular change in the eccentricity of the orbit of any other satellite.

The problem is thus simplified by the consideration that Diana and the moon need only be regarded as distinct as far as regards epoch and perigee, and that they are ultimately to be made identical.

Before carrying out the procedure above sketched, it will be well to consider what sort of approximations are to be made, for the subsequent labour will be thus largely abridged.

From the preceding sketch it is clear that all the terms of the X-Y-Z functions corresponding with Diana's tide-generating potential are of the form

$$(a+be+ce^2+de^3+fe^4+\&c.)\cos[l\chi+m(\Omega t+\epsilon)+n\varpi+\delta].$$

From this it follows that all the terms of the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions are of the form

$$\mathrm{F}(a+be+ce^2+de^3+fe^4+\&c.)\cos[l\chi+m(\Omega t+\epsilon)+n\varpi+\delta-f].$$

Also by symmetry all the terms of the X′-Y′-Z′ functions are of the form

$$(a+be+ce^2+de^3+fe^4+\&c.)\cos[l\chi'+m(\Omega t+\epsilon')+n\varpi'+\delta],$$

and in the present problem the accent to χ may be omitted.

The products of the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions multiplied by the X′-Y′-Z′ functions occur in such a way that when they are added together in the required manner (as for example in $\mathrm{Y'Z'}\,\mathfrak{YZ}+\mathrm{X'Z'}\,\mathfrak{XZ}$) only differences of arguments occur, and χ disappears from the disturbing function. Also secular changes can only arise in the satellite's eccentricity and mean distance from such terms in the disturbing function as are independent of $\Omega t+\epsilon$ and ϖ, when we put $\epsilon'=\epsilon$ and $\varpi'=\varpi$. Hence we need only select from the

complete products the products of terms of the like argument in the two sets of functions.

Whence it follows that all the part of the disturbing function, which is here important, consists of terms of the form

$$F(a+be+ce^2+de^3+fe^4+\&c.)^2\cos\left[m(\epsilon-\epsilon')+n(\varpi-\varpi')-f\right]$$

or

$$F(a^2+2abe+(2ac+b^2)e^2+(2ad+2bc)e^3+(2af+2bd+c^2)e^4+\&c.)$$
$$\cos\left[m(\epsilon-\epsilon')+n(\varpi-\varpi')-f\right]$$

Now it is intended to develop the disturbing function rigorously with respect to the obliquity of the ecliptic, and as far as the fourth power of the eccentricity.

The question therefore arises, what terms will it be necessary to retain in developing the X-Y-Z functions, so as to obtain the disturbing function correct to e^4.

In the X-Y-Z functions (and in their constituent functions $\Phi(\alpha)$, $\Psi(\alpha)$, R) those terms in which a is not zero will be said to be of the order zero; those in which a is zero, but b not zero, of the first order; those in which $a=b=0$, but c not zero, of the second order, and so on.

Then, by considering the typical term in the disturbing function, we have the following—

Rule of approximation for the development of the X-Y-Z functions and of $\Phi(\alpha)$, $\Psi(\alpha)$, R: develop terms of order zero to e^4; terms of the first order to e^3; terms of the second order to e^2; and drop terms of the third and fourth orders.

To obtain further rules of approximation, and for the subsequent developments, we now require the following theorem.

Expansion of $\cos(k\theta+\beta)$ *in powers of the eccentricity.*

θ is the true longitude of the satellite, $\Omega t+\epsilon$ the mean longitude, and ϖ the longitude of the perigee. For the present I shall write simply Ω in place of $\Omega t+\epsilon$.

By the theory of elliptic motion

$$\Omega=\theta-2e\sin(\theta-\varpi)+\tfrac{3}{4}e^2(1+\tfrac{1}{6}e^2)\sin 2(\theta-\varpi)-\tfrac{1}{3}e^3\sin 3(\theta-\varpi)+\tfrac{5}{32}e^4\sin 4(\theta-\varpi)$$

If this series be inverted, it will be found that*

$$\theta=\Omega+2e(1-\tfrac{1}{8}e^2)\sin(\Omega-\varpi)+\tfrac{5}{4}e^2(1-\tfrac{11}{30}e^2)\sin 2(\Omega-\varpi)+\tfrac{13}{12}e^3\sin 3(\Omega-\varpi)$$
$$+\tfrac{103}{96}e^4\sin 4(\Omega-\varpi)$$

* See TAIT and STEELE'S 'Dynamics,' art. 118, or any other work on elliptic motion.

By differentiation we find that, when e=0,

$$\frac{d\theta}{de}=2\sin(\Omega-\varpi),\quad \frac{d^2\theta}{de^2}=\tfrac{5}{2}\sin 2(\Omega-\varpi),\quad \frac{d^3\theta}{de^3}=-\tfrac{3}{2}\sin(\Omega-\varpi)+\tfrac{13}{2}\sin 3(\Omega-\varpi)$$

$$\frac{d^4\theta}{de^4}=-11\sin 2(\Omega-\varpi)+\tfrac{103}{4}\sin 4(\Omega-\varpi),\quad \left(\frac{d\theta}{de}\right)^2=2-2\cos(\Omega-\varpi)$$

$$\left(\frac{d\theta}{de}\right)^3=6\sin(\Omega-\varpi)-2\sin 3(\Omega-\varpi),\quad \left(\frac{d\theta}{de}\right)^4=6-8\cos 2(\Omega-\varpi)+2\cos 4(\Omega-\varpi)$$

$$\frac{d\theta}{de}\frac{d^2\theta}{de^2}=\tfrac{5}{2}\cos(\Omega-\varpi)-\tfrac{5}{2}\cos 3(\Omega-\varpi),\quad \left(\frac{d\theta}{de}\right)^2\frac{d^2\theta}{de^2}=5\sin 2(\Omega-\varpi)-\tfrac{5}{2}\sin 4(\Omega-\varpi)$$

$$\left(\frac{d^2\theta}{de^2}\right)^2=\tfrac{25}{8}-\tfrac{25}{8}\cos 4(\Omega-\varpi),\quad \frac{d\theta}{de}\frac{d^3\theta}{de^3}=-\tfrac{3}{2}+8\cos 2(\Omega-\varpi)-\tfrac{13}{2}\cos 4(\Omega-\varpi)$$

To expand $\cos(k\theta+\beta)$ by means of MACLAURIN'S theorem, we require the values of the following differentials when e=0 and $\theta=\Omega$:—

$$\frac{d}{de}\cos(k\theta+\beta)=-k\sin(k\theta+\beta)\frac{d\theta}{de}$$

$$\frac{d^2}{de^2}\cos(k\theta+\beta)=-k^2\cos(k\theta+\beta)\left(\frac{d\theta}{de}\right)^2-k\sin(k\theta+\beta)\frac{d^2\theta}{de^2}$$

$$\frac{d^3}{de^3}\cos(k\theta+\beta)=k^3\sin(k\theta+\beta)\left(\frac{d\theta}{de}\right)^3-3k^2\cos(k\theta+\beta)\frac{d\theta}{de}\frac{d^2\theta}{de^2}-k\sin(k\theta+\beta)\frac{d^3\theta}{de^3}$$

$$\frac{d^4}{de^4}\cos(k\theta+\beta)=k^4\cos(k\theta+\beta)\left(\frac{d\theta}{de}\right)^4+6k^3\sin(k\theta+\beta)\left(\frac{d\theta}{de}\right)^2\frac{d^2\theta}{de^2}-3k^2\cos(k\theta+\beta)\left(\frac{d^2\theta}{de^2}\right)^2$$
$$-4k^2\cos(k\theta+\beta)\frac{d\theta}{de}\frac{d^3\theta}{de^3}-k\sin(k\theta+\beta)\frac{d^4\theta}{de^4}$$

Now when e=0, $k\theta+\beta=k\Omega+\beta$, and the values of the differentials and functions of differentials of e are given above. Then if we substitute for these functions their values, and express the products of sines and cosines as the sums of sines and cosines, and introduce the abridged notation in which $k\Omega+\beta+s(\Omega-\varpi)$ is written $(k+s)$, we have

$$\Theta_1=\frac{d}{de}\cos(k\theta+\beta)=-k\cos(k-1)+k\cos(k+1)$$

$$\Theta_2=\frac{d^2}{de^2}\cos(k\theta+\beta)=(k^2-\tfrac{5}{4}k)\cos(k-2)-2k^2\cos k+(k^2+\tfrac{5}{4}k)\cos(k+2)$$

$$\left.\begin{aligned}\Theta_3=\frac{d^3}{de^3}\cos(k\theta+\beta)=&-(k^3-\tfrac{15}{4}k^2+\tfrac{13}{4}k)\cos(k-3)+3(k^3-\tfrac{5}{4}k^2+\tfrac{1}{4}k)\cos(k-1)\\&-3(k^3+\tfrac{5}{4}k^2+\tfrac{1}{4}k)\cos(k+1)+(k^3+\tfrac{15}{4}k^2+\tfrac{13}{4}k)\cos(k+3)\end{aligned}\right\} \qquad (268)$$

$$\begin{aligned}\Theta_4=\frac{d^4}{de^4}\cos(k\theta+\beta)=&(k^4-\tfrac{15}{2}k^3+\tfrac{75}{16}k^2+13k^2-\tfrac{103}{8}k)\cos(k-4)\\&-(4k^4-15k^3+16k^2-\tfrac{11}{2}k)\cos(k-2)\\&+3(2k^4-\tfrac{25}{8}k^2+2k^2)\cos(k)-(4k^4+15k^3+16k^2+\tfrac{11}{2}k)\cos(k+2)\\&+(k^4+\tfrac{15}{2}k^3+\tfrac{75}{16}k^2+13k^2+\tfrac{103}{8}k)\cos(k+4)\end{aligned}$$

where the Θ's are merely introduced as an abbreviation.

Then by MACLAURIN's theorem

$$\cos(k\theta+\beta)=\cos(k\Omega+\beta)+e\Theta_1+\tfrac{1}{2}e^2\Theta_2+\tfrac{1}{6}e^3\Theta_3+\tfrac{1}{24}e^4\Theta_4. \quad . \quad . \quad . \qquad (269)$$

In order to obtain further rules of approximation we will now run through the future developments, merely paying attention to the order of the coefficients and to the factors by which $\Omega t+\epsilon$ will be multiplied in the results. From this point of view we may write

$$\Phi(\alpha)=(e^0)\cos(2\theta)+(e)[\cos(3\theta)+\cos(\theta)]+(e^2)[\cos(4\theta)+\cos(0)]$$
$$+(e^3)[\cos(5\theta)+\cos(\theta)]$$

$$\Psi(\alpha)=R=(e^0)\cos(0)+(e)\cos(\theta)+(e^2)\cos(2\theta)+(e^3)\cos(3\theta)$$

The cosines of the multiples of θ have now to be found by the theorem (269) and substituted in the above equations.

In making the developments the following abridged notation is adopted; a term of the form $\cos[(k+s)\Omega+\beta-s\varpi]$ is written $\{k+s\}$.

Consider the series for $\Phi(\alpha)$ first.

We have by successive applications of (269) with $k=1, 2, 3, 4, 5$.

$$(e^0)\cos(2\theta)=(e^0)\{2\}+(e)[\{1\}+\{3\}]+(e^2)[\{0\}+\{2\}+\{4\}]$$
$$+(e^3)[\{-1\}+\{1\}+\{3\}+\{5\}]+(e^4)[\{-2\}+\{0\}+\{2\}+\{4\}+\{6\}]$$

$$(e)\cos(3\theta)=(e)\{3\}+(e^2)[\{2\}+\{4\}]+(e^3)[\{1\}+\{3\}+\{5\}]$$
$$+(e^4)[\{0\}+\{2\}+\{4\}+\{6\}]$$

$$(e)\cos(\theta)=(e)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{-1\}+\{1\}+\{3\}]$$
$$+(e^4)[\{-2\}+\{0\}+\{2\}+\{4\}]$$

$$(e^2)\cos(4\theta)=(e^2)\{4\}+(e^3)[\{3\}+\{5\}]+(e^4)[\{2\}+\{4\}+\{6\}]$$

$$(e^2)\cos(0)=(e^2)\{0\}$$

$$(e^3)\cos(5\theta)=(e^3)\{5\}+(e^4)[\{4\}+\{6\}]$$

$$(e^3)\cos(\theta)=(e^3)\{1\}+(e^4)[\{0\}+\{2\}]$$

In these expressions we have no right, as yet, to assume that $\{-2\}$ and $\{-1\}$ are different from $\{2\}$ and $\{1\}$; and in fact we shall find that in the expansion for $\Phi(\alpha)$ they *are* different, but in that for R they are the same.

Then adding up these, and rejecting terms of the third and fourth orders by the first rule of approximation, we have

$$\Phi(\alpha)=[(e^0)+(e^2)+(e^4)]\{2\}+[(e)+(e^3)][\{1\}+\{3\}]+[(e^2)+(e^4)][\{0\}+\{4\}]$$
$$+(e^3)\{-1\}+(e^4)\{-2\}$$

It will be observed that $\{5\}$ and $\{6\}$ are wanting, and might have been dropped from the expansions. Also $\{0\}$ and $\{4\}$ are terms of the second order, therefore wherever they are multiplied by (e^4) they might have been dropped. Hence $(e^3)\cos(5\theta)$ need not have been expanded at all. A little further consideration is required to show that $(e^3)\cos(\theta)$ need not have been expanded.

$(e^3)\cos(\theta)$ is an abbreviation for $\frac{1}{8}e^3\cos(\theta-\alpha-3\varpi)$, and therefore in this case $\{1\}=\cos(\Omega-\alpha-3\varpi)$ and $\{2\}=\cos(2\Omega-\alpha-4\varpi)$; but in every other case $\{1\}=\cos(\Omega+\alpha+\varpi)$ and $\{2\}=\cos(2\Omega+\alpha)$. Hence the terms $\{1\}$ and $\{2\}$ in $(e^3)\cos(\theta)$ are of the third and fourth orders and may be dropped, and $\{0\}$ may also be dropped. Thus the whole of $(e^3)\cos(\theta)$ may be dropped.

With respect to $\{-2\}$ and $\{-1\}$, observe that $\{2\}$ in the expansion of $\cos(k_1\theta+\beta_1)$ stands for $\cos[2\Omega+(k_1-2)\varpi+\beta_1]$; and $\{-2\}$ in the expansion of $\cos(k_2\theta+\beta_2)$ stands for $\cos[2\Omega-(k_2+2)\varpi-\beta_2]$; and k_1, k_2 are either 1, 2, 3, or 4; and β_1, β_2 are multiples of $\chi+$ a constant. Hence $\{2\}$ and $\{-2\}$ are necessarily different, but if β_1 and β_2 were multiples of ϖ they might be the same, and indeed in the expansion of R necessarily are the same.

In the same way it may be shown that $\{-1\}$ and $\{1\}$ are necessarily different.

Therefore $\{-1\}$ and $\{-2\}$ being terms of the third and fourth orders may be dropped.

It follows from this discussion that, as far as concerns the present problem,

$$(e^0)\cos(2\theta)=(e^0)\{2\}+(e)[\{1\}+\{3\}]+(e^2)[\{0\}+\{2\}+\{4\}]+(e^3)[\{1\}+\{3\}]+(e^4)[\{2\}]$$

$$(e)\cos(3\theta)=(e)\{3\}+(e^2)[\{2\}+\{4\}]+(e^3)[\{1\}+\{3\}]+(e^4)\{2\}$$

$$(e)\cos(\theta)=(e^2)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{1\}+\{3\}]+(e^4)\{2\}$$

$$(e^2)\cos(4\theta)=(e^2)\{4\}+(e^3)\{3\}+(e^4)\{2\}$$

$$(e^2)\cos(0)=(e^2)\{0\}$$

And the sum of these expressions is equal to $\Phi(\alpha)$.

We thus get the following rules for the use of the expansion (269) of $\cos(k\theta+\beta)$ for the determination of $\Phi(\alpha)$:

When $k=2$, omit in Θ_3 terms in $\cos(k-3)$, $\cos(k+3)$
in Θ_4 terms in $\cos(k-4)$, $\cos(k-2)$, $\cos(k+2)$, $\cos(k+4)$

When $k=3$, omit in Θ_2 term in $\cos(k+2)$
in Θ_3 terms in $\cos(k-3)$, $\cos(k+1)$, $\cos(k+3)$
all of Θ_4

When $k=1$, omit in Θ_2 term in $\cos(k-2)$
in Θ_3 term in $\cos(k-3)$, $\cos(k-1)$, $\cos(k+3)$
all of Θ_4

When $k=4$, omit in Θ_1 term in $\cos(k+1)$
in Θ_2 term in $\cos(k)$, $\cos(k+2)$
all of Θ_3, Θ_4

Then following these rules we easily find,

When $k=2$, $\beta=\alpha$

$$\cos(2\theta+\alpha)=(1-4e^2+\tfrac{55}{16}e^4)\cos(2\Omega+\alpha)-2e(1-\tfrac{7}{8}e^2)\cos(\Omega+\alpha+\varpi)$$
$$+2e(1-\tfrac{27}{8}e^2)\cos(3\Omega+\alpha-\varpi)+\tfrac{3}{4}e^2\cos(\alpha+2\varpi)+\tfrac{13}{4}e^2\cos(4\Omega+\alpha-2\varpi)\,. \quad (270)$$

When $k=3$, $\beta=\alpha-\varpi$

$$\cos(3\theta+\alpha-\varpi)=(1-9e^2)\cos(3\Omega+\alpha-\varpi)-3e(1-\tfrac{11}{4}e^2)\cos(2\Omega+\alpha)$$
$$+3e\cos(4\Omega+\alpha-2\varpi)+\tfrac{21}{8}e^2\cos(\Omega+\alpha+\varpi)\,. \quad (271)$$

When $k=1$, $\beta=\alpha+\varpi$

$$\cos(\theta+\alpha+\varpi)=(1-e^2)\cos(\Omega+\alpha+\varpi)+e(1-\tfrac{5}{4}e^2)\cos(2\Omega+\alpha)-e\cos(\alpha+2\varpi)$$
$$+\tfrac{9}{8}e^2\cos(3\Omega+\alpha-\varpi)\,. \quad (272)$$

When $k=4$, $\beta=\alpha-2\varpi$

$$\cos(4\theta+\alpha-2\varpi)=\cos(4\Omega+\alpha-2\varpi)-4e\cos(3\Omega+\alpha-\varpi)+\tfrac{11}{2}e^2\cos(2\Omega+\alpha)\,. \quad (273)$$

These are all the series required for the expression of $\Phi(\alpha)$, since $\cos(\alpha+2\varpi)$ does not involve θ, and by what has been shown above $\cos(5\theta+\alpha-3\varpi)$ and $\cos(\theta-\alpha-3\varpi)$ need not be expanded.

We now return again to the series for R or $\Psi(\alpha)$, and consider the nature of the approximations to be adopted there.

With the same notation

$$(e^0)\cos(0)=(e^0)\{0\}$$
$$(e)\cos(\theta)=(e)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{-1\}+\{1\}+\{3\}]$$
$$+(e^4)[\{-2\}+\{0\}+\{2\}+\{4\}]$$
$$(e^2)\cos(2\theta)=(e^2)\{2\}+(e^3)[\{1\}+\{3\}]+(e^4)[\{0\}+\{2\}+\{4\}]$$
$$(e^3)\cos(3\theta)=(e^3)\{3\}+(e^4)[\{2\}+\{4\}]$$

Since R is a function of $\theta-\varpi$, therefore after expansion it must be a function of $\Omega-\varpi$, and hence $\{1\}$ must be necessarily identical with $\{-1\}$, and $\{2\}$ with $\{-2\}$.

Adding these up, and dropping terms of the third and fourth orders,

$$R=[(e^0)+(e^2)+(e^4)]\{0\}+[(e)+(e^3)]\{1\}+(e^3)\{-1\}+[(e^2)+(e^4)]\{2\}+(e^4)\{-2\}$$

Here $\{0\}$ is a term of the order zero, $\{1\}$ of the first order, and $\{2\}$ of the second. Therefore by the first rule of approximation $\{2\}$ and $\{-2\}$ may be dropped when multiplied by (e^4).

Also $\{3\}$ and $\{4\}$ may be dropped.

Hence as far as concerns the present problem

$$(e^0)\cos(0)=(e^0)\{0\}$$
$$(e)\cos(\theta)=(e)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{-1\}+\{1\}]+(e^4)\{0\}$$
$$(e^2)\cos(2\theta)=(e^2)\{2\}+(e^3)\{1\}+(e^4)\{0\}$$

and $(e^3)\cos(3\theta)$ need not be expanded.

And the sum of these expressions is equal to R.

We thus get the following rules for the use of the expansion of $\cos(k\theta+\beta)$ for the determination of R.

When $k=1$, omit in Θ_2 term in $\cos(k+2)$
in Θ_3 terms in $\cos(k-3)$, $\cos(k+1)$, $\cos(k+3)$
all of Θ_4

When $k=2$, omit in Θ_1 term in $\cos(k+1)$
in Θ_2 terms in $\cos(k)$, $\cos(k+2)$
all of Θ_3, Θ_4.

Then following these rules, we find

When $k=1$, $\beta=-\varpi$

$$\cos(\theta-\varpi)=(1-e^2)\cos(\Omega-\varpi)-e+e\cos 2(\Omega-\varpi) \quad . \quad . \quad . \quad . \quad (274)$$

When $k=2$, $\beta=-2\varpi$

$$\cos 2(\theta-\varpi)=\cos 2(\Omega-\varpi)-2e\cos(\Omega-\varpi)+\frac{3e^2}{4} \quad . \quad . \quad . \quad . \quad (275)$$

These are the only series required for the expansion of R or $\Psi(\alpha)$, since by what is shown above, $\cos 3(\theta-\varpi)$ need not be expanded.

Now multiply (270) by $1+\tfrac{3}{2}e^2$; (271) by $\tfrac{3}{2}e(1+\tfrac{1}{4}e^2)$; (272) by $\tfrac{3}{2}e(1+\tfrac{1}{4}e^2)$; and (273) by $\tfrac{3}{4}e^2$; add the four products together, and add $\tfrac{3}{4}e^2\cos(\alpha+2\varpi)$, and we find from (267) after reduction

$$\Phi(\alpha)=(1-\tfrac{11}{2}e^2+\tfrac{181}{16}e^4)\cos(2\Omega+\alpha)-\tfrac{1}{2}e(1-\tfrac{25}{8}e^2)\cos(\Omega+\alpha+\varpi)$$
$$+\tfrac{7}{2}e(1-\tfrac{291}{56}e^2)\cos(3\Omega+\alpha-\varpi)+\tfrac{17}{2}e^2\cos(4\Omega+\alpha-2\varpi) \quad . \quad . \quad . \quad (276)$$

Next multiply (274) by $3e(1+\tfrac{1}{4}e^2)$; (275) by $\tfrac{9}{2}e^2$; add the two products, and add $1+\tfrac{3}{2}e^2$, and we find from (267) after reduction,

$$R=1-\tfrac{3}{2}e^2+\tfrac{3}{8}e^4+3e(1-\tfrac{15}{8}e^2)\cos(\Omega-\varpi)+\tfrac{9}{2}e^2\cos 2(\Omega-\varpi) \quad . \quad . \quad (277)$$

Now let

$$\left.\begin{array}{l}\mathrm{E}_1=-\tfrac{1}{2}\mathrm{e}(1-\tfrac{25}{8}\mathrm{e}^2);\ \mathrm{E}_2=1-\tfrac{11}{2}\mathrm{e}^2+\tfrac{181}{16}\mathrm{e}^4;\ \mathrm{E}_3=\tfrac{7}{2}\mathrm{e}(1-\tfrac{201}{56}\mathrm{e}^2);\ \mathrm{E}_4=\tfrac{17}{2}\mathrm{e}^2\\ \mathrm{J}_0=1-\tfrac{3}{2}\mathrm{e}^2+\tfrac{3}{8}\mathrm{e}^4;\ \mathrm{J}_1=\tfrac{3}{2}\mathrm{e}(1-\tfrac{15}{8}\mathrm{e}^2);\ \mathrm{J}_2=\tfrac{9}{4}\mathrm{e}^2\end{array}\right\}\quad(278)$$

And we have

$$\left.\begin{array}{l}\Phi(\alpha)=\mathrm{E}_1\cos(\Omega+\alpha+\varpi)+\mathrm{E}_2\cos(2\Omega+\alpha)+\mathrm{E}_3\cos(3\Omega+\alpha-\varpi)\\ \qquad\qquad+\mathrm{E}_4\cos(4\Omega+\alpha-2\varpi)\\ \mathrm{R}=\mathrm{J}_0+2\mathrm{J}_1\cos(\Omega-\varpi)+2\mathrm{J}_2\cos 2(\Omega-\varpi)\\ \text{whence}\\ \Psi(\alpha)=\mathrm{J}_0\cos\alpha+\mathrm{J}_1[\cos(\Omega+\alpha-\varpi)+\cos(\Omega-\alpha-\varpi)]\\ \qquad\qquad+\mathrm{J}_2[\cos(2\Omega+\alpha-2\varpi)+\cos(2\Omega-\alpha-2\varpi)].\end{array}\right\}\quad(279)$$

These three expressions are parts of infinite series which only go as far as terms in e^2, but the terms of the orders e^0 and e have their coefficients developed as far as e^4 and e^3 respectively.

Then substituting from (279) for Φ, Ψ, and R their values in the expressions (265), we find

$$\left.\begin{array}{l}\mathrm{X}^2-\mathrm{Y}^2=P^4[\mathrm{E}_1\cos(2\chi-\Omega-\varpi)+\mathrm{E}_2\cos(2\chi-2\Omega)+\mathrm{E}_3\cos(2\chi-3\Omega+\varpi)\\ \qquad\qquad+\mathrm{E}_4\cos(2\chi-4\Omega+2\varpi)]\\ \qquad+2P^2Q^2[\mathrm{J}_0\cos 2\chi+\mathrm{J}_1\{\cos(2\chi-\Omega+\varpi)+\cos(2\chi+\Omega-\varpi)\}\\ \qquad\qquad+\mathrm{J}_2\{\cos(2\chi-2\Omega+2\varpi)+\cos(2\chi+2\Omega-2\varpi)\}]\\ \qquad+Q^4[\mathrm{E}_1\cos(2\chi+\Omega+\varpi)+\mathrm{E}_2\cos(2\chi+2\Omega)+\mathrm{E}_3\cos(2\chi+3\Omega-\varpi)\\ \qquad\qquad+\mathrm{E}_4\cos(2\chi+4\Omega-2\varpi)]\\ -2\mathrm{XY}=\text{The same, with sines for cosines}\\ \quad\mathrm{YZ}=\text{The same as }\mathrm{X}^2-\mathrm{Y}^2\text{, but with }-P^3Q\text{ for }P^4,\ PQ(P^2-Q^2)\text{ for}\\ \qquad 2P^2Q^2,\ PQ^3\text{ for }Q^4\text{ and with }\chi\text{ for }2\chi\\ \quad\mathrm{XZ}=\text{The same as the last, but with sines for cosines}\\ \tfrac{1}{3}(\mathrm{X}^2+\mathrm{Y}^2-2\mathrm{Z}^2)=\tfrac{1}{3}(P^4-4P^2Q^2+Q^4)[\mathrm{J}_0+2\mathrm{J}_1\cos(\Omega-\varpi)+2\mathrm{J}_2\cos 2(\Omega-\varpi)]\\ \qquad+2P^2Q^2[\mathrm{E}_1\cos(\Omega+\varpi)+\mathrm{E}_2\cos 2\Omega+\mathrm{E}_3\cos(3\Omega-\varpi)\\ \qquad\qquad+\mathrm{E}_4\cos(4\Omega-2\varpi)]\end{array}\right\}\quad(280)$$

Then if we regard ϖ as constant, and remember that $\chi=nt$, and that Ω stands for $\Omega t+\epsilon$, and if we look through the above functions we see that there are trigonometrical

terms of 22 different speeds, viz.: 9 in the first pair all involving $2nt$, 9 in the second pair all involving nt, and 4 in the last.

Then since these five functions correspond to Diana's tide-generating potential, therefore we are going to consider the effects of 22 different tides, nine being semi-diurnal, nine diurnal, and the last four may be conveniently called monthly, since their periods are $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$ of a month and one month.

We next have to form the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions. We found that in the X-Y-Z functions there were terms of 22 different speeds; hence we shall now have to introduce 44 symbols indicating the reduction in the height of tide below its equilibrium height, and the retardation of phase. The notation adopted is analogous to that used in the preceding problem, and the following schedule gives the symbols.

Semi-diurnal tides.

speed	$2n-4\Omega$	$2n-3\Omega$	$2n-2\Omega$	$2n-\Omega$	$2n$	$2n+\Omega$	$2n+2\Omega$	$2n+3\Omega$	$2n+4\Omega$
height	F^{iv}	F^{iii}	F^{ii}	F^{i}	F	F_{i}	F_{ii}	F_{iii}	F_{iv}
lag	$2f^{iv}$	$2f^{iii}$	$2f^{ii}$	$2f^{i}$	$2f$	$2f_{i}$	$2f_{ii}$	$2f_{iii}$	$2f_{iv}$

Diurnal tides.

speed	$n-4\Omega$	$n-3\Omega$	$n-2\Omega$	$n-\Omega$	n	$n+\Omega$	$n+2\Omega$	$n+3\Omega$	$n+4\Omega$
height	G^{iv}	G^{iii}	G^{ii}	G^{i}	G	G_{i}	G_{ii}	G_{iii}	G_{iv}
lag	g^{iv}	g^{iii}	g^{ii}	g^{i}	g			g_{iii}	g_{iv}

*Monthly tides.**

speed	Ω	2Ω	3Ω	4Ω
height	H^{i}	H^{ii}	H^{iii}	H^{iv}
lag	h^{i}	$2h^{ii}$	$3h^{iii}$	$4h^{iv}$

The $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions might now be easily written out; for each term of the X-Y-Z functions is to be multiplied, according to its *speed* by the corresponding *height*, and the corresponding *lag* subtracted from the argument of the trigonometrical term. For example, the first term of $\mathfrak{X}^2-\mathfrak{Y}^2$ is $F^{i}E_1P^4\cos(2\chi-\Omega-\varpi-2f^{i})$. It will however be unnecessary to write out these long expressions.

In order to form the disturbing function W, the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions have now to be multiplied by the X′-Y′-Z′ functions according to the formula (31). Now the X′-Y′-Z′ functions only differ from the X-Y-Z functions in the accentuation of Ω and ϖ, because Diana is to be ultimately identical with the moon.

Then in the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions Ω is an abbreviation for $\Omega t+\epsilon$, and in the X′-Y′-Z′ functions Ω' for $\Omega t+\epsilon'$; hence wherever in the products we find $\Omega-\Omega'$, we may replace it by $\epsilon-\epsilon'$.

* With periods of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and one month.

Again, since we are only seeking to find the secular changes in the ellipticity and mean distance, therefore (as before pointed out) we need only multiply together terms whose arguments only differ by the lag. Secular *inequalities*, in the sense in which the term is used in the planetary theory, will indeed arise from the cross-multiplication of certain terms of like *speeds* but of different *arguments*,—for example, the product of the term $\mathrm{F}^{\mathrm{ii}}P^4\mathrm{E}_2\cos(2\chi-2\Omega-2\mathrm{f}^{\mathrm{ii}})$ in $\mathfrak{X}^2-\mathfrak{Y}^2$ multiplied by the term $2P^2Q^2\mathrm{J}_2\cos(2\chi-2\Omega'+2\varpi')$ in $\mathrm{X}'^2-\mathrm{Y}'^2$, when added to the similar cross-product in $4\mathrm{X}'\mathrm{Y}'\mathfrak{X}\mathfrak{Y}$ (which only differs in having sines for cosines) will give a term $2\mathrm{F}^{\mathrm{ii}}P^6Q^2\mathrm{E}_2\mathrm{J}_2\cos[2(\epsilon'-\epsilon)-2\varpi'-2\mathrm{f}^{\mathrm{ii}}]$. This term in the disturbing function will give a long inequality, but it is of no present interest.

The products may now be written down without writing out in full either the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions or the X'-Y'-Z' functions. In order that the results may form the constituent terms of W, the factor $\frac{1}{2}$ is introduced in the first pair of products, the factor 2 in the second pair, and the factor $\frac{3}{2}$ in the last. Then from (280) we have

$$
\begin{aligned}
&2\,\frac{\mathrm{X}'^2-\mathrm{Y}'^2}{2}\,\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}+2\mathrm{X}'\mathrm{Y}'\mathfrak{X}\mathfrak{Y}\\
&\quad=\tfrac{1}{2}P^8\{\mathrm{F}^{\mathrm{i}}\mathrm{E}_1{}^2\cos[(\epsilon'-\epsilon)+(\varpi'-\varpi)-2\mathrm{f}^{\mathrm{i}}]+\mathrm{F}^{\mathrm{ii}}\mathrm{E}_2{}^2\cos[2(\epsilon'-\epsilon)-2\mathrm{f}^{\mathrm{ii}}]\\
&\qquad\qquad+\mathrm{F}^{\mathrm{iii}}\mathrm{E}_3{}^2\cos[3(\epsilon'-\epsilon)-(\varpi'-\varpi)-2\mathrm{f}^{\mathrm{iii}}]+\mathrm{F}^{\mathrm{iv}}\mathrm{E}_4{}^2\cos[4(\epsilon'-\epsilon)-2(\varpi'-\varpi)-2\mathrm{f}^{\mathrm{iv}}]\}\\
&\qquad+2P^4Q^4\{\mathrm{F}\mathrm{J}_0{}^2\cos 2\mathrm{f}\\
&\qquad\qquad+\mathrm{F}^{\mathrm{i}}\mathrm{J}_1{}^2\cos[(\epsilon'-\epsilon)-(\varpi'-\varpi)-2\mathrm{f}^{\mathrm{i}}]+\mathrm{F}_{\mathrm{i}}\mathrm{J}_1{}^2\cos[(\epsilon'-\epsilon)-(\varpi'-\varpi)+2\mathrm{f}_{\mathrm{i}}]\\
&\qquad\qquad+\mathrm{F}^{\mathrm{ii}}\mathrm{J}_2{}^2\cos[2(\epsilon'-\epsilon)-2(\varpi'-\varpi)-2\mathrm{f}^{\mathrm{ii}}]+\mathrm{F}_{\mathrm{ii}}\mathrm{J}_2{}^2\cos[2(\epsilon'-\epsilon)-2(\varpi'-\varpi)+2\mathrm{f}_{\mathrm{ii}}]\}\\
&\qquad+\tfrac{1}{2}Q^8\{\mathrm{F}_{\mathrm{i}}\mathrm{E}_1{}^2\cos[(\epsilon'-\epsilon)+(\varpi'-\varpi)+2\mathrm{f}_{\mathrm{i}}]+\mathrm{F}_{\mathrm{ii}}\mathrm{E}_2{}^2\cos[2(\epsilon'-\epsilon)+2\mathrm{f}_{\mathrm{ii}}]\\
&\qquad\qquad+\mathrm{F}_{\mathrm{iii}}\mathrm{E}_3{}^2\cos[3(\epsilon'-\epsilon)-(\varpi'-\varpi)+2\mathrm{f}_{\mathrm{iii}}]+\mathrm{F}_{\mathrm{iv}}\mathrm{E}_4{}^2\cos[4(\epsilon'-\epsilon)-2(\varpi'-\varpi)+2\mathrm{f}_{\mathrm{iv}}]\}
\end{aligned}
\tag{281}
$$

$2\mathrm{Y}'\mathrm{Z}'\mathfrak{Y}\mathfrak{Z}+2\mathrm{X}'\mathrm{Z}'\mathfrak{X}\mathfrak{Z}=$ the same, when $2P^6Q^2$ replaces $\frac{1}{2}P^8$; $2P^2Q^2(P^2-Q^2)^2$ replaces $2P^4Q^4$; $2P^2Q^6$ replaces $\frac{1}{2}Q^8$; and G's and g's replace F's and 2f's . (282)

$$
\begin{aligned}
&\tfrac{3}{2}\,\frac{\mathrm{X}'^2+\mathrm{Y}'^2-2\mathrm{Z}'^2}{3}\,\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3}\\
&\quad=\tfrac{1}{6}(P^4-4P^2Q^2+Q^4)^2\{\mathrm{J}_0{}^2+2\mathrm{H}^{\mathrm{i}}\mathrm{J}_1{}^2\cos[(\epsilon'-\epsilon)-(\varpi'-\varpi)+\mathrm{h}^{\mathrm{i}}]\\
&\qquad\qquad+2\mathrm{H}^{\mathrm{ii}}\mathrm{J}_2{}^2\cos[2(\epsilon'-\epsilon)-2(\varpi'-\varpi)+2\mathrm{h}^{\mathrm{ii}}]\}\\
&\qquad+3P^4Q^4\{\mathrm{H}^{\mathrm{i}}\mathrm{E}_1{}^2\cos[(\epsilon'-\epsilon)+(\varpi'-\varpi)+\mathrm{h}^{\mathrm{i}}]+\mathrm{H}^{\mathrm{ii}}\mathrm{E}_2{}^2\cos[2(\epsilon'-\epsilon)+2\mathrm{h}^{\mathrm{ii}}]\\
&\qquad\qquad+\mathrm{H}^{\mathrm{iii}}\mathrm{E}_3{}^2\cos[3(\epsilon'-\epsilon)-(\varpi'-\varpi)+3\mathrm{h}^{\mathrm{iii}}]+\mathrm{H}^{\mathrm{iv}}\mathrm{E}_4{}^2\cos[4(\epsilon'-\epsilon)-2(\varpi'-\varpi)+4\mathrm{h}^{\mathrm{iv}}]\}
\end{aligned}
\tag{283}
$$

The sum of these three last expressions (281–3) when multiplied by $\frac{\tau^2}{g}\frac{1}{(1-e^2)^6}$ is equal to W the disturbing function.

§ 24. *Secular changes in eccentricity and mean distance.*

Before proceeding to the differentiation of W, it is well to note the following coincidences between the coefficients and arguments, viz.: E_1^2 occurs with $(\epsilon'-\epsilon)+(\varpi'-\varpi)$, E_2^2 with $2(\epsilon'-\epsilon)$, E_3^2 with $3(\epsilon'-\epsilon)-(\varpi'-\varpi)$, E_4^2 with $4(\epsilon'-\epsilon)-2(\varpi'-\varpi)$, J_1^2 with $(\epsilon'-\epsilon)-(\varpi'-\varpi)$, J_2^2 with $2(\epsilon'-\epsilon)-2(\varpi'-\varpi)$, and the terms in J_0^2 do not involve ϵ, ϵ', ϖ, ϖ'. In consequence of these coincidences it will be possible to arrange the results in a highly symmetrical form.

By equations (11) and (12)

$$-\frac{\xi}{k}\frac{d}{dt}\log\eta=\left(\frac{d}{d\epsilon'}+\gamma\frac{d}{d\varpi'}\right)W, \text{ when } \gamma=\frac{1}{\eta}$$

and

$$\frac{1}{k}\frac{d\xi}{dt}=\left(\frac{d}{d\epsilon'}+\gamma\frac{d}{d\varpi'}\right)W, \text{ when } \gamma=0$$

Hence the single operation $d/d\epsilon'+\gamma d/d\varpi'$ will enable us by proper choice of the value of γ to find either $\xi d\log\eta/kdt$ or $d\xi/kdt$.

Perform this operation; then putting $\epsilon'=\epsilon$, $\varpi'=\varpi$, and collecting the terms according to their respective E's and J's, we have

$$\left(\frac{dW}{d\epsilon'}+\gamma\frac{dW}{d\varpi'}\right)\div\frac{\tau^2}{g}\frac{1}{(1-e^2)^6}$$
$$=E_1^2(1+\gamma)\{\tfrac{1}{2}P^8F^i\sin 2f^i+2P^6Q^2G^i\sin g^i-2P^2Q^6G_i\sin g_i-\tfrac{1}{2}Q^8F_i\sin 2f_i$$
$$-3P^4Q^4H^i\sin h^i\}$$
$$+E_2^2(2)\{\text{the same with ii for i, and } 2h^{ii} \text{ for } h^i\}$$
$$+E_3^2(3-\gamma)\{\text{the same with iii for i, and } 3h^{iii} \text{ for } h^i\}$$
$$+E_4^2(4-2\gamma)\{\text{the same with iv for i, and } 4h^{iv} \text{ for } h^i\}$$
$$+J_1^2(1-\gamma)\{2P^4Q^4(F^i\sin 2f^i-F_i\sin 2f_i)+2P^2Q^2(P^2-Q^2)^2(G^i\sin g^i-G_i\sin g_i)$$
$$-\tfrac{1}{8}(P^4-4P^2Q^2+Q^4)^2H^i\sin h^i\}$$
$$+J_2^2(2-2\gamma)\{\text{the same with ii for i, and } 2h^{ii} \text{ for } h^i\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (284)$$

The functions of P and Q, which appear here, will occur hereafter so frequently that it will be convenient to adopt an abridged notation for them. Let x then represent either i, ii, iii or iv, and let

$$\left.\begin{aligned}\phi(\mathrm{x})&=\tfrac{1}{2}P^8\mathrm{F}^{\mathrm{x}}\sin 2\mathrm{f}^{\mathrm{x}}+2P^6Q^2\mathrm{G}^{\mathrm{x}}\sin \mathrm{g}^{\mathrm{x}}-2P^2Q^6\mathrm{G}_{\mathrm{x}}\sin \mathrm{g}_{\mathrm{x}}-\tfrac{1}{2}Q^8\mathrm{F}_{\mathrm{x}}\sin 2\mathrm{f}_{\mathrm{x}}\\&\qquad-3P^4Q^4\mathrm{H}^{\mathrm{x}}\sin(\mathrm{xh}^{\mathrm{x}})\\\psi(\mathrm{x})&=2P^4Q^4(\mathrm{F}^{\mathrm{x}}\sin 2\mathrm{f}^{\mathrm{x}}-\mathrm{F}_{\mathrm{x}}\sin 2\mathrm{f}_{\mathrm{x}})+2P^2Q^2(P^2-Q^2)^2(\mathrm{G}^{\mathrm{x}}\sin \mathrm{g}^{\mathrm{x}}-\mathrm{G}_{\mathrm{x}}\sin \mathrm{g}_{\mathrm{x}})\\&\qquad-\tfrac{1}{2}(P^4-4P^2Q^2+Q^4)^2\mathrm{H}^{\mathrm{x}}\sin(\mathrm{xh}^{\mathrm{x}})\end{aligned}\right\}\quad(285)$$

And the generalised definition of the F's, G's, H's, &c., is contained in the following schedule

$$\left.\begin{array}{llllll}\text{speed} & 2n-\mathrm{x}\Omega, & n-\mathrm{x}\Omega, & \mathrm{x}\Omega, & n+\mathrm{x}\Omega, & 2n+\mathrm{x}\Omega\\ \text{height} & \mathrm{F}^{\mathrm{x}} & \mathrm{G}^{\mathrm{x}} & \mathrm{H}^{\mathrm{x}} & \mathrm{G}_{\mathrm{x}} & \mathrm{F}_{\mathrm{x}}\\ \text{lag} & 2\mathrm{f}^{\mathrm{x}} & \mathrm{g}^{\mathrm{x}} & (\mathrm{xh}^{\mathrm{x}}) & \mathrm{g}_{\mathrm{x}} & 2\mathrm{f}_{\mathrm{x}}\end{array}\right\}\quad .\ .\ .\quad(286)$$

We must now substitute for the E's and J's their values, and as the ellipticity is chosen as the variable they must be expressed in terms of η instead of e. Also each of the E^2's and J^2's must be divided by $(1-\mathrm{e}^2)^6$.

Then since $\sqrt{1-\mathrm{e}^2}=1-\eta$, therefore

$$\mathrm{e}^2=2\eta-\eta^2 \text{ and } (1-\mathrm{e}^2)^{-6}=(1-\eta)^{-12}=1+12\eta+78\eta^2$$

Then by (278)

$$\left.\begin{aligned}
&\mathrm{E}_1^2=\tfrac{1}{4}\mathrm{e}^2(1-\tfrac{25}{4}\mathrm{e}^2)=\tfrac{1}{2}\eta(1-13\eta) &&, \text{ and } \frac{\mathrm{E}_1^2}{(1-\eta)^{12}}=\tfrac{1}{2}\eta(1-\eta)\\
&\mathrm{E}_2^2=1-11\mathrm{e}^2+\tfrac{423}{8}\mathrm{e}^4=1-22\eta+\tfrac{445}{2}\eta^2, &&\text{ and } \frac{\mathrm{E}_2^2}{(1-\eta)^{12}}=1-10\eta+\tfrac{73}{2}\eta^2\\
&\mathrm{E}_3^2=\tfrac{49}{4}\mathrm{e}^2(1-\tfrac{291}{28}\mathrm{e}^2)=\tfrac{49}{2}\eta(1-\tfrac{149}{7}\eta) &&, \text{ and } \frac{\mathrm{E}_3^2}{(1-\eta)^{12}}=\tfrac{49}{2}\eta(1-\tfrac{65}{7}\eta)\\
&\mathrm{E}_4^2=\tfrac{289}{4}\mathrm{e}^4=289\eta^2 &&, \text{ and } \frac{\mathrm{E}_4^2}{(1-\eta)^{12}}=289\eta^2\\
&\mathrm{J}_0^2=1-3\mathrm{e}^2+3\mathrm{e}^4=1-6\eta+15\eta^2 &&, \text{ and } \frac{\mathrm{J}_0^2}{(1-\eta)^{12}}=1+6\eta+21\eta^2\\
&\mathrm{J}_1^2=\tfrac{9}{4}\mathrm{e}^2(1-\tfrac{15}{4}\mathrm{e}^2)=\tfrac{9}{2}\eta(1-8\eta) &&, \text{ and } \frac{\mathrm{J}_1^2}{(1-\eta)^{12}}=\tfrac{9}{2}\eta(1+4\eta)\\
&\mathrm{J}_2^2=\tfrac{81}{16}\mathrm{e}^4=\tfrac{81}{4}\eta^2 &&, \text{ and } \frac{\mathrm{J}_2^2}{(1-\eta)^{12}}=\tfrac{81}{4}\eta^2
\end{aligned}\right\}\quad(287)$$

When γ is put equal to $\frac{1}{\eta}$ we shall also require the following :—

$$\begin{aligned}
&\frac{\mathrm{E}_1^2(1+\eta)}{\eta(1-\eta)^{12}}=\tfrac{1}{2}; &&\frac{\mathrm{E}_2^2(2)}{(1-\eta)^{12}}=2(1-10\eta);\\
&\frac{\mathrm{E}_3^2(3\eta-1)}{\eta(1-\eta)^{12}}=-\tfrac{49}{2}(1-\tfrac{86}{7}\eta); &&\frac{\mathrm{E}_4^2(4\eta-2)}{\eta(1-\eta)^{12}}=-578\eta;\\
&\frac{\mathrm{J}_1^2(\eta-1)}{\eta(1-\eta)^{12}}=-\tfrac{9}{2}(1+3\eta); &&\frac{\mathrm{J}_2^2(2\eta-2)}{\eta(1-\eta)^{12}}=-\tfrac{81}{2}\eta
\end{aligned}\quad(288)$$

Therefore by putting $\gamma=\frac{1}{\eta}$ in equation (284) we have

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{2}\phi(\text{i})+2(1-10\eta)\phi(\text{ii})-\tfrac{49}{2}(1-\tfrac{86}{7}\eta)\phi(\text{iii})-578\eta\phi(\text{iv})$$
$$-\tfrac{9}{2}(1+3\eta)\psi(\text{i})-\tfrac{81}{2}\eta\psi(\text{ii})$$

and by putting $\gamma=0$ in (284)

$$\frac{\mathfrak{g}}{\tau^2}\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\eta(1-\eta)\phi(\text{i})+2(1-10\eta+\tfrac{73}{2}\eta^2)\phi(\text{ii})+\tfrac{147}{2}\eta(1-\tfrac{65}{7}\eta)\phi(\text{iii})+1156\eta^2\phi(\text{iv})$$
$$+\tfrac{9}{2}\eta(1+4\eta)\psi(\text{i})+\tfrac{81}{2}\eta^2\psi(\text{ii})$$

The equations may be also arranged in the following form :—

$$\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{2}[\phi(\text{i})+4\phi(\text{ii})-49\phi(\text{iii})-9\psi(\text{i})]$$
$$+\eta[-20\phi(\text{ii})+301\phi(\text{iii})-578\phi(\text{iv})-\tfrac{27}{2}\psi(\text{i})-\tfrac{81}{2}\psi(\text{ii})] \quad . \quad (289)$$

$$\frac{\mathfrak{g}}{\tau^2}\frac{1}{k}\frac{d\xi}{dt}=2\phi(\text{ii})$$
$$+\eta[\tfrac{1}{2}\phi(\text{i})-20\phi(\text{ii})+\tfrac{147}{2}\phi(\text{iii})+\tfrac{9}{2}\psi(\text{i})]$$
$$+\eta^2[-\tfrac{1}{2}\phi(\text{i})+73\phi(\text{ii})-\tfrac{1365}{2}\phi(\text{iii})+1156\phi(\text{iv})+18\psi(\text{i})+\tfrac{81}{2}\psi(\text{ii})] \quad . \quad . \quad (290)$$

The former of these apparently stops with the first power of η, but it will be observed that we have $d\log\eta/dt$ on the left-hand side so that $d\eta/dt$ is developed as far as η^2.

These equations give the required solutions of the problem.

§ 25. *Application to the case where the planet is viscous.*

If the planet or earth be viscous, we have, as in § 7, $\text{F}^{\text{x}}=\cos 2\text{f}^{\text{x}}$, $\text{G}^{\text{x}}=\cos \text{g}^{\text{x}}$, $\text{H}^{\text{x}}=\cos(\text{xh}^{\text{x}})$; $\text{G}_{\text{x}}=\cos \text{g}_{\text{x}}$, $\text{F}_{\text{x}}=\cos 2\text{f}_{\text{x}}$.

When these values are substituted in (289) we have the equation giving the rate of change of ellipticity in the case of viscosity. The equation is however so long and complex that it does not present to the mind any physical meaning, and I shall therefore illustrate it graphically.

The case taken is the same as that in § 7, where the planet rotates 15 times as fast as the satellite revolves.

The eccentricity or ellipticity is supposed to be small, so that only the first line of (289) is taken.

I took as five several standards of viscosity of the planet, such viscosities as would

make the lag f^{ii} of the principal slow semi-diurnal tide, of speed $2n-2\Omega$, equal to 10°, 20°, 30°, 40°, 44°. (The curves thus correspond to the same cases as in §§ 7 and 10). Values of $\sin 4f^x$, $\sin 2g^x$, $\sin 2xh^x$, $\sin 2g_x$, $\sin 4f_x$, when $x=$ i, ii, iii were then computed, according to the theory of viscous tides.

These values were then taken for computing values of ϕ(i), ϕ(ii), ϕ(iii), ψ(i) with values of $i=0°$, 15°, 30°, 45°, 60°, 75°, 90°. The results were then combined so as to give a series of values of $d \log\eta/dt$ or de/edt, and these values were set out graphically in the accompanying fig. 8.

Fig. 8.

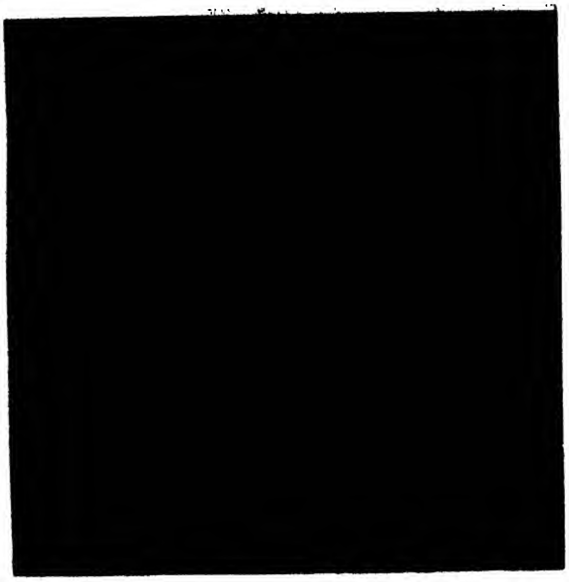


Diagram showing the rate of change in the eccentricity of the orbit of the satellite for various obliquities and viscosities of the planet $\left(\frac{1}{e}\frac{de}{dt}\text{, when } e \text{ is small}\right)$.

In the figure the ordinates are proportional to de/edt, and the abscissæ to i the obliquity; each curve corresponds to one degree of viscosity.

From the figure we see that, unless the viscosity be so great as to approach rigidity (when $f^{ii}=45°$), the eccentricity will increase for all values of the obliquity, except values approaching 90°.

The rate of increase is greatest for zero obliquity unless the viscosity be very large, and in that case it is a little greater for about 35° of obliquity.

It appears from the paper on "Precession" that if the obliquity be very nearly 90°, the satellite's distance from the planet decreases with the time. Hence it follows from this figure that in general the eccentricity of the orbit increases or diminishes with the mean distance; this is however not true if the viscosity approaches very near rigidity, for then the eccentricity will diminish for zero obliquity, whilst the mean distance will increase.

If the viscosity be very small, the equations (289-90) admit of reduction to very simple forms.

In this case the sines of twice the angles of lagging are proportional to the speeds of the several tides, and we have (as in previous cases)—

$$\frac{\sin 4f^x}{\sin 4f}=1-\tfrac{1}{2}x\lambda,\ \frac{\sin 2g^x}{\sin 4f}=\tfrac{1}{2}-\tfrac{1}{2}x\lambda,\ \frac{\sin 2xh^x}{\sin 4f}=\tfrac{1}{2}x\lambda,\ \frac{\sin 2g_x}{\sin 4f_x}=\tfrac{1}{2}+\tfrac{1}{2}x\lambda,\ \frac{\sin 4f_x}{\sin 4f}=1+\tfrac{1}{2}x\lambda.$$

Therefore

$$\phi(x)=\tfrac{1}{4}\sin 4f\left[P^8+2P^6Q^2-2P^2Q^6-Q^8-\tfrac{1}{2}x\lambda(P^8+4P^6Q^2+4P^2Q^6+Q^6+6P^4Q^4)\right]$$
$$=\tfrac{1}{4}\sin 4f\,(\cos i-\tfrac{1}{2}x\lambda)$$
$$\psi(x)=\tfrac{1}{2}\sin 4f\left[-2P^4Q^4x\lambda-2P^2Q^2(P^2-Q^2)^2x\lambda-\tfrac{1}{6}x\lambda(P^4-4P^2Q^2+Q^4)^2\right]$$
$$=-\tfrac{1}{4}\sin 4f\,(\tfrac{1}{2}x\lambda)(\tfrac{8}{3})$$

And

$$\phi(\mathrm{i})+4\phi(\mathrm{ii})-49\phi(\mathrm{iii})-9\psi(\mathrm{i})=-\sin 4f(11\cos i-18\lambda)$$
$$-20\phi(\mathrm{ii})+301\phi(\mathrm{iii})-578\phi(\mathrm{iv})-\tfrac{27}{2}\psi(\mathrm{i})-\tfrac{81}{2}\psi(\mathrm{ii})=-\tfrac{1}{4}\sin 4f(297\cos i-756\lambda)$$

Whence from (289)

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=-\tfrac{1}{2}\sin 4f\{11\cos i(1+\tfrac{27}{2}\eta)-18\lambda(1+21\eta)\}$$

or

$$\frac{\xi}{k}\frac{d}{dt}\log\eta=\frac{\tau^2}{\mathfrak{g}}(1+\tfrac{27}{2}\eta)\tfrac{11}{2}\sin 4f\left\{\cos i-\tfrac{18}{11}\frac{\Omega}{n}(1+\tfrac{15}{2}\eta)\right\}\quad . \quad . \quad . \quad (291)$$

From this we see that, in the case of small viscosity, tidal reaction is in general competent to cause the eccentricity of the orbit of a satellite to increase. But if 18 sidereal days of the planet be greater than 11 sidereal months of the satellite the eccentricity will decrease. Wherefore a circular orbit for the satellite is only dynamically stable provided 18 such days is greater than 11 such months.

Now if we treat the equation (290) for $\frac{d\xi}{dt}$ in the same way, we find—

The first line $=\tfrac{1}{2}\sin 4f(\cos i-\lambda)$.

The second $=\tfrac{1}{8}\eta\sin 4f(27\cos i-46\lambda)$.

The third $=\tfrac{1}{8}\eta^2\sin 4f(273\cos i-697\lambda)$

Therefore

$$\frac{\mathfrak{g}}{\tau^2}\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\sin 4\mathrm{f}\left[(1+27\eta+273\eta^2)\cos i-\lambda(1+46\eta+697\eta^2)\right]$$

or

$$\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}(1+27\eta+273\eta^2)\sin 4\mathrm{f}\left[\cos i-\frac{\Omega}{n}(1+19\eta-89\eta^2)\right] \quad . \quad . \quad (292)$$

From this it follows that the rate of tidal reaction is greater if the orbit be eccentric than if it be circular. Also for zero obliquity the tidal reaction vanishes when

$$\frac{\Omega}{n}=1-19\eta+450\eta^2$$

Hence if a satellite were to separate from a planet in such a way that, at the moment after separation, its mean motion were equal to the angular velocity of the planet, then if its orbit were eccentric it must fall back into the planet; but if its orbit were circular an infinitesimal disturbance would decide whether it should approach or recede from the planet.*

Now suppose that the viscosity is very large, and that the obliquity is zero.

Then

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{8}(\sin 4\mathrm{f}^{\mathrm{I}}+4\sin 4\mathrm{f}^{\mathrm{II}}-49\sin 4\mathrm{f}^{\mathrm{III}}+6\sin 2\mathrm{h}^{\mathrm{I}})$$

and the sines are reciprocally proportional to the speeds of the tides, from which they take their origin. As to the term in $\sin 2\mathrm{h}^{\mathrm{I}}$, which takes its origin from the elliptic monthly tide, the viscosity must make a close approach to absolute rigidity for this term to be reciprocally proportional to the speed of that tide; for the present, therefore, $\sin 2\mathrm{h}^{\mathrm{I}}$ will be left as it is.

Then the equation becomes, on this hypothesis,

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{8}\sin 4\mathrm{f}^{\mathrm{II}}\left[\frac{1-\lambda}{1-\frac{1}{2}\lambda}+4-\frac{49(1-\lambda)}{1-\frac{3}{2}\lambda}\right]+\tfrac{6}{8}\sin 2\mathrm{h}^{\mathrm{I}}$$

$$\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{8}\sin 4\mathrm{f}^{\mathrm{II}}\frac{44-63\lambda+20\lambda^2}{(1-\frac{1}{2}\lambda)(1-\frac{3}{2}\lambda)}-\tfrac{6}{8}\sin 2\mathrm{h}^{\mathrm{I}}. \quad . \quad . \quad . \quad . \quad . \quad (293)$$

The numerator of the first term on the right is always positive for values of λ less than unity, and the denominator is always positive if λ be less than $\frac{2}{3}$. Hence if the viscosity be not so great but that the last term is small, the eccentricity always increases if λ lies between zero and $\frac{2}{3}$.

* See Appendix (p. 886) for further considerations on this subject.

If however λ be not small, then even though the viscosity be not great enough to approach perfect rigidity, we must have $\sin 2h' = 2(1-\lambda)\sin 4f''/\lambda$. And of course, by supposing the viscosity great enough, this relation may be fulfilled whatever be λ.

Then our equation becomes

$$\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta = -\tfrac{1}{8}\sin 4f''\frac{12-80\lambda+96\lambda^2-29\lambda^3}{\lambda(1-\frac{1}{2}\lambda)(1-\frac{3}{2}\lambda)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (294)$$

The numerator on the right-hand side is always positive for values of λ less than unity, and the denominator is positive for values of λ less than $\frac{2}{3}$.

Since

$$\frac{1}{k}\frac{d\xi}{dt} = \frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f''$$

we have

$$\xi\frac{d}{d\xi}\log\eta = -\frac{1}{4}\frac{12-80\lambda+96\lambda^2-29\lambda^3}{\lambda(1-\frac{1}{2}\lambda)(1-\frac{3}{2}\lambda)}$$

From this we see that, for very large viscosity,—

For values of λ between 1 and ·6667, the eccentricity increases per unit increase of ξ, and the rate of increase tends to become infinite when $\lambda = \cdot 6667$.

The remarks concerning the physical absurdity of this class of result in § 21 may be repeated in this case.

And for values of λ between ·6667 and 0, the eccentricity diminishes.

A similar treatment of the case of small viscosity shows that—

For values of λ between 1 and ·6111 the eccentricity decreases, and for values of λ between ·6111 and 0 the eccentricity increases.

Thus it is only between $\lambda = \cdot 6111$ and ·6667 that the two cases agree.

Hence in the course of evolution of a satellite revolving about a purely viscous planet :—

For small viscosity the orbit will remain circular until 11 months of the satellite are equal to 18 days of the planet, then the eccentricity will increase until this relationship is again fulfilled, when the eccentricity will again diminish.*

And for very large viscosity the orbit will at once become eccentric, and the eccentricity will increase very rapidly until two months of the satellite are equal to three days of the planet. The eccentricity will then diminish until this relationship is again fulfilled, after which the eccentricity will again increase.

We shall consider later which of these views seems the more probable with regard to the history of the moon.

* See "On the Analytical Expressions, &c.," Proc. Roy. Soc., No. 202, 1880.

§ 26. *Secular change in the obliquity and diurnal rotation of the planet, when the satellite moves in an eccentric orbit.*

The method of treating this problem will be the same as that of § 12, to which the reader is referred.

In the complete development of the disturbing function $\chi-\chi'$ would occur wherever the F's and G's occur, but never with the H's.

If we put $\gamma=1$ in (284), we have

$$\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}=\frac{2\tau^2}{\mathfrak{g}(1-\eta)^{12}}\Sigma \mathrm{E}_x{}^2\phi(x). \qquad (295)$$

Where Σ means summation for i, ii, iii, iv.

This result follows from the fact that in all the E-terms of W, ϵ' and ϖ' enter in the form $l\epsilon'+m\varpi'$, where $l+m=2$.

In the F^x-terms χ' enters in the form $2\chi'$, and is of the opposite sign from $l+m$; in the F_x-terms it enters in the form $2\chi'$, and is of the same sign as $l+m$; in the G^x-terms it enters in the form χ', and is of the opposite sign from $l+m$; in the G_x-terms it enters in the form χ', and is of the same sign as $l+m$.

Hence as far as regards the E-terms of W, we have

$$\begin{aligned}
&\text{in the F}^x\text{-terms } \frac{dW}{d\chi'} &&= -\left(\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\right)\\
&\text{in the F}_x\text{-terms} &&= \frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\\
&\text{in the G}^x\text{-terms} &&= -\tfrac{1}{2}\left(\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\right)\\
&\text{in the G}_x\text{-terms} &&= \tfrac{1}{2}\left(\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\right)\\
&\text{in the H-terms} &&= 0
\end{aligned}$$

In the J-terms of W, χ' enters with coefficient 2 in the F^x- and F_x-terms, and with the coefficient 1 in the G^x- and G_x-terms, and is always of the same sign as the corresponding lag.

Hence for the J-terms

$$\frac{dW}{d\chi'}=\Sigma\left(\frac{dW}{df^x}+\frac{dW}{dg^x}\right)$$

Where Σ means summation for the cases where x is zero and both upper and lower i and ii.

From this we have

$$\frac{dn}{dt}=\frac{d\text{W}}{d\chi}$$

$$=-\frac{\tau^2}{\mathfrak{g}}\frac{1}{(1-\eta)^{12}}[\Sigma \text{E}_x^2\{P^8\text{F}^x \sin 2\text{f}^x+2P^6Q^2\text{G}^x \sin \text{g}^x+2P^2Q^6\text{G}_x \sin \text{g}_x+Q^8\text{F}_x \sin 2\text{f}_x\}$$

$$+\text{J}_0^2\{4P^4Q^4\text{F} \sin 2\text{f}+2P^2Q^2(P^2-Q^2)^2\text{G} \sin \text{g}\}$$

$$+\Sigma \text{J}_x^2\{4P^4Q^4(\text{F}^x \sin 2\text{f}^x+\text{F}_x \sin 2\text{f}_x)+2P^2Q^2(P^2-Q^2)^2(\text{G}^x \sin \text{g}^x+\text{G}_x \sin \text{g}_x)\}] \quad (296)$$

The first Σ being from iv to i, and the last only for ii and i.

This is a partial solution for the tidal friction, and corresponds only to the action of the moon on her own tides; that of the sun on his tides may be obtained by symmetry.

It is easy to see that for the joint effect of the two bodies we have

$$\frac{dn}{dt}=-\frac{2\tau\tau'}{\mathfrak{g}}\frac{1}{(1-\eta)^6(1-\eta')^6}\text{J}_0\text{J}_0'\{4P^4Q^4\text{F} \sin 2\text{f}+2P^2Q^2(P^2-Q^2)^2\text{G} \sin \text{g}\} \quad . \quad (297)$$

From (296-7) and (287-8) the complete solution may be collected.

In order to find the secular change in the obliquity, we must consider how ψ' would enter in W.

Now in the development of W, $\Omega't+\epsilon'$ stands for $\Omega't+\epsilon'-\psi'$, and ϖ' stands for $\varpi'-\psi'$. Hence from (295)

$$\frac{d\text{W}}{d\psi'}=-\left(\frac{d\text{W}}{d\epsilon'}+\frac{d\text{W}}{d\varpi'}\right)$$

$$=-\frac{2\tau^2}{\mathfrak{g}}\frac{1}{(1-\eta)^{12}}\Sigma \text{E}_x^2\phi(\text{x}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (298)$$

Now by (18)

$$n \sin i\frac{di}{dt}=\frac{d\text{W}}{d\chi}\cos i-\frac{d\text{W}}{d\psi'}$$

Then substituting for $\frac{d\text{W}}{d\chi}$ from (296) and for $\frac{d\text{W}}{d\psi'}$ from (298), we find

$$n\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}}\frac{1}{(1-\eta)^{12}}\{\Sigma \text{E}_x^2[P^7Q\text{F}^x \sin 2\text{f}^x+P^5Q(P^2+3Q^2)\text{G}^x \sin \text{g}^x$$

$$-PQ^5(3P^2+Q^2)\text{G}_x \sin \text{g}_x-PQ^7\text{F}_x \sin 2\text{f}_x-3P^3Q^3\text{H}^x \sin (\text{xh}^x)]$$

$$-\text{J}_0^2[2P^3Q^3(P^2-Q^2)\text{F} \sin 2\text{f}+PQ(P^2-Q^2)^3\text{G} \sin \text{g}]$$

$$-\Sigma \text{J}_x^2[2P^3Q^3(P^2-Q^2)(\text{F}^x \sin 2\text{f}^x+\text{F}_x \sin 2\text{f}_x)$$

$$+PQ(P^2-Q^2)^3(\text{G}^x \sin \text{g}^x+\text{G}_x \sin \text{g}_x)]\} \quad . \quad (299)$$

The first Σ being from iv to i, and the last only for ii and i.

This is only a partial solution, and gives the result of the action of the moon on her own tides; that for the sun on his tides may be obtained by symmetry.

It is easy to see that for the joint effect

$$n\frac{di}{dt}=-\frac{2\tau\tau'}{\mathfrak{g}}\frac{1}{(1-\eta)^6(1-\eta')^6}\mathrm{J}_0\mathrm{J}_0'[2P^3Q^3(P^2-Q^2)\mathrm{F}\sin 2\mathrm{f}+PQ(P^2-Q^2)^3\mathrm{G}\sin\mathrm{g}] \quad . \quad (300)$$

From (299, 300) and (287–8) the complete solution may be collected.

Then if these solutions be applied to the case where the earth is viscous and where the viscosity is small, it will be found after reduction as in previous cases that

$$\begin{aligned}-\frac{dn}{dt}=\frac{\sin 4\mathrm{f}}{2\mathfrak{g}}\Big[&\tau^2(1-\tfrac{1}{2}\sin^2 i)(1+15\eta+\tfrac{105}{2}\eta^2)+\tau'^2(1-\tfrac{1}{2}\sin^2 i)(1+15\eta'+\tfrac{105}{2}\eta'^2)\\&-\tau^2\frac{\Omega}{n}\cos i(1+27\eta+273\eta^2)-\tau'^2\frac{\Omega'}{n}\cos i(1+27\eta'+273\eta'^2)\\&+\tau\tau'\tfrac{1}{2}\sin^2 i(1+3\eta+3\eta'+6\eta^2+9\eta\eta'+6\eta'^2)\Big] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)\end{aligned}$$

$$\begin{aligned}n\frac{di}{dt}=\frac{\sin 4\mathrm{f}}{4\mathfrak{g}}\sin i\cos i\Big[&\tau^2(1+15\eta+\tfrac{105}{2}\eta^2)+\tau'^2(1+15\eta'+\tfrac{105}{2}\eta'^2)\\&-2\tau^2\frac{\Omega}{n}\sec i(1+27\eta+273\eta^2)-2\tau'^2\frac{\Omega'}{n}\sec i(1+27\eta'+273\eta'^2)\\&-\tau\tau'(1+3\eta+3\eta'+6\eta^2+9\eta\eta'+6\eta'^2)\Big] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (302)\end{aligned}$$

These results give the tidal friction and rate of change of obliquity due both to the sun and moon; η is the ellipticity of the lunar orbit, and η' of the solar (or terrestrial) orbit.

If η and η' be put equal to zero they agree with the results obtained in the paper on "Precession."

§ 27. *Verification of analysis, and effect of evectional tides.*

The analysis of this part of the paper has been long and complex, and therefore a verification is valuable.

The moment of momentum of the orbital motion of the moon and earth round their common centre of inertia is proportional to the square root of the *latus rectum* of the orbit, according to the ordinary theory of elliptic motion. In the present notation this moment of momentum is equal to $\mathrm{C}\xi(1-\eta)/k$. Let us suppose the obliquity of the ecliptic to be zero. Then the whole moment of momentum of the system (supposing only one satellite to exist) is

$$C\left\{\frac{\xi}{k}(1-\eta)+n\right\}$$

Therefore we ought to find, if the analysis has been correctly worked, that

$$\frac{\xi}{k}\frac{d\eta}{dt}=(1-\eta)\frac{1}{k}\frac{d\xi}{dt}+\frac{dn}{dt}$$

This test will be only applied in the case where the viscosity is small, because the analysis is pretty short; but it may also be applied in the general case.

When $i=0$, we have from (292), after multiplying both sides by $1-\eta$,

$$\frac{2}{\sin 4f}\frac{\mathfrak{g}}{\tau^2}(1-\eta)\frac{1}{k}\frac{d\xi}{dt}=1+26\eta+246\eta^2-\lambda(1+45\eta+651\eta^2)$$

And when $i=0$ and $\tau'=0$, from (301)

$$-\frac{2}{\sin 4f}\frac{\mathfrak{g}}{\tau^2}\frac{dn}{dt}=1+15\eta+\tfrac{195}{2}\eta^2-\lambda(1+27\eta+273\eta^2)$$

Hence

$$(1-\eta)\frac{1}{k}\frac{d\xi}{dt}+\frac{dn}{dt}=\tfrac{1}{2}\sin 4f\frac{\tau^2}{\mathfrak{g}}\left[11\eta(1+\tfrac{27}{2}\eta)-18\lambda\eta(1+21\eta)\right]$$

$$=\frac{\xi}{k}\frac{d\eta}{dt}\text{ from (291)}$$

Thus the above formulas satisfy the condition of the constancy of the moment of momentum of the system.

The most important lunar inequality after the Equation of the centre is the Evection. The effects of lagging evectional tides may be worked out on the same plan as that pursued above for the Equation of the centre.

I will not give the analysis, but will merely state that, in the case of small viscosity of the earth, the equation for the rate of change of ellipticity, inclusive of the evectional terms, becomes

$$\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{11}{2}(1+\tfrac{27}{2}\eta)\sin 4f\frac{\tau^2}{\mathfrak{g}}\left\{1-\tfrac{18}{11}\frac{\Omega}{n}-\tfrac{675}{352}\left(\frac{\Omega'}{\Omega}\right)^2\right\}$$

where Ω' is the earth's mean motion in its orbit round the sun.

From this we see that, even at the present time, the evectional tides will only reduce the rate of increase of the ellipticity by $\frac{1}{88}$th part of the whole. In the integrations to be carried out in Part VI. this term will sink in importance, and therefore it will be entirely neglected.

The Variation is another lunar inequality of slightly less importance than the Evection; but it may be observed that the Evection was only of any importance because its argument involved the lunar perigee, and its coefficient the eccentricity. Now neither of these conditions are fulfilled in the case of the Variation. Moreover in the retrospective integration the coefficients will degrade far more rapidly than those of the evectional terms, because they will depend on $(\Omega'/\Omega)^4$. Hence the secular effects of the variational tides will not be given, though of course it would be easy to find them if they were required.

VI.

INTEGRATION FOR CHANGES IN THE ECCENTRICITY OF THE ORBIT.

§ 28. *Integration in the case of small viscosity.*

By (291-2), we have approximately

$$\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{11}{2}\sin 4\mathrm{f}\frac{\tau^2}{\mathrm{g}}(1+\tfrac{27}{2}\eta)[\cos i-\tfrac{18}{11}\lambda]$$

$$\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\sin 4\mathrm{f}\frac{\tau^2}{\mathrm{g}}(1+27\eta)[\cos i-\lambda]$$

Therefore

$$(1+\tfrac{27}{2}\eta)\frac{d}{d\xi}\log\eta=\frac{11}{\xi}\,\frac{1-\frac{18}{11}\lambda\sec i}{1-\lambda\sec i}$$

$$=\frac{11}{\xi}-7\frac{\Omega}{\xi n}\sec i\text{ approximately}$$

The last transformation assumes that λ or Ω/n is small compared with unity; this will be the case in the retrospective integration for a long way back.

Then as a first approximation we have

$$\eta=\eta_0\xi^{11}$$

Therefore

$$\int_1^\xi \tfrac{27}{2}\eta d\log\eta=\tfrac{27}{2}(\eta-\eta_0)=-\tfrac{27}{2}\eta_0(1-\xi^{11})\text{ approximately}$$

And for a second approximation

$$\log_e\left(\frac{\eta}{\eta_0\xi^{11}}\right)=\tfrac{27}{2}\eta_0(1-\xi^{11})-7\Omega_0\int_1^\xi\frac{\sec i}{\xi^4 n}d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad (303)$$

The integral in this expression is very small, and therefore to evaluate it we may

assign to i an average value, say I, and neglect the solar tidal friction in assigning a value to n.

Then

$$n=n_0+\frac{1}{k}(1-\xi)$$

Let

$$kn_0+1=\kappa, \text{ so that } n=\frac{1}{k}(\kappa-\xi)$$

Hence the last term in (303) is approximately equal to

$$=-7k\Omega_0 \sec \mathrm{I}\int_1^\xi \frac{d\xi}{\xi^4(\kappa-\xi)}=7k\Omega_0 \sec \mathrm{I}\left[\frac{1}{3\kappa}\left(\frac{1}{\xi^3}-1\right)+\frac{1}{2\kappa^2}\left(\frac{1}{\xi^2}-1\right)+\frac{1}{\kappa^3}\left(\frac{1}{\xi}-1\right)\right]$$
$$-\frac{7k\Omega_0}{\kappa^4}\sec \mathrm{I} \log\left(\frac{\xi n_0}{n}\right)$$

In the last term n has been written for $(\kappa-\xi)/k$.

Now let

$$\mathrm{K}=\left[\frac{1}{3\kappa}\left(\frac{1}{\xi^3}-1\right)+\frac{1}{2\kappa^2}\left(\frac{1}{\xi^2}-1\right)+\frac{1}{\kappa^3}\left(\frac{1}{\xi}-1\right)\right]7k\Omega_0 \sec \mathrm{I}+\tfrac{27}{2}\eta_0(1-\xi^{11})$$

Then

$$\eta=\eta_0\xi^{11}\left(\frac{n}{n_0\xi}\right)^{\frac{7k\Omega_0}{\kappa^4}\sec \mathrm{I}}e^{\mathrm{K}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (304)$$

This formula will now be applied to trace the changes in the eccentricity of the lunar orbit.

The integration will be made over a series of "periods" which cover the same ground as those in the paper on "Precession;" and the numerical results of that paper will be used for assigning the values to n and I.

kn_0 is equal to $1/\mu$ of that paper, and therefore κ is $(1+\mu)/\mu$.

First period of integration.

From $\xi=1$ to $\cdot 88$.

I is taken as 22°. In "Precession" μ was $4\cdot 0074$, therefore $kn_0=\cdot 24954$ and $\kappa=1\cdot 24954$. Also $k\Omega_0=kn_0\Omega_0/n_0$, and $\Omega_0/n_0=1/27\cdot 32$.

In computing for § 17 of "Precession" I found at the end of the period $\log n/n_0=\cdot 18971$.

Using these values I find

$$\log_{10}\left(\frac{n}{\xi n_0}\right)^{\frac{7k}{\kappa^4}\Omega_0 \sec \mathrm{I}}=\cdot 00692$$

Also

$$\mathrm{K}=\cdot 01980+\tfrac{27}{2}\eta_0(1-\xi^{11})$$

Now e_0, the present eccentricity of the lunar orbit, is ·054908.
Whence

$$\eta_0 = 1 - \sqrt{1 - e_0^2} = \cdot 001509$$

And

$$\tfrac{27}{2}\eta_0(1-\xi^{11}) = \cdot 015375.$$

Using these values I find

$$\log_{10}\eta = 6\cdot 59007 - 10, \text{ and the first approximation gave } \log_{10}\eta = 6\cdot 56788 - 10$$

Then $\eta = \cdot 00038911$
Whence $e = \cdot 02789$, at the end of the first period of integration.

Second period of integration.

From $\xi = 1$ to ·76. I was taken as 18° 45′.
A similar calculation gives

$$\log\left(\frac{n}{n_0\xi}\right)^{\frac{7k\Omega_0}{\kappa^4}\sec I} = \cdot 00817, \text{ the first part of } K = \cdot 06998, \ \tfrac{27}{2}\eta_0(1-\xi^{11}) = \cdot 00500$$

Whence

$$\log\eta = 5\cdot 31758 - 10, \text{ and the first approximation gave } \log\eta = 5\cdot 27902 - 10$$

Therefore $\eta = \cdot 000020777$ and $e = \cdot 006446$, at the end of the second period of integration.

Third period of integration.

From $\xi = 1$ to 76. I was taken as 16° 13′.
Then a similar calculation gave

$$\log\left(\frac{n}{n_0\xi}\right)^{\frac{7k}{\kappa^4}\Omega_0\sec I} = \cdot 00566, \text{ first part of } K = \cdot 12355, \ \tfrac{27}{2}\eta_0(1-\xi^{11}) = \cdot 00027$$

Whence

$$\log\eta = 4\cdot 06584 - 10, \text{ and the first approximation gave } \log\eta = 4\cdot 00653 - 10$$

Therefore $\eta = \cdot 0000011637$, and $e = \cdot 001526$ at the end of the third period of integration.

Fourth period of integration.

The procedure is now changed in the same way, and for the same reason, as in the fourth period of § 17 of "Precession."

Let $N=\frac{n}{n_0}$ (as in that paper). Then the equation of tidal friction is

$$-\frac{dN}{dt}=\tfrac{1}{2}\sin 4\mathrm{f}\,\frac{\tau^2}{\mathrm{g}n_0}(1-\lambda)$$

and the equation for the change in η may be written approximately

$$-\frac{d}{dN}\log\eta=\frac{n_0k}{\xi}\,\frac{11-18\lambda}{1-\lambda}$$

Since λ or Ω/n is no longer small, this expression will be integrated by quadratures. Using the numerical values given in § 17 of "Precession," I find the following corresponding values.

$N=$	1·000	1·107	1·214	1·321
$\frac{kn_0}{\xi}\,\frac{11-18\lambda}{1-\lambda}=$	15·469	17·665	19·465	11·994

Then integrating by quadratures with the common difference dN equal to ·107, we find the integral equal to 5·5715.

Whence $\eta=44{\cdot}273\times10^{-10}$, and $e=\cdot00009411$.

The results of the whole integration are given in the following table, of which the first two columns are taken from the paper on "Precession."

TABLE XVI.

Day in m. s. hours and minutes.		Moon's sidereal period in m. s. days.	Eccentricity of lunar orbit.
h.	m.	Days.	
23	56	27·32	·054908
15	28	18·62	·027894
9	55	8·17	·006446
7	49	3·59	·001526
5	55	12 hours	·000094

Beyond this the eccentricity would decrease very little more, because this integration stops where λ is about $\frac{1}{2}$, and the eccentricity ceases to diminish when λ is $\frac{11}{18}$. The final eccentricity in the above table is only $\frac{1}{580}$th of the initial eccentricity, and the orbit is very nearly circular.

§ 29. *The change of eccentricity when the viscosity is large.*

I shall not integrate the equations in the case where the viscosity is large, because the solution depends so largely on the exact degree of viscosity.

If the viscosity were infinitely large, then in the retrospective integration the eccentricity would be found getting larger and larger and finally would become infinite, when λ is equal to $\frac{2}{3}$. This result is of course physically absurd. If on the other hand the viscosity were large, we might find the eccentricity diminishing, then stationary, and finally increasing until $\lambda=\frac{2}{3}$, after which it would diminish again. Thus by varying the viscosity, supposed always large, we might get considerable diversity of results.

VII.

SUMMARY AND DISCUSSION OF RESULTS.

§ 30. *Explanation of problem.—Summary of Parts I. and II.*

In considering the changes in the orbit of a satellite due to frictional tides, very little interest attaches to those elements of the orbit which are to be specified, in order to assign the position which the satellite would occupy at a given instant of time. We are rather here merely concerned with those elements which contain a description of the nature of the orbit.

These elements are the mean distance, inclination, and eccentricity. Moreover all those inequalities in these three elements, which are periodic in time, whether they fall into the class of "secular" or "periodic" inequalities, have no interest for us, and what we require is to trace their *secular changes.*

Similarly, in the case of the planet we are only concerned to discover the secular changes in the period of its rotation, and in the obliquity of its equator to a fixed plane.

It has unfortunately been found impossible to direct the investigation strictly according to these considerations. Amongst the ignored elements are the longitudes of the nodes of the orbit and equator upon the fixed plane, and it was found in one part of the investigation, viz.: Part III., that secular inequalities (in the ordinary acceptance of the term) had to be taken into consideration both in the five elements which define the nature of the orbit, and the planet's mode of motion, and also in the motion of the two nodes.

In the paper on "Precession" I considered the secular changes in the mean distance of the satellite, and the obliquity and rotation-period of the planet, but the satellite's orbit was there assumed to be circular and confined to the fixed plane. In the present paper the inclination and eccentricity are specially considered, but the introduction of these elements has occasioned a modification of the results attained in the previous

paper. For convenience of diction I shall henceforth speak of the planet as the earth, and of the satellites as the moon and sun; for, as far as regards tides, the sun may be treated as a satellite of the earth. The investigation has been kept as far as possible general, so as to be applicable to any system of tides in the earth; but it has been directed more especially towards the conception of a bodily distortion of the earth's mass, and all the actual applications are made on the hypothesis that the earth is a viscous body. A very slight modification would however make the results applicable to frictional oceanic tides on a rigid nucleus (see § 1 immediately after (15)).

I thought it sufficient to consider the problem as divisible into the two following cases :—

1st. Where the moon's orbit is circular, but inclined to the ecliptic. (Parts I., II., III., IV.)

2nd. Where the orbit is eccentric, but always coincident with the ecliptic. (Parts I., V., VI.)

Now that these problems are solved, it would not be difficult, although laborious, to unite the two investigations into a single one; but the additional interest of the results would hardly repay one for the great labour, and besides this division of the problem makes the formulas considerably shorter, and this conduces to intelligibility.

For the present I only refer to the first of the above problems.

It appears that the problem requires still further subdivision, for the following reasons :—

It is a well-known result of the theory of perturbed elliptic motion, that the orbit of a satellite, revolving about an oblate planet and perturbed by a second satellite, always maintains a constant inclination to a certain plane, which is said to be *proper* to the orbit; the nodes also of the orbit revolve with a uniform motion on that plane, apart from "periodic" inequalities.

If then the moon's proper plane be inclined at a very small angle to the ecliptic, the nodes revolve very nearly uniformly on the ecliptic, and the orbit is inclined at very nearly a constant angle thereto. In this case the equinoctial line revolves also nearly uniformly, and the equator is inclined at nearly a constant angle to the ecliptic.

Here then any inequalities in the motion of the earth and moon, which depend on the longitudes of the nodes or of the equinoctial line, are harmonically periodic in time (although they are "secular inequalities"), and cannot lead to any cumulative effects which will alter the elements of the earth or moon.

Again, suppose that the moon and earth are the only bodies in existence. Here the axis of resultant moment of momentum of the system, or the normal to the invariable plane, remains fixed in space. The component moments of momentum are those of the earth's rotation, and of the moon's and earth's orbital revolution round their common centre of inertia. Hence the earth's axis and the normal to the lunar orbit must always be coplanar with the normal to the invariable plane, and therefore the orbit and equator must have a common node on the invariable plane. This node

revolves with a uniform precessional motion, and (so long as the earth is rigid) the inclinations of the orbit and equator to the invariable plane remain constant.

Here also inequalities, which depend on the longitude of the common node, are harmonically periodic in time, and can lead to no cumulative effects.

But if the lunar proper plane be not inclined at a small angle to the ecliptic, the nodes of the orbit may either revolve with much irregularity, or may oscillate about a mean position* on the ecliptic. In this case the inclinations of the orbit and equator to the ecliptic may oscillate considerably.

Here then inequalities, which depend on the longitudes of the node and of the equinoctial line, are not simply periodic in time, and may and will lead to cumulative effects.

This explains what was stated above, namely, that we cannot entirely ignore the motion of the two nodes.

Our problem is thus divisible into three cases:—

(i.) Where the nodes revolve uniformly on the ecliptic, and where there is a second disturbing satellite, viz.: the sun.

(ii.) Where the earth and moon are the only two bodies in existence.

(iii.) Where the nodes either oscillate, or do not revolve uniformly.

The cases (i.) and (ii.) are distinguished by our being able to ignore the nodes. They afford the subject matter for the whole of Part II.

It is proved in § 5 that the tides raised by any one satellite can produce directly no secular change in the mean distance of any other satellite. This is true for all three of the above cases.

It is also shown that, in cases (i.) and (ii.), the tides raised by any one satellite can produce directly no secular change in the inclination of the orbit of any other satellite to the plane of reference. This is not true for case (iii.).

The change of inclination of the moon's orbit in case (i.) is considered in § 6. The equation expressive of the rate of change of inclination is given in (61) and (62). In § 7 this is applied in the case where the earth is viscous. Fig. 4 illustrates the physical meaning of the equation, and the reader is referred to § 7 for an explanation of the figure. From this figure we learn that the effect of the frictional tides is in general to diminish the inclination of the lunar orbit to the ecliptic, unless the obliquity of the ecliptic be large, when the inclination will increase. The curves also show that for moderate viscosities the rate of decrease of inclination is most rapid when the obliquity of the ecliptic is zero, but for larger viscosities the rate of decrease has a maximum value, when the obliquity is between 30° and 40°.

If the viscosity be small the equation for the rate of decrease of inclination is reducible to a very simple form; this is given in (64) § 7.

In §§ 8, 9, is found the law of increase of the square root of the moon's distance from the earth under the influence of tidal reaction. The law differs but little from

* It is true that this mean position will itself have a slow precessional motion.

that found and discussed in the paper on "Precession," where the plane of the lunar orbit was supposed to be coincident with the ecliptic. If the viscosity be small the equation reduces to a very simple form; this is given in (70). In § 10 I pass to case (ii.), where the earth and moon are the only bodies. The equation expressive of the rate of change of inclination of the lunar orbit to the invariable plane is given in (71). Fig. 5 illustrates the physical meaning of the equation, and an explanation of it is given in § 10. From it we learn that the effect of the tides is always to cause a diminution of the inclination—at least so long as the periodic time of the satellite, as measured in rotations of the planet, is pretty long. The following considerations show that this must generally be the case. It appears from the paper on "Precession" that the effect of tidal friction is to cause a continual transference of moment of momentum from that of terrestrial rotation to that of orbital motion; hence it follows that the normal to the lunar orbit must continually approach the normal to the invariable plane. It is true that the rate of this approach will be to some extent counteracted by a parallel increase in the inclination of the earth's axis to the same normal. It will appear later that if the moon were to revolve very rapidly round the earth, and if the viscosity of the earth were great, then this counteracting influence might be sufficiently great to cause the inclination to increase.* This possible increase of inclination is not exhibited in fig. 5, because it illustrates the case where the sidereal month is 15 days long.

In § 11 it is shown that, for case (ii.), the rate of variation of the mean distance, obliquity, and terrestrial rotation follow the laws investigated in "Precession," but that the angle, there called the obliquity of the ecliptic, must be interpreted as the angle between the plane of the lunar orbit and the equator.

In § 12 I return again to case (i.) and find the laws governing the rate of increase of the obliquity of the ecliptic, and of decrease of the diurnal rotation of the earth. The results differ so little from those discussed in "Precession" that they need not be further referred to here.

Up to this point no approximation has been admitted with regard to smallness either in the obliquity or the inclination of the orbit, but mathematical difficulties have rendered it expedient to assume their smallness in the following part of the paper.

§ 31. *Summary of Part III.*

Part III. is devoted to case (iii.) of our first problem. It was found necessary in the first instance to consider the theory of the secular inequalities in the motion of a moon revolving about an oblate rigid earth, and perturbed by a second satellite, the sun. The sun being large and distant, the ecliptic is deemed sensibly unaffected, and is taken as the fixed plane of reference.

The proper plane of the lunar orbit has been already referred to, but I was here led

* See the abstract of this paper, Proc. R.S., No. 200, 1879, for certain general considerations bearing on this case.

to introduce a new conception, viz.: that of a second proper plane to which the motion of the earth is referred. It is proved that the motion of the system may then be defined as follows:—

The two proper planes intersect one another on the ecliptic, and their common node regredes on the ecliptic with a slow precessional motion. The lunar orbit and the equator are respectively inclined at constant angles to their proper planes, and their nodes on their respective planes also regrede uniformly and at the same speed. The motions are timed in such a way that when the inclination of the orbit to the ecliptic is at the maximum, the obliquity of the equator to the ecliptic is at the minimum, and *vice versâ*.

Now let us call the angular velocity with which the nodes of the orbit would regrede on the ecliptic, if the earth were spherical, *the nodal velocity*.

And let us call the angular velocity with which the common node of the orbit and equator would regrede on the invariable plane of the system, if the sun did not exist, *the precessional velocity*.

If the various obliquities and inclinations be not large, the precessional velocity is in fact the purely lunar precession.

Then if the nodal velocity be large compared with the precessional velocity, the lunar proper plane is inclined at a small angle to the ecliptic, and the equator is inclined at a small angle to the earth's proper plane.

This is the case with the earth, moon, and sun at present, because the nodal period is about $18\frac{1}{2}$ years, and the purely lunar precession would have a period of between 20,000 and 30,000 years. It is not usual to speak of a proper plane of the earth, because it is more simple to conceive a mean equator, about which the true equator nutates with a period of about $18\frac{1}{2}$ years.

Here the precessional motion of the two proper planes is the whole luni-solar precession, and the regression of the nodes on the proper planes is practically the same as the regression of the lunar nodes on the ecliptic.

A comparison of my result with the formula ordinarily given will be found at the end of § 13, and in a note to § 18.

Secondly, if the nodal velocity be small compared with the precessional velocity, the lunar proper plane is inclined at a small angle to the earth's proper plane.

Also the inclination of the equator to the earth's proper plane bears very nearly the same ratio to the inclination of the orbit to the moon's proper plane as the orbital moment of momentum of the two bodies bears to that of the rotation of the earth.

In the planets of the solar system, on account of the immense mass of the sun, the nodal velocity is never small compared with the precessional velocity, unless the satellite moves with a very short periodic time round its planet, or unless the satellite be very small; and if either of these be the case the ratio of the two moments of momentum is small.

Hence it follows that in our system, if the nodal velocity be small compared with

the precessional velocity, the proper plane of the satellite is inclined at a small angle to the equator of the planet. The rapidity of motion of the satellites of Mars, Jupiter, and of some of the satellites of Saturn, and their smallness compared with their planets, necessitates that their proper planes should be inclined at small angles to the equators of the planets. A system may, however, be conceived in which the two proper planes are inclined at a small angle to one another, but where the satellite's proper plane is not inclined at a small angle to the planet's equator.

In the case now before us the regression of the common node of the two proper planes is a sort of compound solar precession of the planet with its attendant moon, and the regression of the two nodes on their respective proper planes is very nearly the same as the purely lunar precession on the invariable plane of the system. Thus there are two precessions, the first of the system as a whole, and the second going on within the system, almost as though the external precession did not exist.

If the nodal velocity be of nearly equal speed with the precessional velocity, the regression of the proper planes and of the nodes on those planes are each a compound phenomenon, which it is rather hard to disentangle without the aid of analysis. Here none of the angles are necessarily small.

It appears from the investigation in "Precession" that the effect of tidal friction is that, on tracing the changes of the system backwards in time, we find the moon getting nearer and nearer to the earth. The result of this is that the ratio of the nodal velocity to the precessional velocity continually diminishes retrospectively; it is initially very large, it decreases, then becomes equal to unity, and finally is very small. Hence it follows that a retrospective solution will show us the lunar proper plane departing from its present close proximity to the ecliptic, and gradually passing over until it becomes inclined at a small angle to the earth's proper plane.

Therefore the problem, involved in the history of the obliquity of the ecliptic and in the inclination of the lunar orbit, is to trace the secular changes in the pair of proper planes, and in the inclinations of the orbit and equator to their respective proper planes.

The four angles involved in this system are however so inter-related, that it is only necessary to consider the inclination of one proper plane to the ecliptic, and of one plane of motion to its proper plane, and afterwards the other two may be deduced. I chose as the two, whose motions were to be traced, the inclination of the lunar orbit to its proper plane, and the inclination of the earth's proper plane to the ecliptic; and afterwards deduced the inclination of the moon's proper plane to the ecliptic, and the inclination of the equator to the earth's proper plane.

The next subject to be considered (§ 14 to end of Part III.) was the rate of change of these two inclinations, when both moon and sun raise frictional tides in the earth. The change takes place from two sets of causes :—

First because of the secular changes in the moon's distance and periodic time, and in the earth's rotation and ellipticity of figure—for the earth must always remain a figure of equilibrium.

The nodal velocity varies directly as the moon's periodic time, and it will decrease as we look backwards in time.

The precessional velocity varies directly as the ellipticity of the earth's figure (the earth being homogeneous) and inversely as the cube of the moon's distance, and inversely as the earth's diurnal rotation ; it will therefore increase retrospectively. The ratio of these two velocities is the quantity on which the position of the proper planes principally depends.

The *second* cause of disturbance is due directly to the tidal interaction of the three bodies.

The most prominent result of this interaction is, that the inclination of the lunar orbit to its proper plane in general diminishes as the time increases, or increases retrospectively. This statement may be compared with the results of Part II., where the ecliptic was in effect the proper plane. The retrospective increase of inclination may be reversed however, under special conditions of tidal disturbance and lunar periodic time.

Also the inclination of the earth's proper plane to the ecliptic in general increases with the time, or diminishes retrospectively. This is exemplified by the results of the paper on "Precession," where the obliquity of the ecliptic was found to diminish retrospectively. This retrospective decrease may be reversed under special conditions.

It is in determining the effects of this second set of causes, that we have to take account of the effects of tidal disturbance on the motions of the nodes of the orbit and equator on the ecliptic.

After a long analytical investigation, equations are found in (224), which give the rate of change of the positions of the proper planes, and of the inclinations thereto.

It is interesting to note how these equations degrade into those of case (i.) when the nodal velocity is very large compared with the precessional velocity, and into those of case (ii.) when the same ratio is very small.

In order completely to define the rate of change of the configuration of the system, there are two other equations, one of which gives the rate of increase of the square root of the moon's distance (which I called in a previous paper the equation of tidal reaction), and the other gives the rate of retardation of the earth's diurnal rotation (which I called before the equation of tidal friction). For the latter of these we may however substitute another equation, in which the time is not involved, and which gives a relationship between the diurnal rotation and the square root of the moon's distance. It is in fact the equation of conservation of moment of momentum of the moon-earth system, as modified by the solar tidal friction. This is the equation which was extensively used in the paper on "Precession."

Except for the solar tidal friction and for the obliquity of the orbit and equator, this equation would be rigorously independent of the kind of frictional tides existing in the earth. If the obliquities are taken as small, they do not enter in the equation, and in the present case the degree of viscosity of the earth only enters to an imperceptible

degree, at least when the day is not very nearly equal to the sidereal month. When that relation between the day and month is very nearly fulfilled, the equation may become largely affected by the viscosity; and I shall return to this point later, while for the present I shall assume the equation to give satisfactory results.

This equation of conservation of moment of momentum enables us to compute as many parallel values of the day and month as may be desired.

Now we have got the time-rates of change of the inclinations of the lunar orbit to its proper plane, and of the earth's proper plane to the ecliptic, and we have also the time-rate of change of the square root of the moon's distance. Hence we may obtain the square-root-of-moon's-distance-rate (or shortly the distance-rate) of change of the two inclinations.

The element of time is thus entirely eliminated; and as the period of time required for the changes has been adequately considered in the paper on "Precession," no further reference will here be made to time.

In a precisely similar manner the equations giving the time-rate in the cases (i.) and (ii.) of our first problem, may be replaced by equations of distance-rate.

Up to this point terrestrial phraseology has been used, but there is nothing which confines the applicability of the results to our own planet and satellite.

§ 32. *Summary of Part IV.*

We now, however, pass to Part IV., which contains a retrospective integration of the differential equations, with special reference to the earth, moon, and sun. The mathematical difficulties were so great that a numerical solution was the only one found practicable.* The computations made for the paper on "Precession" were used as far as possible.

The general plan followed was closely similar to that of the previous paper, and consists in arbitrarily choosing a number of values for the distance of the moon from the earth (or what amounts to the same thing for the sidereal month), and then computing all the other elements of the system by the method of quadratures.

The first case considered is where the earth has a small viscosity. And here it may be remarked that although the solution is only rigorous for infinitely small viscosity, yet it gives results which are very nearly true over a considerable range of viscosity. This may be seen to be true by a comparison of the results of the integrations in §§ 15 and 17 of "Precession," in the first of which the viscosity was not at all small; also by observing that the curves in fig. 2 of "Precession" do not differ materially from the curve of sines until ϵ (the f of this paper) is greater than 25°; also by noting a similar peculiarity in figs. 4 and 5 of this paper. The hypothesis of large viscosity does not cover nearly so wide a field.

* An analytical solution in the case of a single satellite, where the viscosity of the planet is small, is given in Proc. Roy. Soc., No. 202, 1880.

That which we here call a small viscosity is, when estimated by terrestrial standards, very great (see the summary of "Precession").

To return, however, to the case in hand :—We begin with the present configuration of the three bodies, when the moon's proper plane is almost identical with the ecliptic, and when the inclination of the equator to its proper plane is very small. This is the case (i.) of the first problem :—

It appears that the solution of "Precession" is sufficiently accurate for this stage of the solution, and accordingly the parallel values of the day, month, and obliquity of the earth's proper plane (or mean equator) are taken from § 17 of that paper; but the change in the new element, the inclination of the lunar orbit, has to be computed.

The results of the solution are given in Table I., § 18, to which the reader is referred.

This method of solution is not applicable unless the lunar proper plane is inclined at a small angle to the ecliptic, and unless the equator is inclined at a small angle to its proper plane. Now at the beginning of the integration, that is to say with a homogenous earth, and with the moon and sun in their present configuration, the moon's proper plane is inclined to the ecliptic at 13″, and the equator is inclined to the earth's proper plane at 12″ (for the heterogeneous earth these angles are about 8″·3 and 9″·0); and at the end of this integration, when the day is 9 hrs. 55 m. and the month 8·17 m. s. days, the former angle has increased to 57′ 31″, and the latter to 22′ 42″. These last results show that the nutations of the system have already become considerable, and although subsequent considerations show that this method of solution has not been overstrained, yet it here becomes advisable to carry out the solution into the more remote past by the methods of Part III.

It was desirable to postpone the transition as long as possible, because the method used up to this point does not postulate the smallness of the inclinations, whereas the subsequent procedure does make that supposition.

In § 19 the solution is continued by the new method, the viscosity of the earth still being supposed to be small. After laborious computations results are obtained, the physical meaning of which is embodied in Table VIII. The last two columns give the periods of the two precessional motions by which the system is affected. The precession of the pair of proper planes is, as it were, the ancestor of the actual luni-solar precession, and the revolution of the two nodes on their proper planes is the ancestor of the present revolution of the lunar nodes on the ecliptic, and of the 19-yearly nutation of the earth's axis.

This table exhibits a continued approach of the two proper planes to one another, so that at the point where the integration is stopped they are only separated by 1° 18′; at the present time they are of course separated by 23° 28′.

The most remarkable feature in this table is that (speaking retrospectively) the inclination of the lunar orbit to its proper plane first increases, then diminishes, and then increases again.

If it were desired to carry the solution still further back, we might without much

error here make the transition to the method of case (ii.) of the first problem, and neglecting the solar influence entirely, refer the motion to the invariable plane of the moon-earth system. This invariable plane would have to be taken as somewhere between the two proper planes, and therefore inclined to the ecliptic at about 11° 45′; the invariable plane would then really continue to have a precessional motion due to the solar influence on the system formed by the earth and moon together, but this would not much affect the treatment of the plane as though it were fixed in space.

We should then have to take the obliquity of the equator to the invariable plane as about 8°, and the inclination of the lunar orbit to the same plane as about 5° 30′.

In the more remote past the obliquity of the equator to the invariable plane would go on diminishing, but at a slower and slower rate, until the moon's period is 12 hours and the day is 6 hours, when it would no longer diminish; and the inclination of the orbit to the invariable plane would go on increasing, until the day and month come to an identity, and at an ever increasing rate.

It follows from this, that if we continued to trace the changes backwards, until the day and month are identical, we should find the lunar orbit inclined at a considerable angle to the equator. If this were necessarily the case, it would be difficult to believe that the moon is a portion of the primeval planet detached by rapid rotation, or by other causes. But the previous results are based on the hypothesis that the viscosity of the earth is small, and it therefore now became important to consider how a different hypothesis concerning the constitution of the earth might modify the results.

In § 20 the solution of the problem is resumed, at the point where the methods of Part III. were first applied, but with the hypothesis that the viscosity of the earth is very large, instead of very small. The results for any intermediate degree of viscosity must certainly lie between those found before and those to be found now.

Then having retraversed the same ground, but with the new hypothesis, I found the results given in Table XV.

The inclinations of the two proper planes to the ecliptic are found to be very nearly the same as in the case of small viscosity. But the inclination of the lunar orbit to its proper plane increases at first and then continues diminishing, without the subsequent reversal of motion found in the previous solution.

If the solution were carried back into the more remote past, the motion being referred to the invariable plane, we should find both the obliquity of the equator and the inclination of the orbit diminishing at a rate which tends to become *infinite*, if the viscosity is *infinitely* great. Infinite viscosity is of course the same as perfect rigidity, and if the earth were perfectly rigid the system would not change at all. The true interpretation to put on this result is that the rate of change of inclination becomes large, if the viscosity be large. This diminution would continue until the day was 6 hours and the month 12 hours. For an analysis of the state of things further back than this, the reader is referred to § 20.

From this it follows, that by supposing the viscosity large enough we may make the obliquity and inclination to the invariable plane as small as we please, by the time that state is reached in which the month is equal to twice the day.

Hence, on the present hypothesis, we trace the system back until the lunar orbit is sensibly coincident with the equator, and the equator is inclined to the ecliptic at an angle of 11° or 12°.

It is probable that in the still more remote past the plane of the lunar orbit would not have a tendency to depart from that of the equator. It is not, however, expedient to attempt any detailed analysis of the changes further back, for the following reason. Suppose a system to be unstable, and that some infinitesimal disturbance causes the equilibrium to break down; then after some time it is moving in a certain way. Now suppose that from a knowledge of the system we endeavour to compute backwards from the observed mode of its motion at that time, and so find the condition from which the observed state of motion originated. Then our solution will carry us back to a state very near to that of instability, from which the system really departed, but as the calculation can take no account of the infinitesimal disturbance, which caused the equilibrium to break down, it can never bring us back to the state which the system really had. And if we go on computing the preceding state of affairs, the solution will continue to lead us further and further astray from the truth. Now this, I take it, is likely to have been the case with the earth and moon; at a certain period in the evolution (viz.: when the month was twice the day) the system probably became dynamically unstable, and the equilibrium broke down. Thus it seems more likely that we have got to the truth, if we cease the solution at the point where the lunar orbit is nearly coincident with the equator, than by going still further back.

In § 21, fig. 7, is given a graphical illustration of the distance-rate of change in the inclinations of the lunar orbit to its proper plane, and of the earth's proper plane to the ecliptic; the dotted curves refer to the hypothesis of large viscosity, and the firm-curves to that of small viscosity.

The figure is explained and discussed in that section; I will here only draw attention to the wideness apart of the two curves illustrative of the rate of change of the inclination of the lunar orbit. This shows how much influence the degree of viscosity of the earth must have had on the present inclination of the lunar orbit to the ecliptic.

It is particularly interesting to observe that in the case of small viscosity this curve rises above the horizontal axis. If this figure is to be interpreted retrospectively, along with our solution, it must be read from left to right, but if we go with the time, instead of against it, from right to left.

Now if the earth had had in its earlier history infinitely small viscosity, and if the moon had moved primitively in the equator, then until the evolution had reached the point represented by P, the lunar orbit would have always remained sensibly coincident with its proper plane. Then in passing from P to Q the inclination of the

orbit to its proper plane would have increased, but the whole increase could not have amounted to more than a few minutes of arc. At the point P the day is 7 hrs. 47 m. in length, and the month 3·25 m. s. days in length; at the point Q the day is 8 hrs. 36 m., and the month 5·20 m. s. days. From Q down to the present state this small inclination would have always decreased.

If then the earth had had small viscosity throughout its evolution, the lunar orbit would at present be only inclined at a very small angle to the ecliptic. But it is actually inclined at about 5° 9′, hence it follows that while the hypothesis of small viscosity is competent to explain *some* inclination, it cannot explain the actually existing inclination.

It was shown in the papers on "Tides" and "Precession" that, if the earth be not at present perfectly rigid or perfectly elastic, its viscosity must be very large. And it was shown in "Precession" that if the viscosity be large, the obliquity of the ecliptic must at present be decreasing. Now it will be observed that in resuming the integration with the hypothesis of large viscosity, the solution of the first method with the hypothesis of small viscosity was accepted as the basis for continuing the integration with large viscosity. This appears at first sight somewhat illogical, and to be strictly correct, we ought to have taken as the initial inclination of the earth's proper plane to the ecliptic, at the beginning of the application of the methods of Part III. to the hypothesis of large viscosity, some angle probably a little less than 23½°* instead of 17°. This would certainly disturb the results, but I have not thought it advisable to take this course for the following reasons.

It is probable that at the present time the greater part, if not the whole of the tidal friction is due to oceanic tides, and not to bodily tides. If the ocean were frictionless, it would be low tide under the moon; consequently the effects of fluid friction must be to accelerate, not retard, the ocean tides.† Then in order to apply our present analysis to the case of oceanic tidal friction, that angle which has been called the lag of the tide must be interpreted as the acceleration of the tide.

We know that the actual friction in water is small, and hence the tides of long period will be less affected by friction than those of short period; thus the effects of fluid tidal friction will probably be closely analogous to those resulting from the hypothesis of small viscosity of the whole earth and bodily tides. On the other hand, it is probable that the earth was once more plastic than at present, either superficially or throughout its mass, and therefore it seems probable that the bodily tides, even if small at present, were once more considerable. I think therefore that on the whole

* In the present configuration of the earth, moon, and sun, the obliquity will decrease, if the viscosity be very large. But if we integrate backwards this retrospective increase of obliquity would soon be converted into a decrease. Thus at the end of "the first period of integration," the obliquity would be a little greater than 23½°, but by the end of the "second period" it would probably be a little less than 23½°. It is at the end of the "second period" that the method of Part III. is first applied.

† Otherwise the lunar attraction on the tides would accelerate the earth's rotation—a clear violation of the principles of energy.

we shall be more nearly correct in supposing that the terrestrial nucleus possessed a high degree of stiffness in the earliest times, and that it will be best to apply the hypothesis of small viscosity to the more modern stages of the evolution, and that of large viscosity to the more ancient.

At any rate this appears to be a not improbable theory, and one which accords very well with the present values of the obliquity of the ecliptic, and of the inclination of the lunar orbit.

§ 33. *On the initial condition of the earth and moon.*

It was remarked above that the equation of conservation of moment of momentum, as modified by the effects of solar tidal friction, could only be regarded as practically independent of the degree of viscosity of the earth, so long as the moon's sidereal period was not nearly equal to the day; and that if this relationship were nearly satisfied, the equation which we have used throughout might be considerably in error.

Now in the paper on "Precession" the system was traced backwards, in much the same way as has been done here, until the moon's tide-generating influence was very large compared with that of the sun; the solar influence was then entirely neglected, and the equation of conservation of moment of momentum was used for determining that initial condition, where the month and day were identical, from which the system started its course of development.* The period of revolution of the system in its initial configuration was found to be about $5\frac{1}{2}$ hours. I now however see reason to believe that the solar tidal friction will make the numerical value assigned to this period of revolution considerably in error, whilst the general principle remains almost unaffected. This subject is considered in § 22.

The necessity of correction arises from the assumption that because the moon is retrospectively getting nearer and nearer to the earth, therefore the effects of lunar tidal friction must more and more preponderate over those of solar tidal friction, so that if the solar tidal friction were once negligeable it would always remain so. But tidal friction depends on two elements, viz.: the magnitude of the tide-generating influence, and the relative motion of the two bodies. Now whilst the tide-generating influence of the moon *does* become larger and larger, as we approach the critical state, yet the relative motion of the moon and earth becomes smaller and smaller; on the other hand the tide-generating influence of the sun remains sensibly constant, whilst the relative motion of the earth and sun slightly increases.†

From this it follows that the solar tidal friction must ultimately become actually more important than the lunar, notwithstanding the close proximity of the moon to the earth.

* See also a paper on "The Determination of the Secular Effects of Tidal Friction by a Graphical Method," Proc. Roy. Soc., No. 197, 1879.

† In the paper on "Precession" it was stated in § 18 that this must be the case, but I did not at that time perceive the importance of this consideration

The complete investigation of this subject involves considerations which will require special treatment. In § 22 it is only so far considered as to show that, when there is identity of the periods of revolution of the moon and earth, the angular velocity of the system must be much greater than that given by the solution in § 18 of "Precession."

When the earth rotates in $5\frac{1}{2}$ hours, the motion of the moon relatively to the earth's surface would already be pretty slow. If the system were traced into the more remote past, the earth's rotation would be found getting more and more rapid, and the moon's orbital angular velocity also continually increasing, but ever approximating to identity with the earth's rotation.

When the surfaces of the two bodies are almost in contact, the motion of the moon relatively to the earth's surface would be almost insensible. This appears to point to the break-up of the primeval planet into two parts, in consequence of a rotation so rapid as to be inconsistent with an ellipsoidal form of equilibrium.

Is it then a mere coincidence that the shortest period of revolution, with which a spheroid of the same mean density as the earth could subsist in the ellipsoidal form, is 2 hrs. 24 m.; whilst if KEPLER's law were to hold true, and if the moon were to revolve round the earth in the same period, the surfaces of the two bodies would just graze one another?

§ 34. *Summary of Parts V. and VI.*

I now come to the second of the two problems, where the moon moves in an eccentric orbit, always coincident with the ecliptic.

In § 23 it is shown that the tides raised by any one satellite can produce no secular change in the eccentricity of the orbit of any other satellite; thus the eccentricity and the mean distance are in this respect on the same footing.

It was found to be more convenient to consider the ellipticity of the orbit instead of the eccentricity. In § 24 (289) and (290), are given the time-rates of increase of the ellipticity and of the square root of mean distance. In § 25 the result for the ellipticity is applied to the case where the earth is viscous, and its physical meaning is graphically illustrated in fig. 8.

This figure shows that in general the ellipticity will increase with the time; but if the obliquity of the ecliptic be nearly 90°, or if the viscosity be so great that the earth is very nearly rigid, the ellipticity will diminish. This last result is due to the rising into prominence of the effects of the elliptic monthly tide.

If the viscosity be very small the equation is reducible to a very simple form, which is given in (291). From (291) we see that if the obliquity of the ecliptic be zero, the ellipticity will either increase or diminish, according as 18 rotations of the planet take a shorter or a longer time than 11 revolutions of the satellite. From this it follows that in the history of a satellite revolving about a planet of small viscosity, the circular orbit is dynamically stable until 11 months of the satellite have become longer than

18 days of the planet. Since the day and month start from equality and end in equality, it follows that the eccentricity will rise to a maximum and ultimately diminish again.

It is also shown that if a satellite be started to move in a circular orbit with the same periodic time as that of the planet's rotation (with maximum energy for given moment of momentum), then if infinitesimal eccentricity be given to the orbit the satellite will ultimately fall into the planet; and if, the orbit being circular, infinitesimal decrease of distance be given the satellite will fall in, whilst if infinitesimal increase of distance be given the satellite will recede from the planet. Thus this configuration, in which the planet and satellite move as parts of a single rigid body, has a complex instability; for there are two sorts of disturbance which cause the satellite to fall in, and one which causes it to recede from the planet.*

If the planet have very large viscosity the case is much more complex, and it is examined in detail in § 25.

It will here only be stated that the eccentricity will diminish if 2 months of the satellite be longer than 3 days of the planet, but will increase if the 2 months be shorter than 3 days; also the rate of increase of eccentricity tends to become infinite, for infinitely great viscosity, if the 2 months are equal to the 3 days.

These results are largely due to the influence of the elliptic monthly tide, and with most of the satellites of the solar system, this is a very slow tide compared with the semi-diurnal tides; therefore it must in general be supposed that the viscosity of the planet makes a close approximation to perfect rigidity, in order that this statement may be true.

The infinite value of the rate of change of eccentricity is due to the speed of the slower elliptic semi-diurnal tide being infinitely slow, when 2 months are equal to 3 days. The result is physically absurd, and its true meaning is commented on in § 25.

In § 26 the time-rate of change of the obliquity of the planet's equator, and of the diurnal rotation is investigated, when the orbits of the tide-raising satellites are eccentric; the only point of general interest in the result is, that the rate of change of obliquity and the tidal friction are both augmented by the eccentricity of the orbit, as was foreseen in the paper on "Precession."

In § 27 it is stated that the effect of the evectional tides is such as to diminish the eccentricity of the orbit, but the formula given shows that the effect cannot have much importance, unless the moon be very distant from the earth.

* Added July, 1880.—This passage appeared to the referee, requested by the R. S. to report on this paper, to be rather obscure, and it has therefore been somewhat modified. To further elucidate the point I have added in an appendix a graphical illustration of the effects of eccentricity, similar to those given in No. 197 of Proc. Roy. Soc., 1879.

See also the abstract of this paper in the Proc. Roy. Soc., No. 200, 1879, for certain general considerations bearing on the problem of the eccentricity.

In Part VI. the equations giving the rate of change of eccentricity are integrated, on the hypothesis that the earth has small viscosity.

The first step is to convert the time-rates of change into distance-rates, and thus to eliminate the time, as in the previous integrations.

The computations made for the paper on "Precession" were here made use of, as far as possible.

The results of the retrospective integration are given in Table XVI., § 28. This table exhibits the eccentricity falling from its present value of $\frac{1}{18}$th down to about $\frac{1}{1000}$th, so that at the end the orbit is very nearly circular.

The integration in the case of large viscosity is not carried out, because the actual degree of viscosity will exercise so very large an influence on the result.

If the viscosity were *infinitely* large, we should find the eccentricity getting larger and larger retrospectively, and ultimately becoming *infinite*, when 2 months were equal to 3 days. This result is of course absurd, and merely represents that the larger the viscosity, the larger would be the eccentricity. On the other hand, if the viscosity were merely large, we might find the eccentricity decreasing at first, then stationary, then increasing until 2 months were equal to 3 days, and then decreasing again.

It follows therefore that various interpretations may be put to the present eccentricity of the lunar orbit.

If, as is not improbable, the more recent changes in the configuration of our system have been chiefly brought about by oceanic tidal friction, whilst the earlier changes were due to bodily tidal friction, with considerable viscosity of the planet, then, supposing the orbit to have been primevally circular, the history of the eccentricity must have been as follows : first an increase to a maximum, then a decrease to a minimum, and finally an increase to the present value. There seems nothing to tell us how large the early maximum, or how small the subsequent minimum of eccentricity may have been.

VIII.

REVIEW OF THE TIDAL THEORY OF EVOLUTION AS APPLIED TO THE EARTH AND THE OTHER MEMBERS OF THE SOLAR SYSTEM.

I will now collect the various results so as to form a sketch of what the previous investigations show as the most probable history of the earth and moon, and in order to indicate how far this history is the result of calculation, references will be given to the parts of my several papers in which each point is especially considered.

We begin with a planet, not very much more than 8,000 miles in diameter,* and probably partly solid, partly fluid, and partly gaseous. This planet is rotating about

* "Precession," § 24.

an axis inclined at about 11° or 12° to the normal to the ecliptic,* with a period of from 2 to 4 hours,† and is revolving about the sun with a period not very much shorter than our present year.‡

The rapidity of the planet's rotation causes so great a compression of its figure that it cannot continue to exist in an ellipsoidal form§ with stability; or else it is so nearly unstable that complete instability is induced by the solar tides.‖

The planet then separates into two masses, the larger being the earth and the smaller the moon. I do not attempt to define the mode of separation, or to say whether the moon was initially more or less annular. At any rate it must be assumed that the smaller mass became more or less conglomerated, and finally fused into a spheroid—perhaps in consequence of impacts between its constituent meteorites, which were once part of the primeval planet. Up to this point the history is largely speculative, for although the limiting ellipticity of form of a rotating mass of fluid is known, yet the conditions of its stability, and *à fortiori* of its rupture, have not as yet been investigated.

We now have the earth and the moon nearly in contact with one another, and rotating nearly as though they were parts of one rigid body.

This is the system which has been made the subject of the present dynamical investigation.

As the two masses are not rigid, the attraction of each distorts the other; and if they do not move rigorously with the same periodic time, each raises a tide in the other. Also the sun raises tides in both.

In consequence of the frictional resistance to these tidal motions, such a system is dynamically unstable.¶ If the moon had moved orbitally a little faster than the earth rotates she must have fallen back into the earth; thus the existence of the moon compels us to believe that the equilibrium broke down by the moon revolving orbitally a little slower than the earth rotates. Perhaps the actual rupture into two masses was the cause of this slower motion; for if the detached mass retained the same moment of momentum as it had initially, when it formed a part of the primeval planet, this would, I think, necessarily be the case.

In consequence of the tidal friction the periodic time of the moon (or the month) increases in length, and that of the earth's rotation (or the day) also increases; but the month increases in length at a much greater rate than the day.

* This at least appears to be the obliquity at the earliest stage to which the system has been traced back in detail, but the effect of solar tidal friction would make the obliquity primevally less than this, to an uncertain and perhaps considerable amount.

† "Precession," § 18, and Part IV., § 22.

‡ "Precession," § 19.

§ "Precession," § 18, and Part IV., § 22.

‖ Summary of "Precession."

¶ "Secular Effects," &c., Proc. Roy. Soc., 197, 1879; and "Precession," § 18.

At some early stage in the history of the system, the moon has conglomerated into a spheroidal form, and has acquired a rotation about an axis nearly parallel with that of the earth. We will now follow the moon itself for a time.

The axial rotation of the moon is retarded by the attraction of the earth on the tides raised in the moon, and this retardation takes place at a far greater rate than the similar retardation of the earth's rotation.* As soon as the moon rotates round her axis with twice the angular velocity with which she revolves in her orbit, the position of her axis of rotation (parallel with the earth's axis) becomes dynamically unstable.† The obliquity of the lunar equator to the plane of the orbit increases, attains a maximum, and then diminishes. Meanwhile the lunar axial rotation is being reduced towards identity with the orbital motion.

Finally her equator is nearly coincident with the plane of her orbit, and the attraction of the earth on a tide, which degenerates into a permanent ellipticity of the lunar equator, causes her always to show the same face to the earth.‡ LAPLACE has shown that this is a necessary consequence of the elliptic form of the lunar equator.

All this must have taken place early in the history of the earth, to which I now return.

As the month increases in length the lunar orbit becomes eccentric, and the eccentricity reaches a maximum when the month occupies about a rotation and a half of the earth. The maximum of eccentricity is probably not large. After this the eccentricity diminishes.§

The plane of the lunar orbit is at first practically identical with the earth's equator, but as the moon recedes from the earth the sun's attraction begins to make itself felt. Here then we must introduce the conception of the two ideal planes (here called the proper planes), to which the motion of the earth and moon must be referred.‖ The lunar proper plane is at first inclined at a very small angle to the earth's proper plane, and the orbit and equator coincide with their respective proper planes.

As soon as the earth rotates with twice the angular velocity with which the moon revolves in her orbit, a new instability sets in. The month is then about 12 of our present hours, and the day is about 6 of our present hours in length.

The inclinations of the lunar orbit and of the equator to their respective proper planes

* "Precession," § 23.

† "Precession," § 17. It is of course possible that the lunar rotation was very rapidly reduced by the earth's attraction on the lagging tides, and was never permitted to be more than twice the orbital motion. In this case the lunar equator has never deviated much from the plane of the orbit.

‡ HELMHOLTZ, I believe, first suggested the reduction of the moon's axial rotation by means of tidal friction.

§ Parts V. and VI. The exact history of the eccentricity is somewhat uncertain, because of the uncertainty as to the degree of viscosity of the earth.

‖ See Parts III. and IV. (and the summaries thereof in Part VII.) for this and what follows about proper planes.

increase. The inclination of the lunar orbit to its proper plane increases to a maximum of 6° or 7°,* and ever after diminishes; the inclination of the equator to its proper plane increases to a maximum of about 2° 45′,† and ever after diminishes. The maximum inclination of the lunar orbit to its proper plane takes place when the day is a little less than 9 of our present hours, and the month a little less than 6 of our present days. The maximum inclination of the equator to its proper plane takes place earlier than this.

Whilst these changes have been going on, the proper planes have been themselves changing in their positions relatively to one another and to the ecliptic. At first they were nearly coincident with one another and with the earth's equator, but they then open out, and the inclination of the lunar proper plane to the ecliptic continually diminishes, whilst that of the terrestrial proper plane continually increases.

At some stage the earth has become more rigid, and oceans have been formed, so that it is probable that oceanic tidal friction has come to play a more important part than bodily tidal friction.‡ If this be the case the eccentricity of the orbit, after passing through a stationary phase, begins to increase again.

We have now traced the system to a state in which the day and month are increasing, but at unequal rates; the inclination of the lunar proper plane to the ecliptic and of the orbit to its proper plane are diminishing; the inclination of the terrestrial proper plane to the ecliptic is increasing, and of the equator to its proper plane is diminishing; and the eccentricity of the orbit is increasing.

No new phase now supervenes,§ and at length we have the system in its present configuration. The minimum time in which the changes from first to last can have taken place is 54,000,000 years.‖

In a previous paper it was shown that there are other collateral results of the viscosity of the earth; for during this course of evolution the earth's mass must have suffered a screwing motion, so that the polar regions have travelled a little from west to east relatively to the equator. This affords a possible explanation of the north and south trend of our great continents.¶ Also a large amount of heat has been generated by friction deep down in the earth, and some very small part of the observed increase of temperature in underground borings may be attributable to this cause.**

* Table XV., Part IV.

† Found from the values in Table XV., and by a graphical construction.

‡ Compare with "Precession," § 14, where the present secular acceleration of the moon's mean motion is considered.

§ Unless the earth's proper plane (or mean equator) be now slowly diminishing in obliquity, as would be the case if the bodily tides are more potent than the oceanic ones. In any case this diminution must ultimately take place in the far future.

‖ "Precession," end of § 18.

¶ "Problems," Part I.

** "Problems," Part II.

The preceding history might vary a little in detail, according to the degree of viscosity which we attribute to the earth's mass, and according as oceanic tidal friction is or is not, now and in the more recent past, a more powerful cause of change than bodily tidal friction.

The argument reposes on the imperfect rigidity of solids, and on the internal friction of semi-solids and fluids; these are *veræ causæ*. Thus changes of the kind here discussed must be going on, and must have gone on in the past. And for this history of the earth and moon to be true throughout, it is only necessary to postulate a sufficient lapse of time, and that there is not enough matter diffused through space to materially resist the motions of the moon and earth in perhaps several hundred million years.

It hardly seems too much to say that granting these two postulates, and the existence of a primeval planet, such as that above described, then a system would necessarily be developed which would bear a strong resemblance to our own.

A theory, reposing on *veræ causæ*, which brings into quantitative correlation the lengths of the present day and month, the obliquity of the ecliptic, and the inclination and eccentricity of the lunar orbit, must, I think, have strong claims to acceptance.

But if this has been the evolution of the earth and moon, then a similar process must have been going on elsewhere. The present investigation has only dealt with a single satellite and the sun, but the theory may of course be extended, with some modification, to planets attended by several satellites. I will now therefore consider some of the other members of the solar system.

A large planet has much more energy of rotation to be destroyed, and moment of momentum to be redistributed than a small one, and therefore a large planet ought to proceed in its evolution more slowly than a small one. Therefore we ought to find the larger planets less advanced than the smaller ones.

The masses of such of the planets as have satellites are, in terms of the earth's mass, as follows: Mars $=\frac{1}{10}$; Jupiter $=301$; Saturn $=90$; Uranus $=14$; Neptune $=16$.

Mars should therefore be furthest advanced in its evolution, and it is here alone in the whole system that we find a satellite moving orbitally faster than the planet rotates. This will also be the ultimate fate of our moon, because, after the moon's orbital motion has been reduced to identity with that of the earth's rotation, solar tidal friction will further reduce the earth's angular velocity, the tidal reaction on the moon will be reversed, and the moon's orbital velocity will increase, and her distance from the earth will diminish. But since the moon's mass is very large, the moon must recede to an enormous distance from the earth, before this reversal will take place. Now the satellites of Mars are very small, and therefore they need only to recede a short distance from the planet before the reversal of tidal reaction.*

* In the graphical method of treating the subject, "the line of momentum" will only just intersect "the curve of rigidity." See Proc. Roy. Soc., No. 197, 1879.

The periodic time of the satellite Deimos is 30 hrs. 18 m.,* and as the period of rotation of Mars is 24hrs. 37m.,† Deimos must be still receding from Mars, but very slowly.

The periodic time of the satellite Phobos in 7 hrs. 39 m.; therefore Phobos must be approaching Mars. It does not seem likely that it has ever been remote from the planet.

The eccentricities of the orbits of both satellites are small, though somewhat uncertain. The eccentricity of the orbit of Phobos appears however to be the larger of the two.

If the viscosity of the planet be small, or if oceanic tidal friction be the principal cause of change, both eccentricities are diminishing; but if the viscosity be large, both are increasing. In any case the rate of change must be excessively slow. As we have no means of knowing whether the eccentricities are increasing or diminishing this larger eccentricity of the orbit of Phobos cannot be a fact of much importance either for or against the present views. But it must be admitted that it is a slightly unfavourable indication.

The position of the proper plane of a satellite is determined by the periodic time of the satellite, the oblateness of the planet, and the sun's distance. The inclination of the orbit of a satellite to its proper plane is not determined by anything in the system. Hence it is only the inclination of the orbit which can afford any argument for or against the theory.

The proper planes of both satellites are necessarily nearly coincident with the equator of the planet; but it is in accordance with the theory that the inclinations of the orbits to their respective proper planes should be small.‡

Any change in the obliquity of the equator of Mars to the plane of his orbit must be entirely due to solar tides. The present obliquity is about 27°, and this points also to an advanced stage of evolution—at least if the axis of the planet was primitively at all nearly perpendicular to the ecliptic.

We now come to the system of Jupiter.

This enormous planet is still rotating in about 10 hours, its axis is nearly perpendicular to the ecliptic, and three of its satellites revolve in 7 days or less, whilst the fourth has a period of 16 days 16 hrs. This system is obviously far less advanced than our own.

The inclinations of the proper planes to Jupiter's equator are necessarily small, but

* 'Observations and Orbits of the Satellites of Mars,' by ASAPH HALL. Washington Government Printing Office, 1878.

† According to KAISER, as quoted by SCHMIDT. 'Ast. Nach.,' vol. 82, p. 333.

‡ For the details of the Martian system, see the paper by Professor ASAPH HALL, above quoted.

With regard to the proper planes, see a paper by Prof. J. C. ADAMS read before the R. Ast. Soc. on Nov. 14, 1879, R. A. S. Month. Not. There is also a paper by Mr. MARTH, 'Ast. Nach.,' No. 2280, vol. 95, Oct., 1879.

the inclinations of the orbits to the proper planes appear to be very interesting from a theoretical point of view. They are as follows :—*

Satellite.	Inclination of orbit to proper plane. ° ′ ″
First	0 0 0
Second	0 27 50
Third	0 12 20
Fourth	0 14 58

Now we have shown above that the orbit of a satellite is at first coincident with its proper plane, that the inclination afterwards rises to a maximum, and finally declines. If then we may assume, as seems reasonable, that the satellites are in stages of evolution corresponding to their distances from the planet, these inclinations accord well with the theory.

The eccentricities of the orbits of the two inner satellites are insensible, those of the outer two small. This does not tell strongly either for or against the theory, because the history of the eccentricity depends considerably on the degree of viscosity of the planet; yet it on the whole agrees with the theory that the eccentricity should be greater in the more remote satellites. It appears that the satellites of Jupiter always present the same face to the planet, just as does our moon.† This was to be expected.

The case of Saturn is not altogether so favourable to the theory. The extremely rapid rotation, the ring, and the short periodic time of the inner satellites point to an early stage of development; whilst the longer periodic time of the three outer satellites, and the high obliquity of the equator indicate a later stage. Perhaps both views may be more or less correct, for successive shedding of satellites would impart a modern appearance to the system. It may be hoped that the investigation of the effects of tidal friction in a planet surrounded by a number of satellites may throw some light on the subject. This I have not yet undertaken, and it appears to have peculiar difficulties. It has probably been previously remarked, that the Saturnian system bears a strong analogy with the solar system, Titan being analgous to Jupiter, Hyperion and Iapetus to Uranus and Neptune, and the inner satellites being analogous to the inner planets. Thus anything which aids us in forming a theory of the one system will throw light on the other.‡

The details of the Saturnian system seem more or less favourable to the theory.

The proper planes of the orbits (except that of Iapetus) are nearly in the plane of the ring, and the inclinations of all the orbits to their proper planes appear not to be large.

* HERSCHEL's 'Astron.' Synoptic Tables in appendix.

† HERSCHEL's 'Astron.' 9th ed., § 546.

‡ An investigation, now (September, 1880) almost completed, seems to show pretty conclusively that tidal friction cannot be in all cases the most important feature in the evolution of such systems as that of Saturn and his satellites, and the solar system itself. I am not however led to reject the views maintained in this paper.

HERSCHEL gives the following eccentricities of orbit:—

Tethys ·04 (?), Dione ·02 (?), Rhea ·02 (?), Titan ·029314, Hyperion "rather large; and he says nothing of the eccentricities of the orbits of the remaining three satellites. If the dubious eccentricities for the first three of the above are of any value, we seem to have some indication of the early maximum of eccentricity to which the analysis points; but perhaps this is pushing the argument too far. The satellite Iapetus appears always to present the same face to the planet.*

Concerning Uranus and Neptune there is not much to be said, as their systems are very little known; but their masses are much larger than that of the earth, and their satellites revolve with a short periodic time. The retrogade motion and high inclination of the satellites of Uranus are, if thoroughly established, very remarkable.

The above theory of the inclination of the orbit has been based on an assumed smallness of inclination, and it is not very easy to see to what results investigation might lead, if the inclination were large. It must be admitted however that the Uranian system points to the possibility of the existence of a primitive planet, with either retrograde rotation, or at least with a very large obliquity of equator.

It appears from this review that the other members of the solar system present some phenomena which are strikingly favourable to the tidal theory of evolution, and none which are absolutely condemnatory. Perhaps by further investigations some light may be thrown on points which remain obscure.

APPENDIX.

(Added July, 1880.)

A graphical illustration of the effects of tidal friction when the orbit of the satellite is eccentric.

In a previous paper (Proc. Roy. Soc., No. 197, 1879†) a graphical illustration of the effects of tidal friction was given for the case of a circular orbit. As this method makes the subject more easily intelligible than the purely analytical method of the present paper, I propose to add an illustration for the case of the eccentric orbit.

Consider the case of a single satellite, treated as a particle, moving in an elliptic orbit, which is co-planar with the equator of the planet.

Let Ch be the resultant moment of momentum of the system. Then with the notation of the present paper, by § 27 the equation of conservation of moment of momentum is

$$n+\frac{\xi}{k}(1-\eta)=h$$

* HERSCHEL'S 'Astron.' 9th ed., § 547.

† The last sentence of this paper contains an erroneous statement; the line of zero eccentricity on the energy surface is not a ridge as there stated. See the figure on p. 890.

Here $\mathrm{C}n$ is the moment of momentum of the planet's rotation, and $\mathrm{C}\xi(1-\eta)/k$ is the moment of momentum of the orbital motion; and the whole moment of momentum is the sum of the two.

By the definitions of ξ and k in § 2, $\mathrm{C}\frac{\xi}{k}=\frac{\mu Mm}{\sqrt{\mu(M+m)}}\sqrt{c}$, where μ is the attraction between unit masses at unit distance.

By a proper choice of units we may make $\mu Mm/\sqrt{\mu(M+m)}$ and C equal to unity.*

Then let x be equal to the square root of the satellite's mean distance c, and the equation of conservation of moment of momentum becomes

$$n+x(1-\eta)=h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

If in (α) η, the *ellipticity* of the orbit, be zero, we have equation (3) of the previous paper, No. 197, 1879.

It is well known that the sum of the potential and kinetic energies in elliptic motion is independent of the eccentricity of the orbit, and depends only on the mean distance.

Hence if CE be the whole energy of the system, we have (as in equations (2) and (4) of the above paper, No. 197), with the present units

$$2\mathrm{E}=n^2-\frac{1}{x^2}$$

Then if z be written for 2E, and if the value of n be substituted from (α), we have

$$z=\{h-x(1-\eta)\}^2-\frac{1}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

This is the equation of energy of the system.

* In the paper above referred to, and in another, Proc. Roy. Soc., No. 202, of 1880, the physical meaning of the units adopted is scarcely adequately explained.

The units are such that C, the planet's moment of inertia, is unity, that $\mu(M+m)$ is unity, and that a quantity called s and defined in (6) of this paper is unity.

From this it may be deduced that the unit length is such a distance that the moment of inertia of planet and satellite when at this distance apart about their common centre of inertia is equal to the moment of inertia of the planet about its own axis. If γ be this unit of length, this condition gives $\frac{Mm}{M+m}\gamma^2=\mathrm{C}$, or $\gamma=\sqrt{\frac{\mathrm{C}(M+m)}{Mm}}$.

The unit of time is the time taken by the satellite to describe an arc of 57°·3 in a circular orbit at distance γ; it is therefore $\left(\frac{\mathrm{C}}{\mu Mm}\right)^{\frac{1}{2}}\left(\mathrm{C}\frac{M+m}{Mm}\right)^{\frac{1}{2}}$. The unit of mass is $\frac{Mm}{M+m}$.

From this it follows that the unit of moment of momentum is the moment of momentum of orbital motion when the satellite moves in a circular orbit at distance γ. The critical moment of momentum of the system, referred to in those two papers and below in this appendix, is $4/3^{\frac{3}{4}}$ of this unit of moment of momentum.

In whatever manner the two bodies may interact on one another, the resultant moment of momentum h must remain constant, and therefore (α) will always give one relation between n, x, and η; a second relation would be given by a knowledge of the nature of the interaction between the two bodies.

The equation (α) might be illustrated by taking n, x, η as the three rectangular co-ordinates of a point, and the resulting surface might be called the surface of momentum, in analogy with the "line of momentum" in the above paper.

This surface is obviously a hyperboloid, which cuts the plane of nx in the straight line $n+x=h$; the planes of $n\eta$ and $\eta=1$ in the straight line determined by $n=h$; and the plane of $x\eta$ in the rectangular hyperbola $x(1-\eta)=h$.

The contour lines of this surface for various values of n are a family of rectangular hyperbolas with common asymptotes, viz.: $\eta=1$ and $x=0$. It does not however seem worth while to give a figure of them.

If the satellite raises frictional tides of any kind in the planet, the system is non-conservative of energy, and therefore in equation (β) x and η must so vary that z may always diminish.

Suppose that equation (β) be represented by a surface the points on which have co-ordinates x, η, z, and suppose that the axis of z be vertical. Then each point on the surface represents by the co-ordinates x and η one configuration of the system, with given moment of momentum h. Then since the energy must diminish, it follows that the point which represents the configuration of the system must always move down hill. To determine the exact path pursued by the point it would be necessary to take into consideration the nature of the frictional tides which are being raised by the satellite.

I will now consider the nature of the surface of energy.

It is clear that it is only necessary to consider positive values of η lying between zero and unity, because values of η greater than unity correspond to a hyperbolic orbit; and the more interesting part of the surface is that for which η is a pretty small fraction.

The curves, formed on the surface by the intersection of vertical planes parallel to x, have maxima and minima points determined by $dz/dx=0$.

This condition gives by differentiation of (β)

$$x^4-\frac{h}{1-\eta}x^3+\frac{1}{(1-\eta)^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

From the considerations adduced in previous papers, namely, those in the Proc. Roy. Soc., No. 197, 1879, and No. 202, 1880, it follows that this equation has either two real roots or no real roots.

When $\eta=0$ the equation has real roots provided h be greater than $4/3^{\frac{3}{4}}$, and since this case corresponds to that of all but one of the satellites of the solar system, I shall

henceforth suppose that h is greater than $4/3^{\frac{3}{4}}$. It will be seen presently that in this case every section parallel to x has a maximum and minimum point, and the nature of the sections is exhibited in the curves of energy in the two previous papers.

Now consider the condition $n=\Omega$, which expresses that the planet rotates in the same period as that in which the satellite revolves, so that if the orbit be circular the two bodies revolve like a single rigid body.

With the present units $\Omega=1/x^3$, and by (α), $n=h-x(1-\eta)$.

Hence the condition $n=\Omega$ leads to the biquadratic

$$x^4-\frac{1}{1-\eta}x^3+\frac{1}{1-\eta}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

If η be zero this equation is identical with (γ), which gives the maxima and minima of energy.

Hence if the orbit be circular the maximum and minimum of energy correspond to two cases in which the system moves as a rigid body. If however the orbit be elliptical, and if $n=\Omega$, there is still relative motion during revolution of the satellite, and the energy must be capable of degradation. The principal object of the present note is to investigate the stability of the circular orbit in these cases, and this question involves a determination of the nature of the degradation when the orbit is elliptical.

In Part V. of the present paper it has been shown that if the planet be a fluid of small viscosity the ellipticity of the satellite's orbit will increase if 18 rotations of the planet be less than 11 revolutions of the satellite, and *vice versâ*. Hence the critical relation between n and Ω is $n=\frac{18}{11}\Omega$. This leads to the biquadratic

$$x^4-\frac{h}{1-\eta}x^3+\frac{18}{11}\frac{1}{1-\eta}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon)$$

This is an equation with two real roots, and when it is illustrated graphically it will lead to a pair of curves. For configurations of the system represented by points lying between these curves the eccentricity increases, and outside it diminishes,—supposing the viscosity of the planet and the eccentricity of the satellite's orbit to be small.

In order to illustrate the surface of energy (β) and the three biquadratics (γ), (δ), and (ϵ), I chose $h=3$, which is greater than $4/3^{\frac{3}{4}}$.

By means of a series of solutions, for several values of η, of the equations (γ), (δ), (ϵ), and a method of graphical interpolation, I have drawn the accompanying figure.

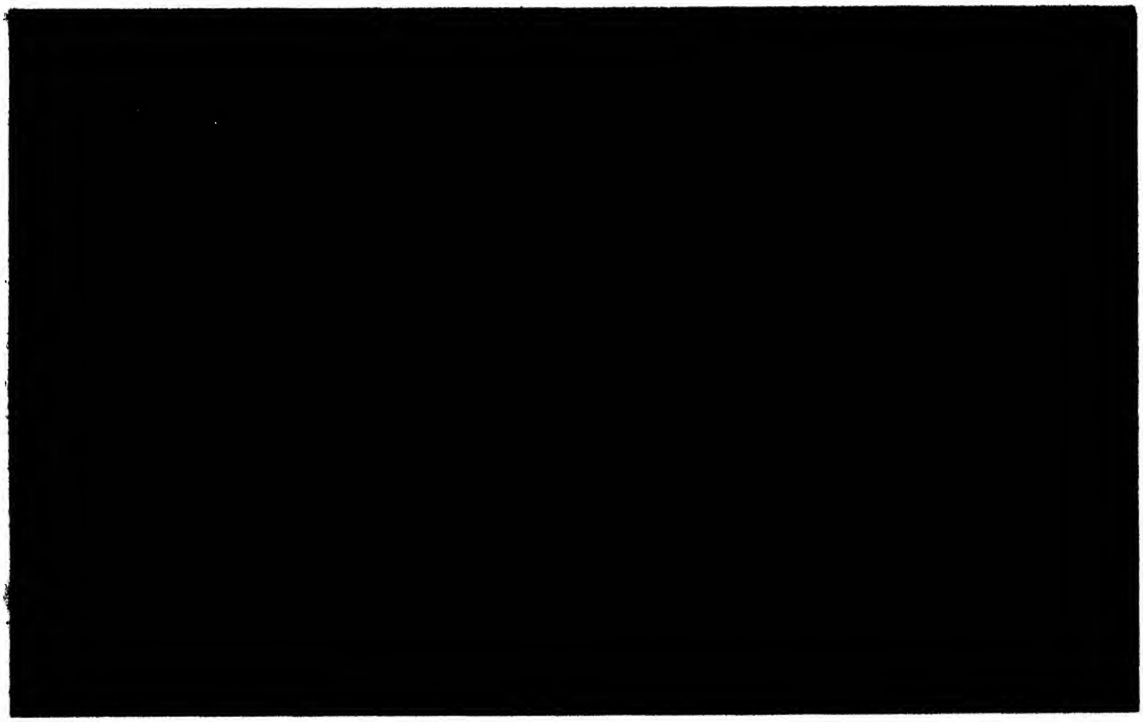
The horizontal axis is that of x, the square root of the satellite's distance, and the numbers written along it are the various values of x. The vertical axis is that of η, and it comprises values of η between 0 and 1. The axis of z is perpendicular to the plane of the paper, but the contour lines for various values of z are projected on to the plane of the paper.

The numbers written on the curves represent the values of z, viz., $z=0, 1, 2, 3, 4, 5$.

The ends of the contour lines on the right are joined by dotted lines, because it would be impossible to draw the curves completely without a very large extension of the figure.

The broken lines (– – –) marked "line of maxima," terminating at A, and "line of minima," terminating at B, represent the two roots of the biquadratic (γ).

The lines marked $n=\Omega$ represent the two roots of (δ), but computation showed that the right-hand branch fell so very near the line of minima, that it was necessary to somewhat exaggerate the divergence in order to show it on the figure.



Contour lines of surface of energy.

The chain-dot lines (– · – · –) C, C, marked $n=\frac{18}{11}\Omega$, represent the two roots of (ϵ). For configurations of the system represented by points lying between these two curves, the ellipticity of orbit will increase; for the regions outside it will decrease. This statement only applies to cases of small ellipticity, and small viscosity of the planet.

Inspection of the figure shows that the line of minima is an infinitely long valley of a hyperbolic sort of shape, with gently sloping hills on each side, and the bed of the valley gently slopes up as we travel away from B.

The line of maxima is a ridge running up from A with an infinitely deep ravine on the left, and the gentle slopes of the valley of minima on the right.

Thus the point B is a true minimum on the surface, whilst the point A is a maximum-minimum, being situated on a saddle-shaped part of the surface.

The lines $n=\Omega$ start from A and B, but one deviates from the ridge of maxima towards the ravine; and the other branch deviates from the valley of minima by going up the slope on the side remote from the origin.

This surface enables us to perfectly determine the stabilities of the circular orbit, when planet and satellite are moving as parts of a rigid body.

The configuration B is obviously dynamically stable in all respects; for any configuration represented by a point near B must degrade down to B.

It is also clear that the configuration A is dynamically unstable, but the nature of the instability is complex. A displacement on the right-hand side of the ridge of maxima will cause the satellite to recede from the planet, because x must increase when the point slides down hill.

If the viscosity be small, the ellipticity given to the orbit will diminish, because A is not comprised between the two chain-dot curves. Thus for this class of tide the *circularity* is stable, whilst the configuration is unstable.

A displacement on the left-hand side of the ridge of maxima will cause the satellite to fall into the planet, because the point will slide down into the ravine. But the circularity of the orbit is again stable.

This figure at once shows that if planet and satellite be revolving with maximum energy as parts of a rigid body, and if, without altering the total moment of momentum, or the equality of the two periods, we impart infinitesimal ellipticity to the orbit, the satellite will fall into the planet. This follows from the fact that the line $n=\Omega$ runs on to the slope of the ravine.

If on the other hand without affecting the moment of momentum, or the circularity, we infinitesimally disturb the relation $n=\Omega$, then the satellite will either recede or approach the planet according to the nature of the disturbance.

These two statements are independent of the nature of the frictional interaction of the two bodies.

The only parts of this figure which postulate anything about the nature of the interaction are the curves $n=\frac{18}{11}\Omega$.

I have not thought it worth while to illustrate the case where h is less than $4/3^{\frac{3}{4}}$, or the negative side of the surface of energy; but both illustrations may easily be carried out.

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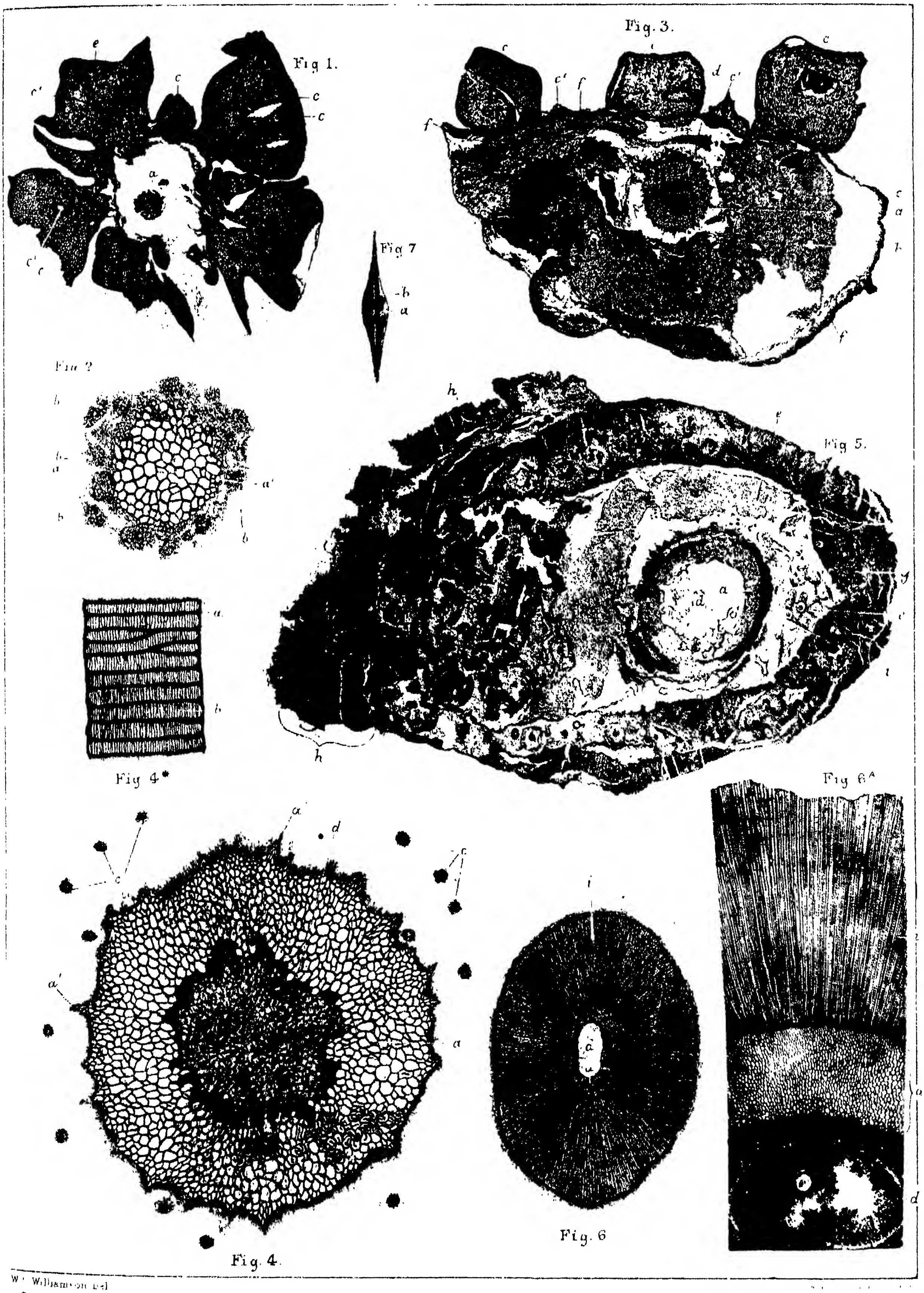

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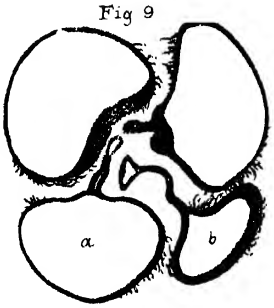

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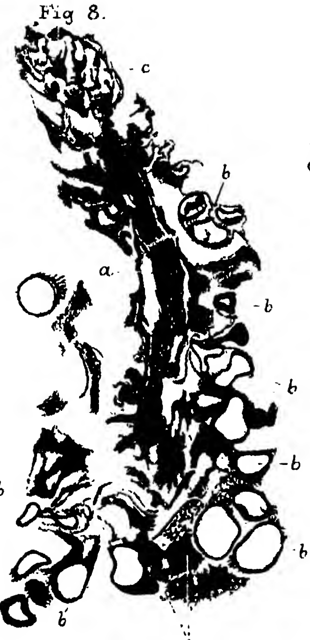

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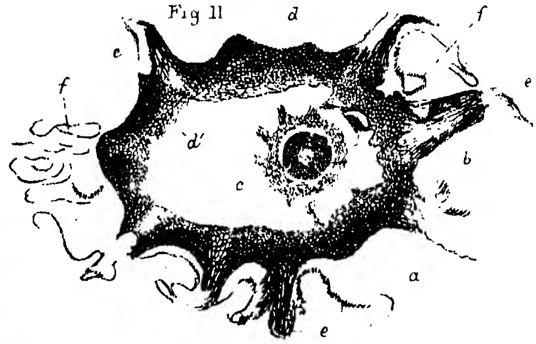

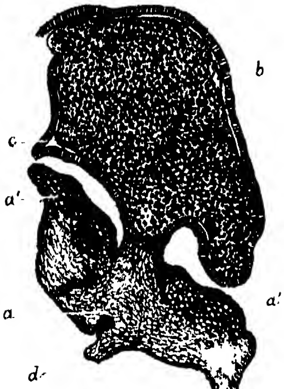

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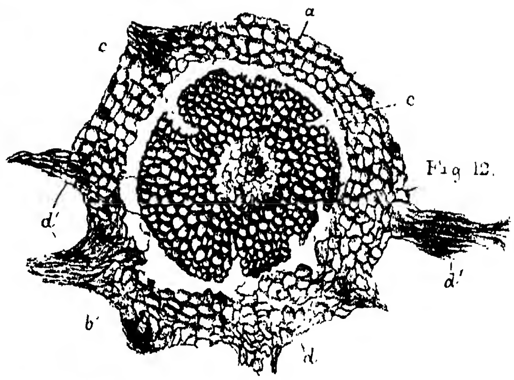

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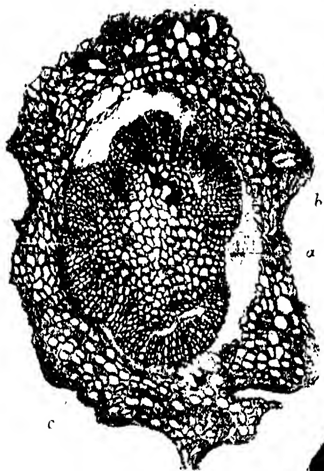

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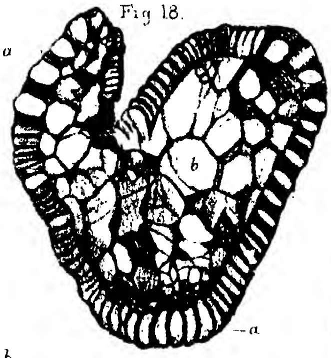

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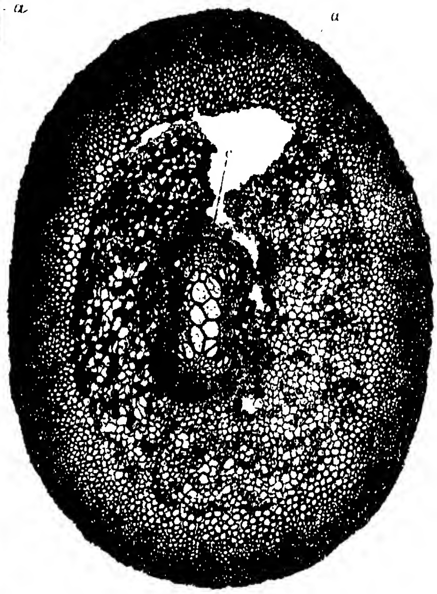

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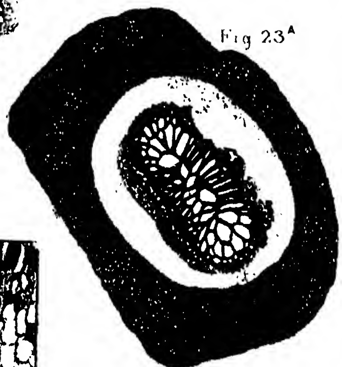

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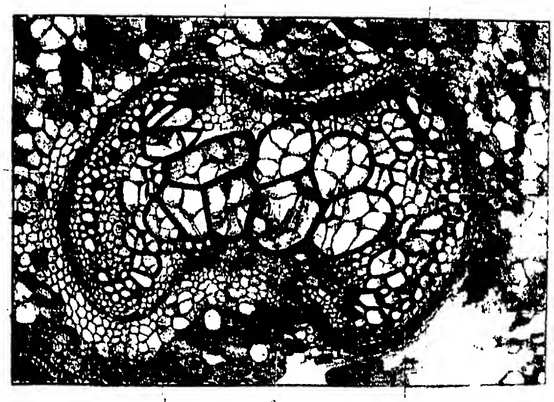

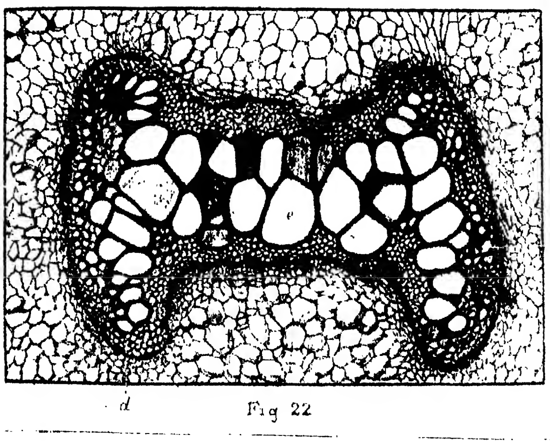

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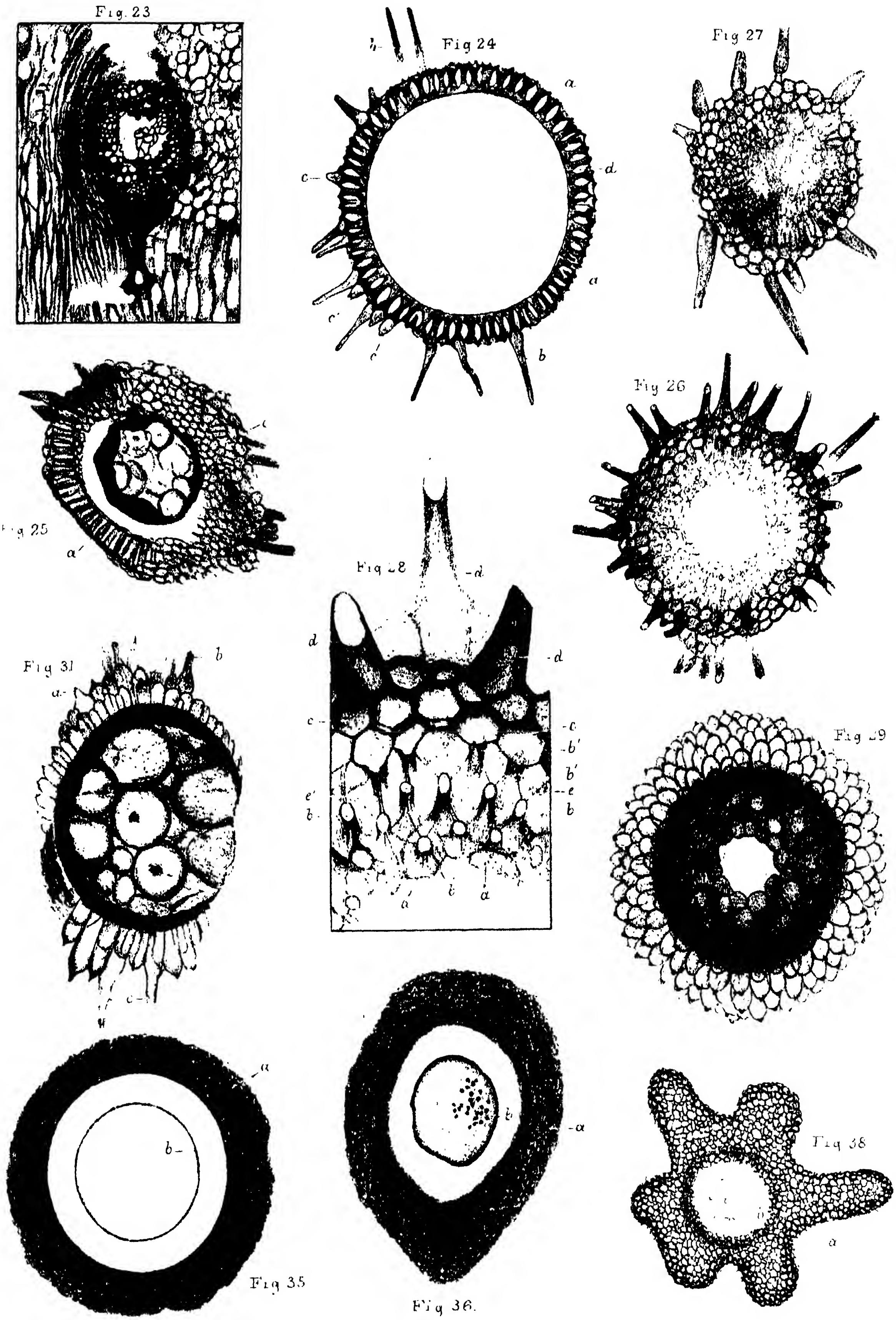
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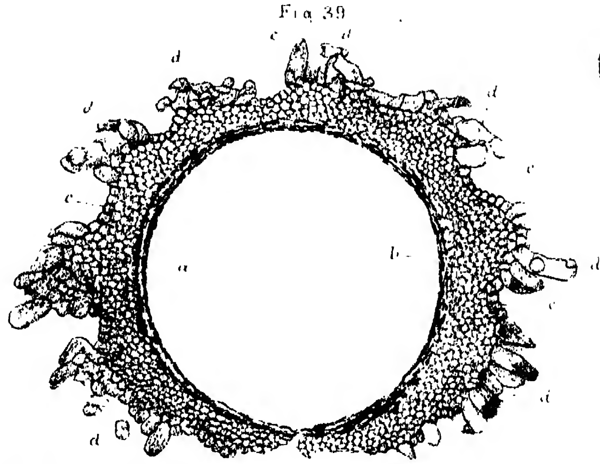

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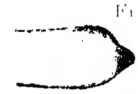

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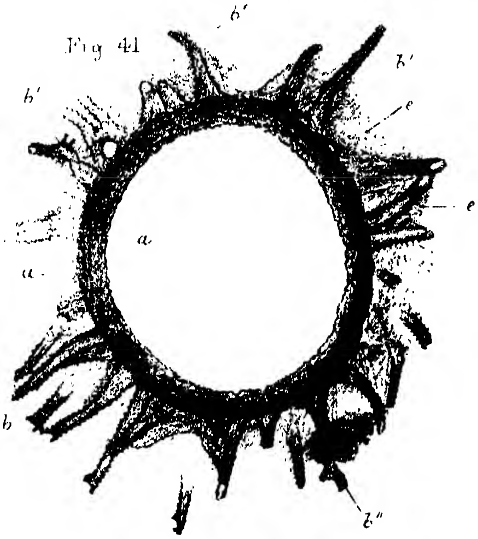

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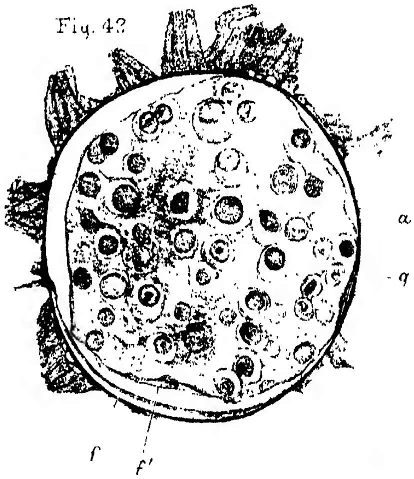

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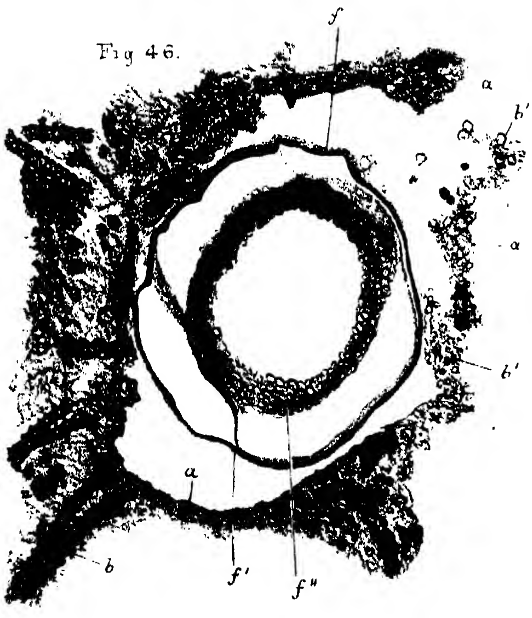

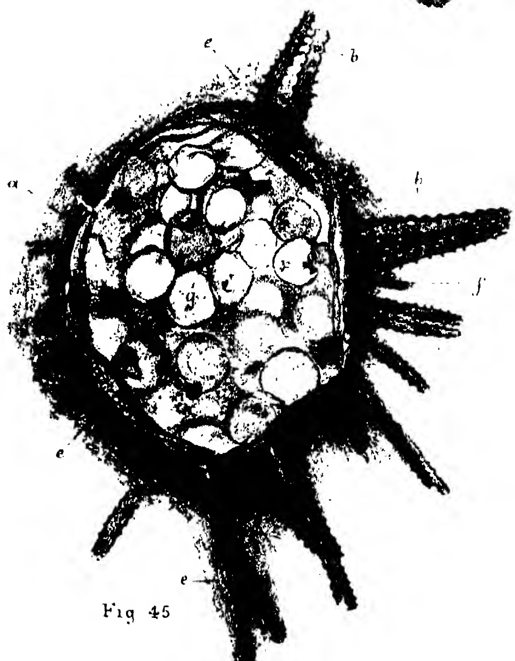

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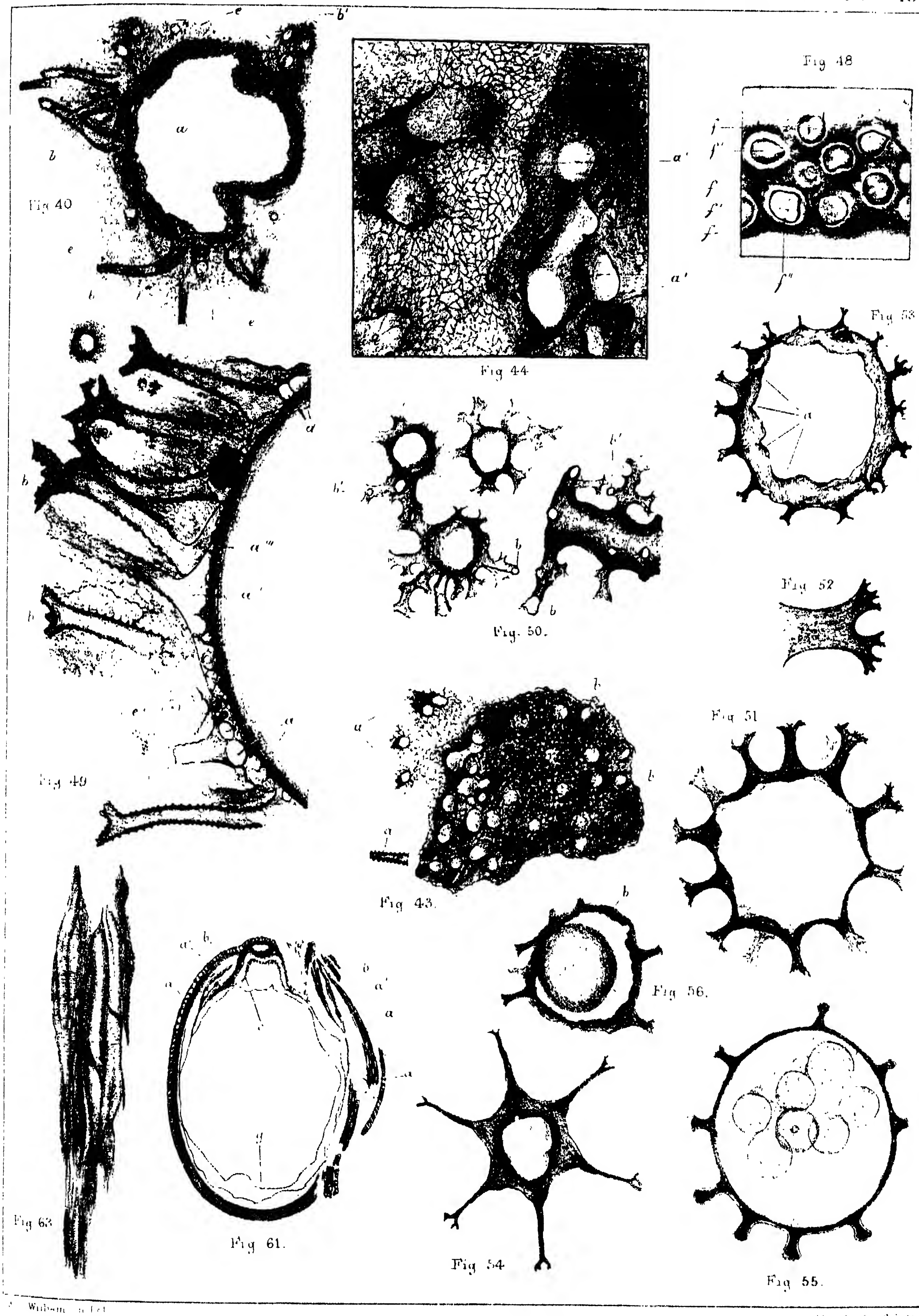
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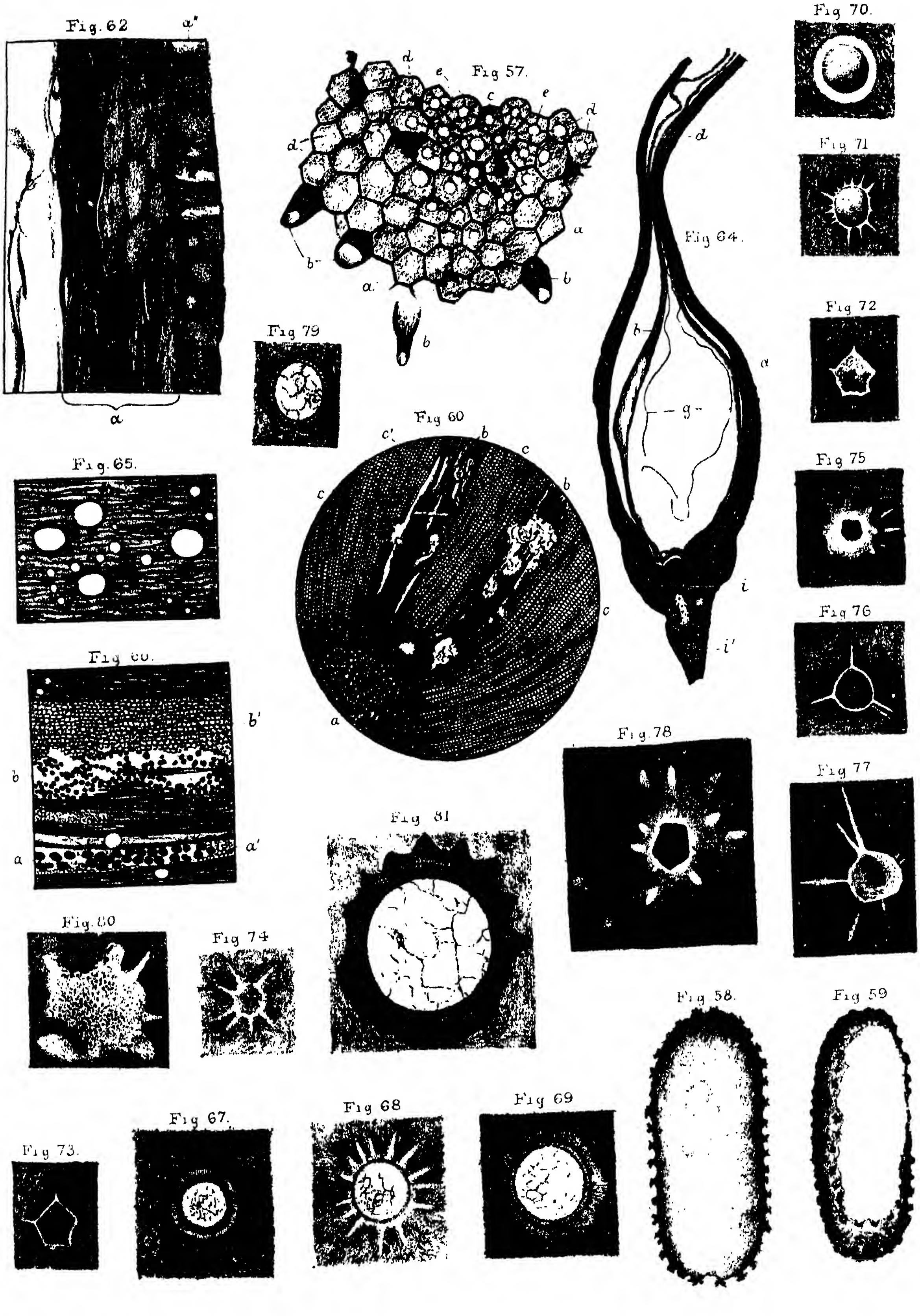

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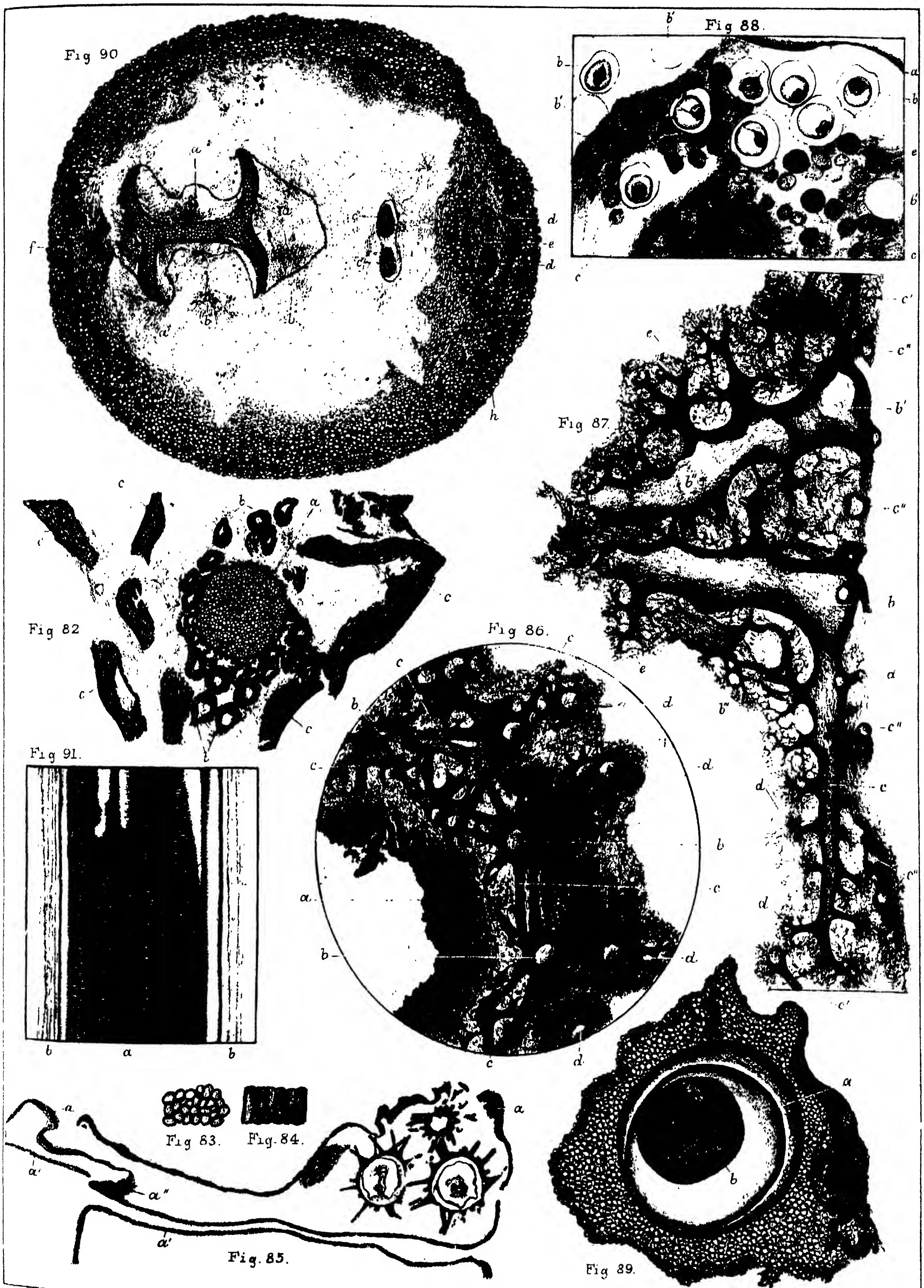

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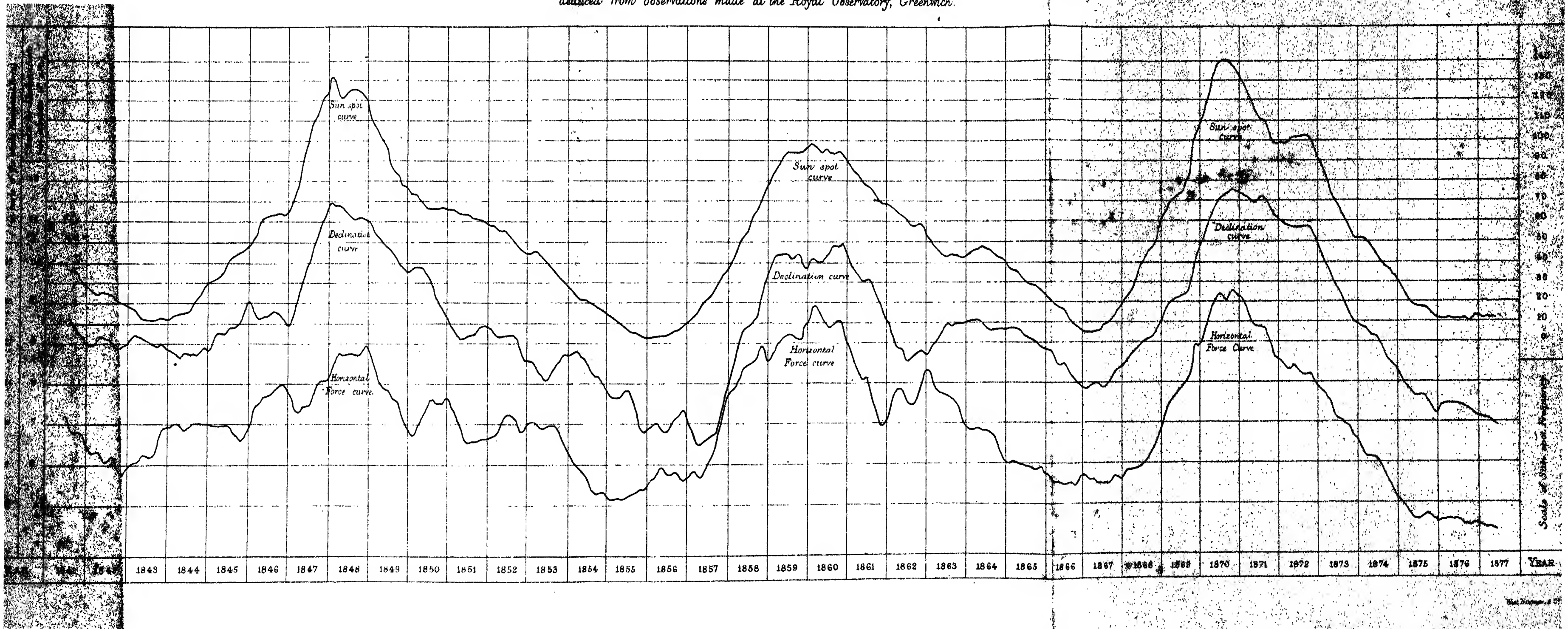
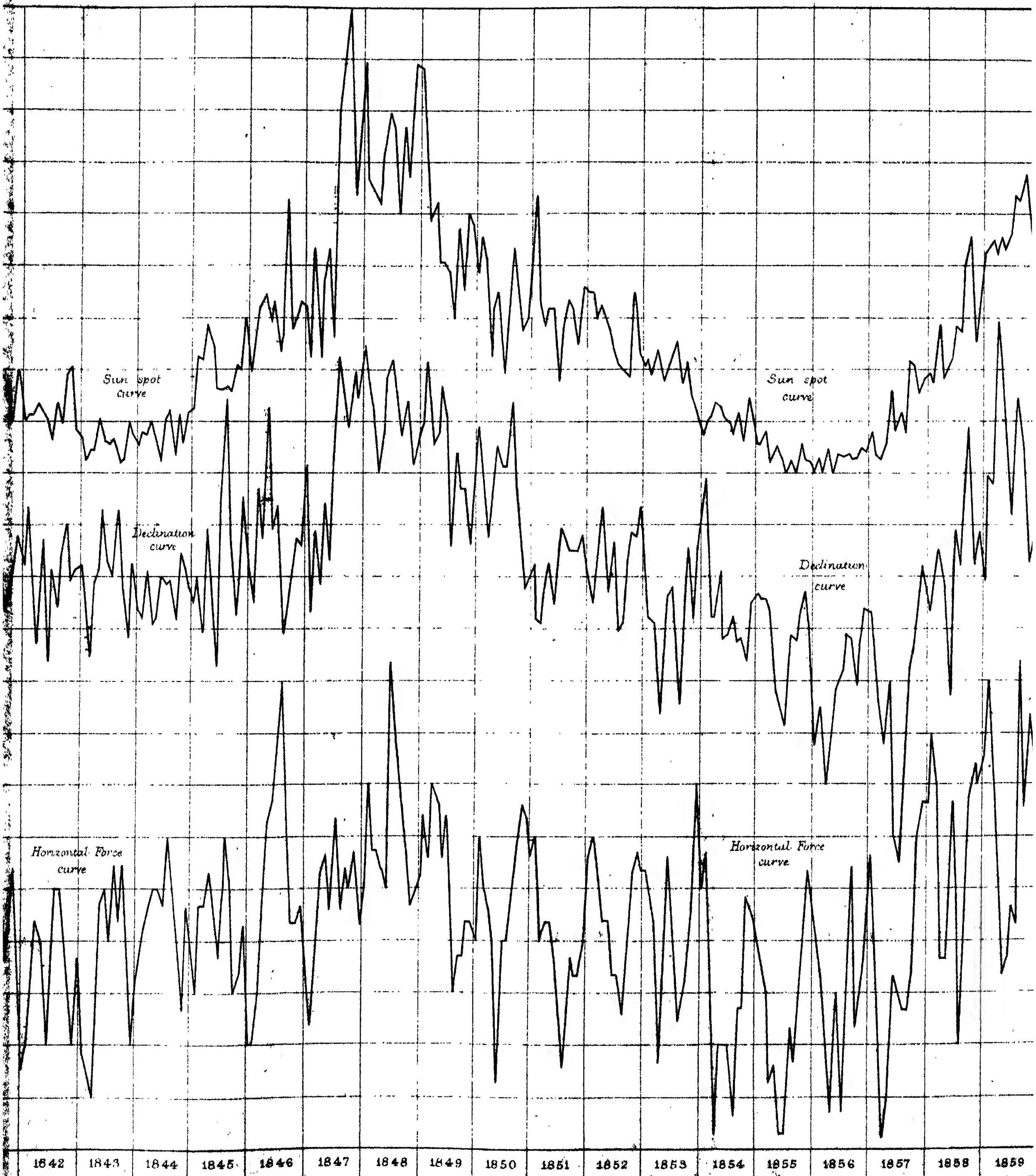
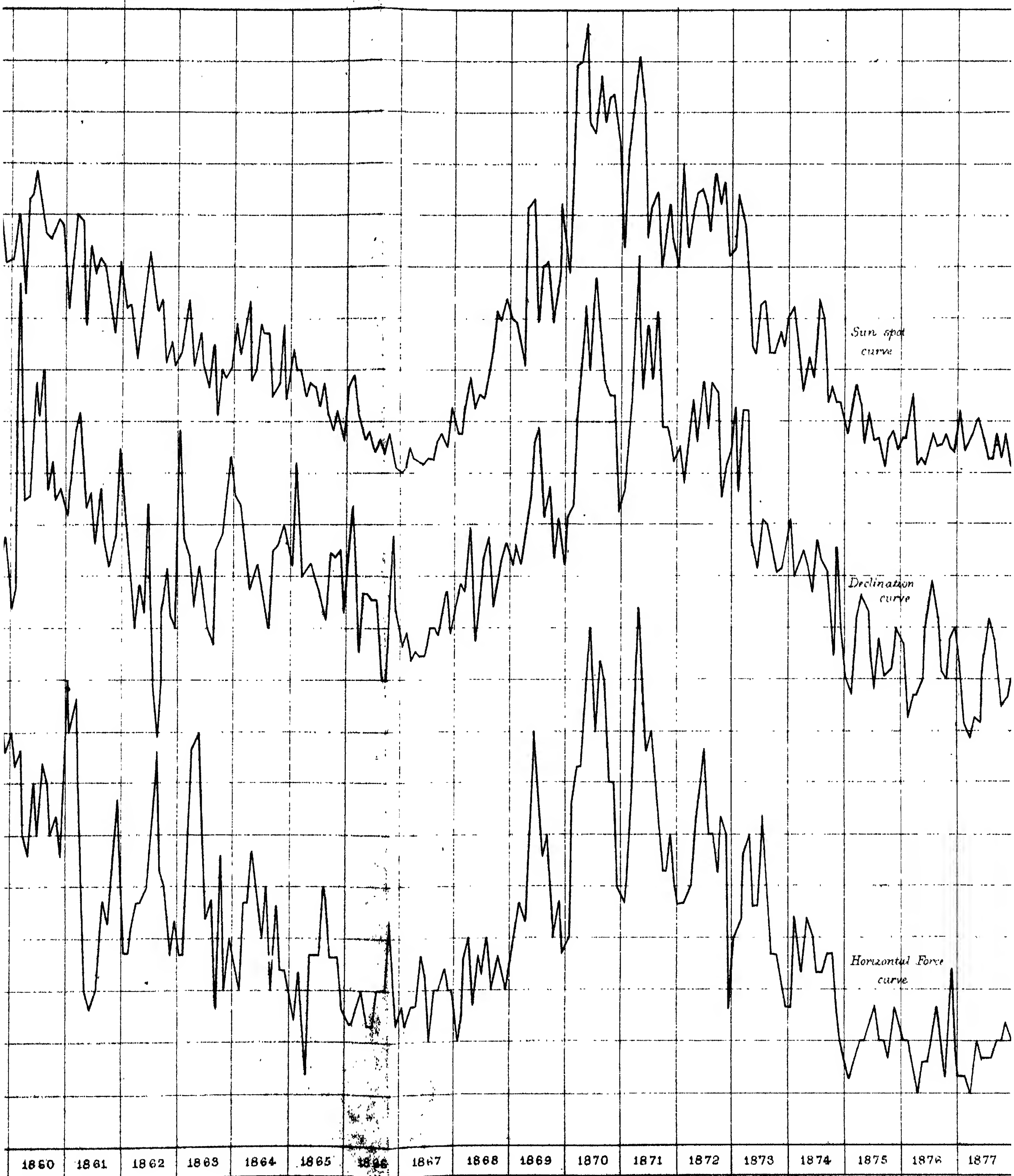
Smoothed Curves of Sun spot Frequency (Wolf) compared with corresponding curves showing the Variation in the Diurnal Range of the Magnetic Elements, Declination, and Horizontal Force, as deduced from observations made at the Royal Observatory, Greenwich.



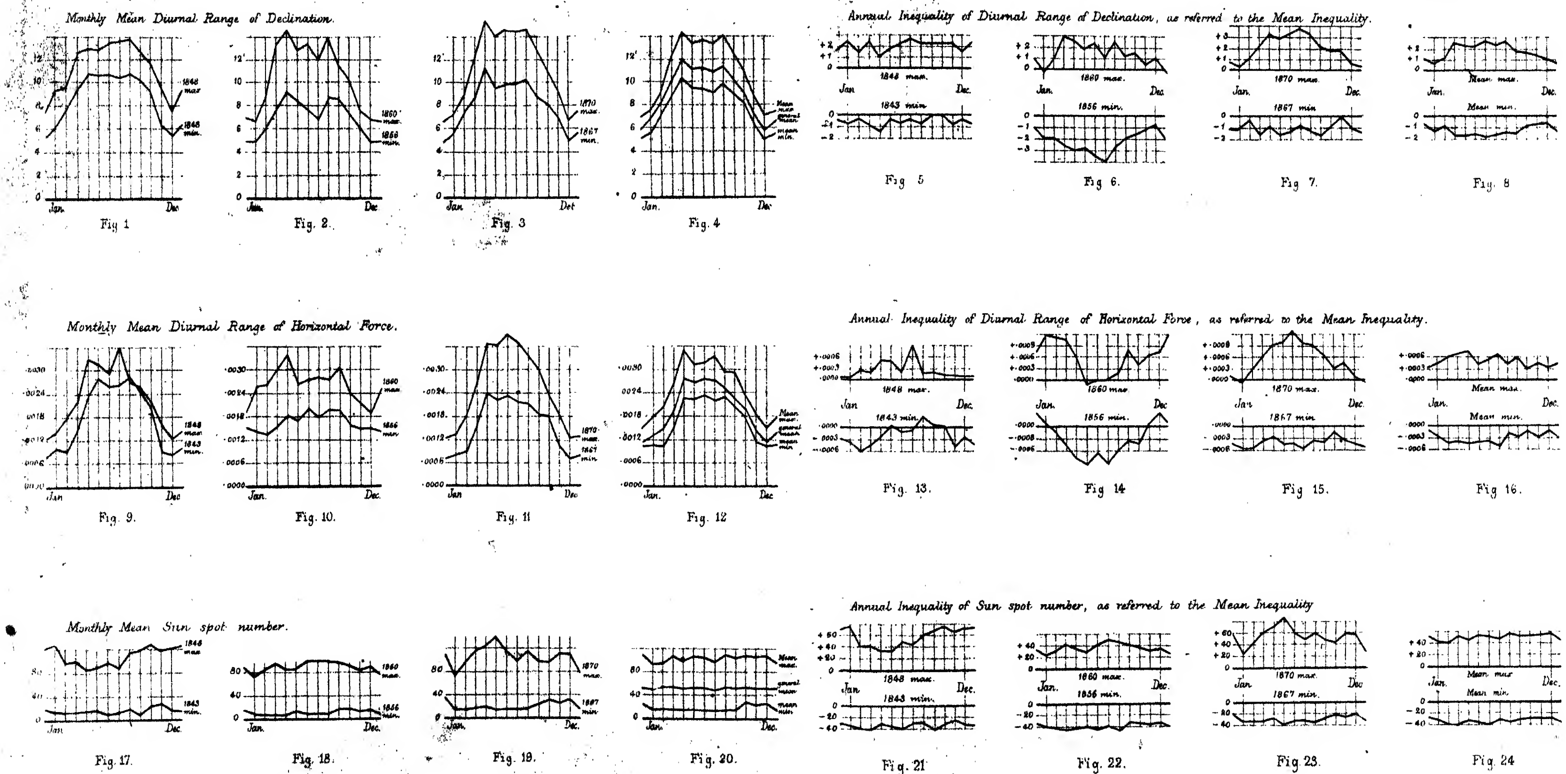
Actual Monthly changes of Sun spot Frequency (Wolf), compared with the corresponding Monthly changes in annual inequality only) as deduced from observations



. the Diurnal Range of the Magnetic Elements. Declination and Horizontal Force (cleared of average made at the Royal Observatory, Greenwich.



Curves showing the Monthly Mean Diurnal Range of the Magnetic Elements, Declination and Horizontal Force, and their Annual Inequalities, as referred to the Mean Inequality, at the Epochs of Sun spot Minimum and Maximum, as deduced from Observations made at the Royal Observatory, Greenwich; also the corresponding curves given by the Sun spot numbers.



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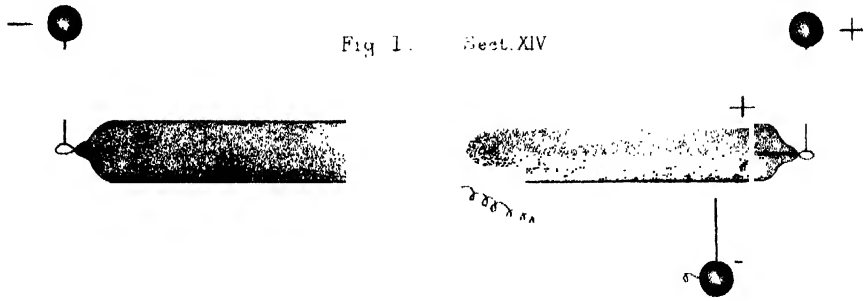

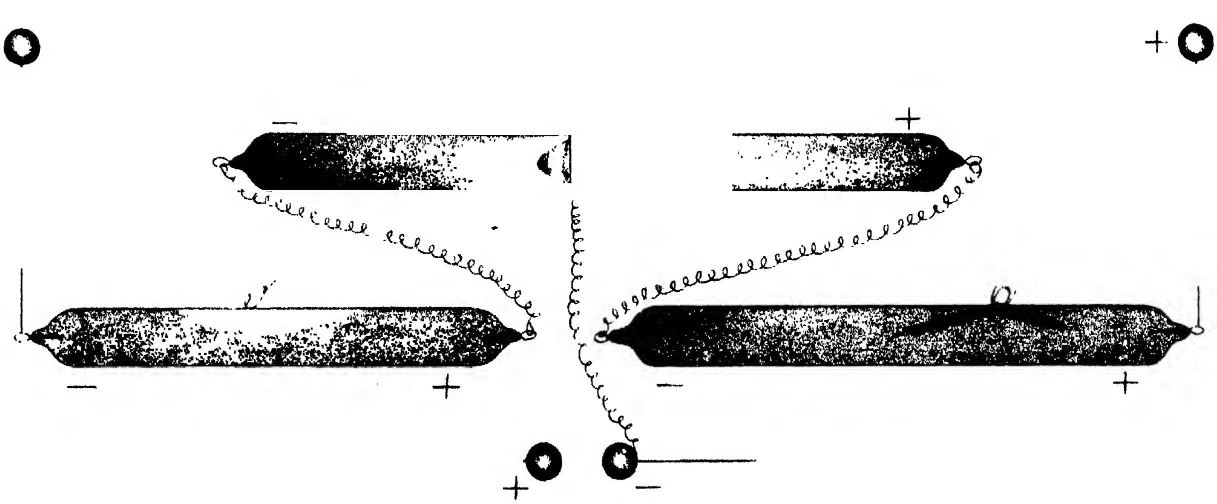

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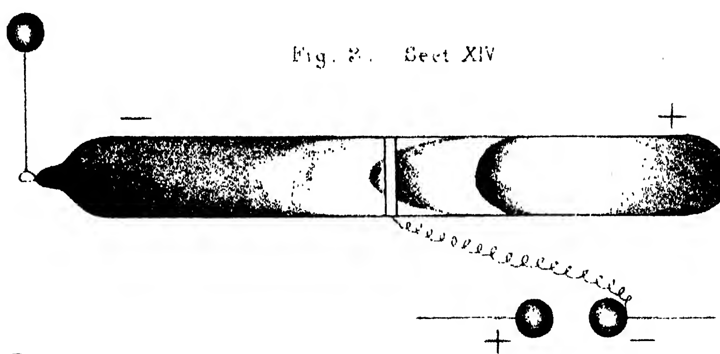

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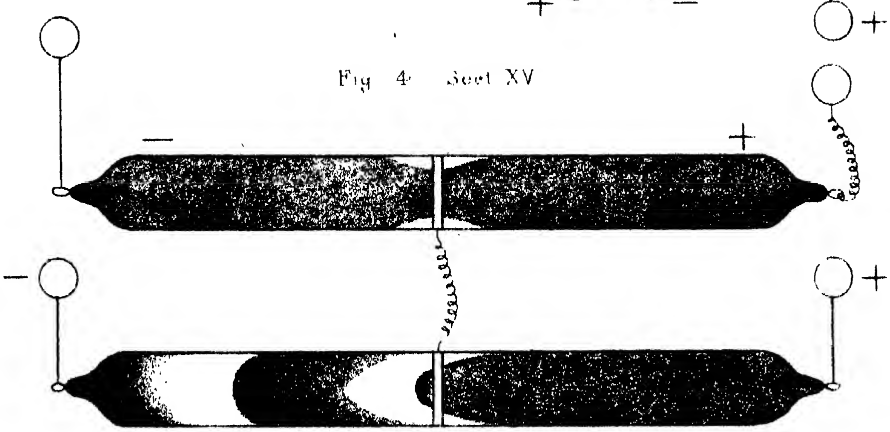

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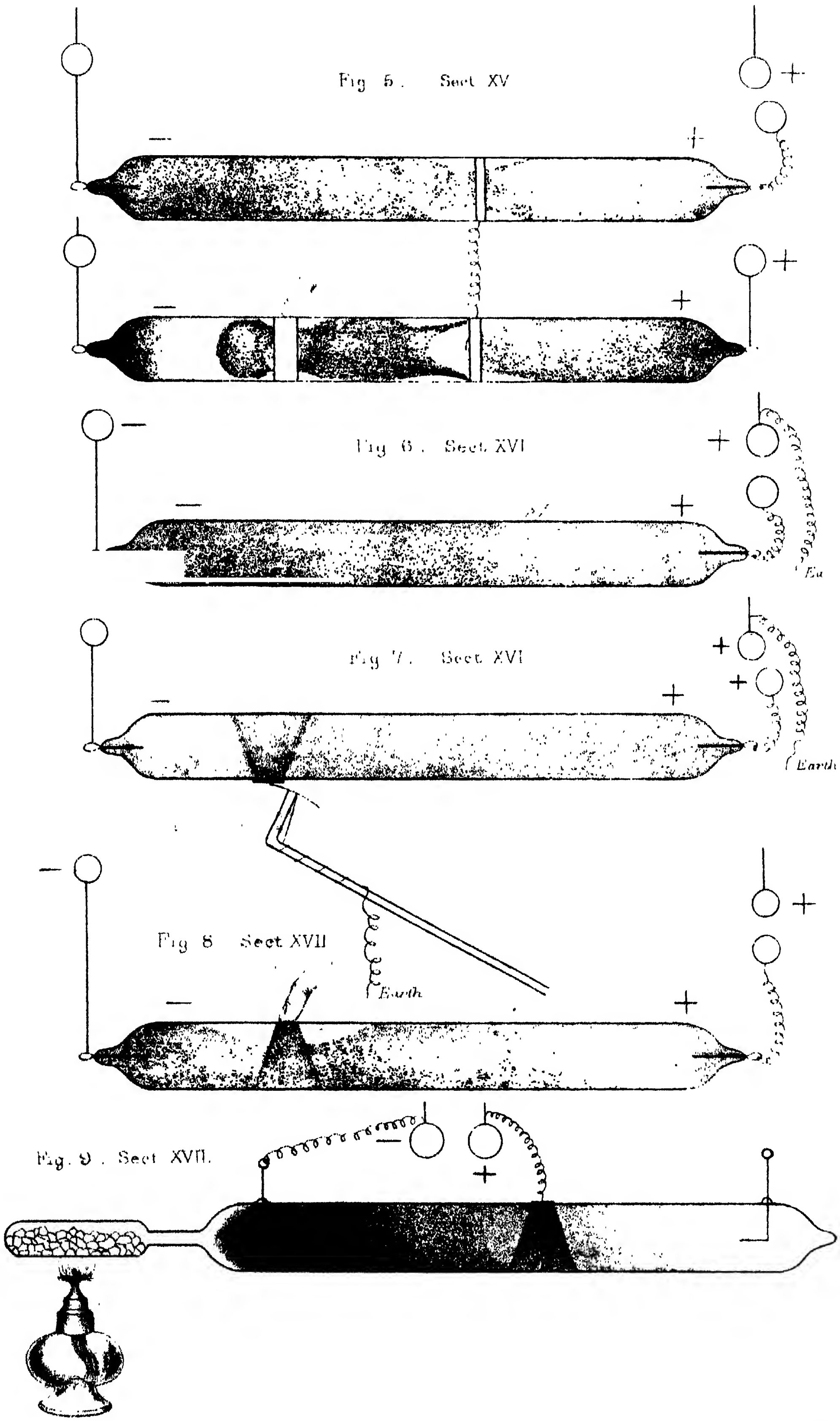
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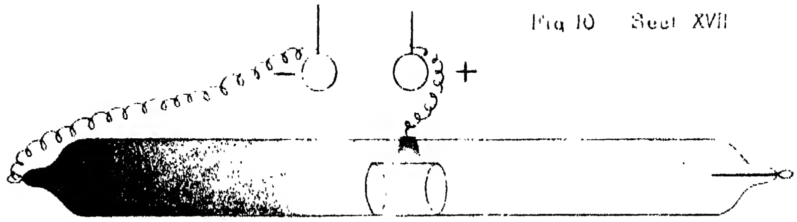

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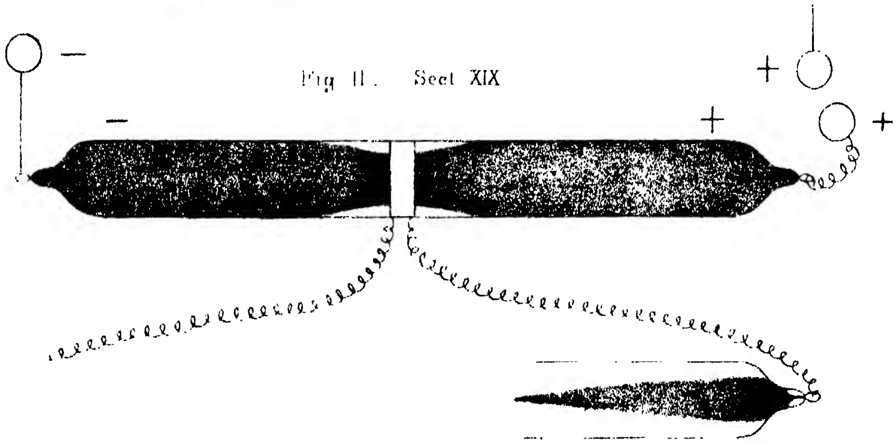

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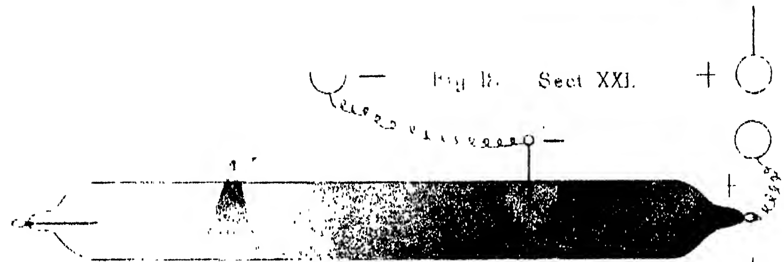

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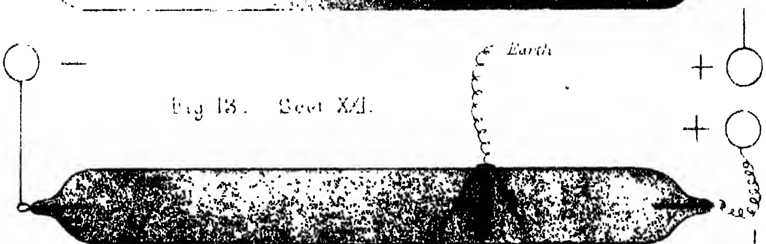

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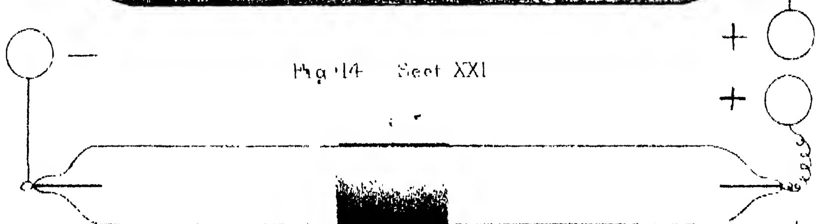

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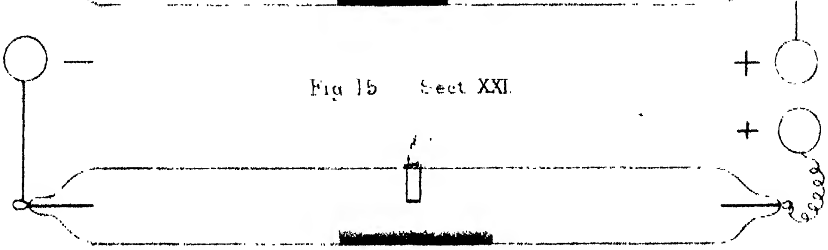

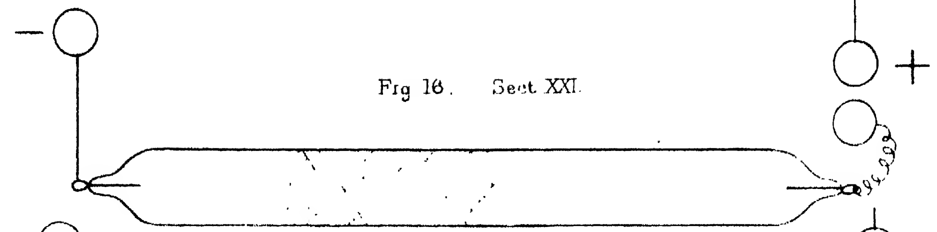

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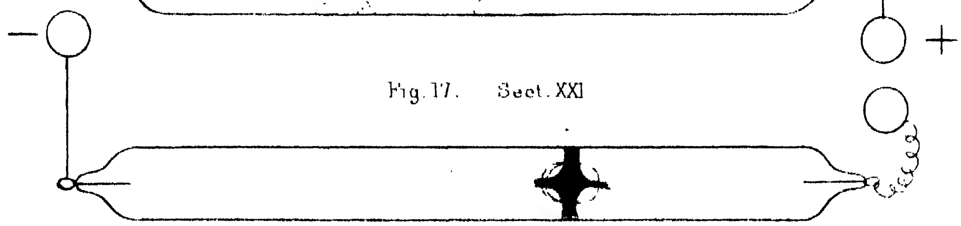

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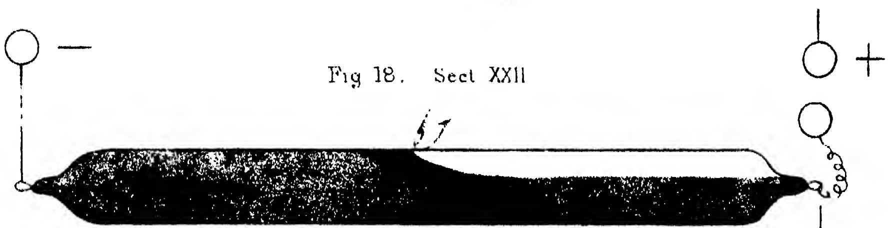

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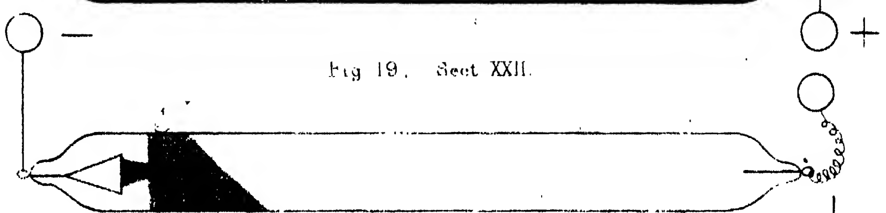

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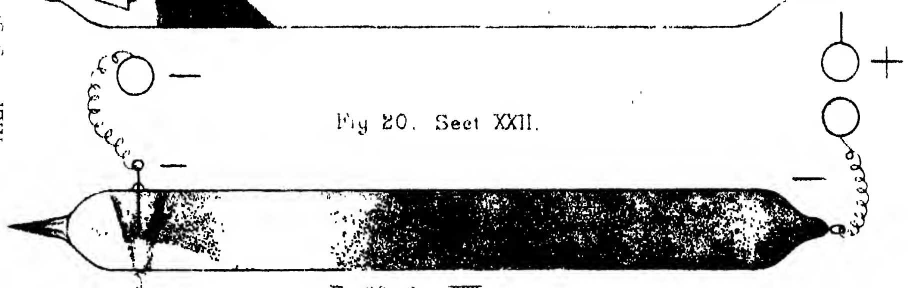

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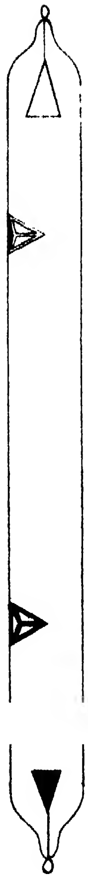
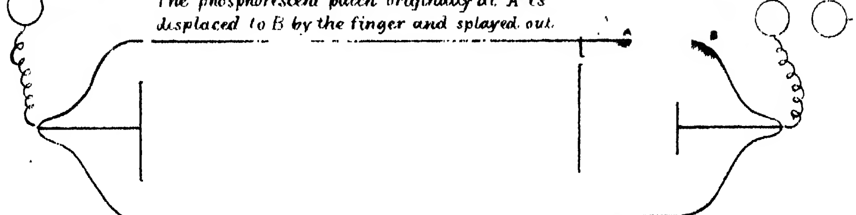


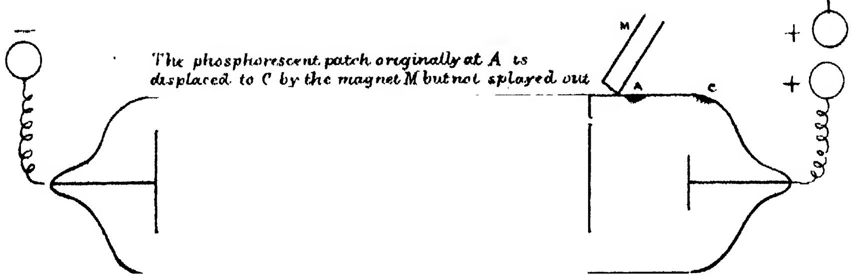
Fig. 21 & 22. Sect. XXII

Fig 23. Sect XXII

The phosphorescent patch originally at A is displaced to B by the finger and splayed out.



The phosphorescent patch originally at A is displaced to C by the magnet M but not splayed out



West, Newman & Co. lith.

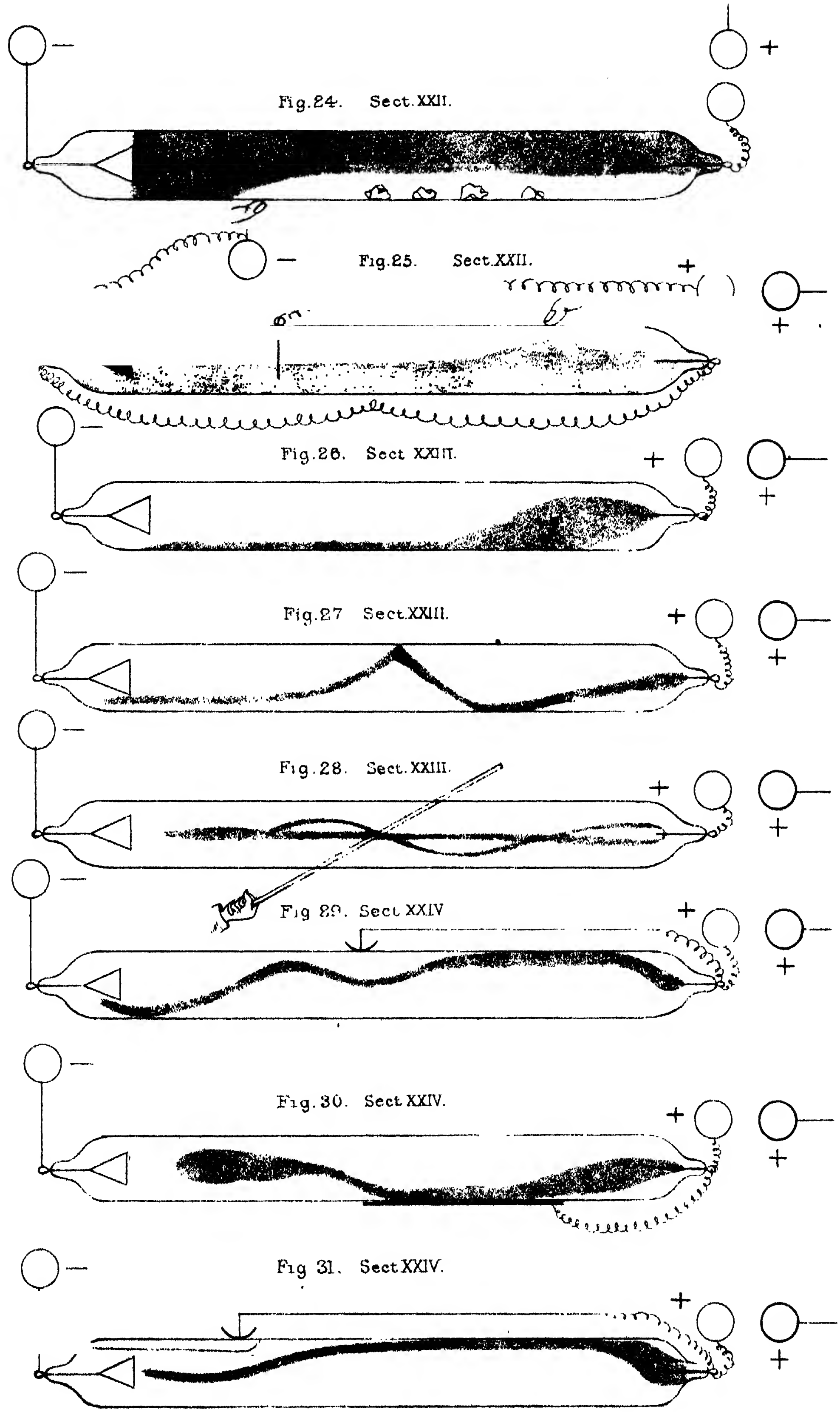


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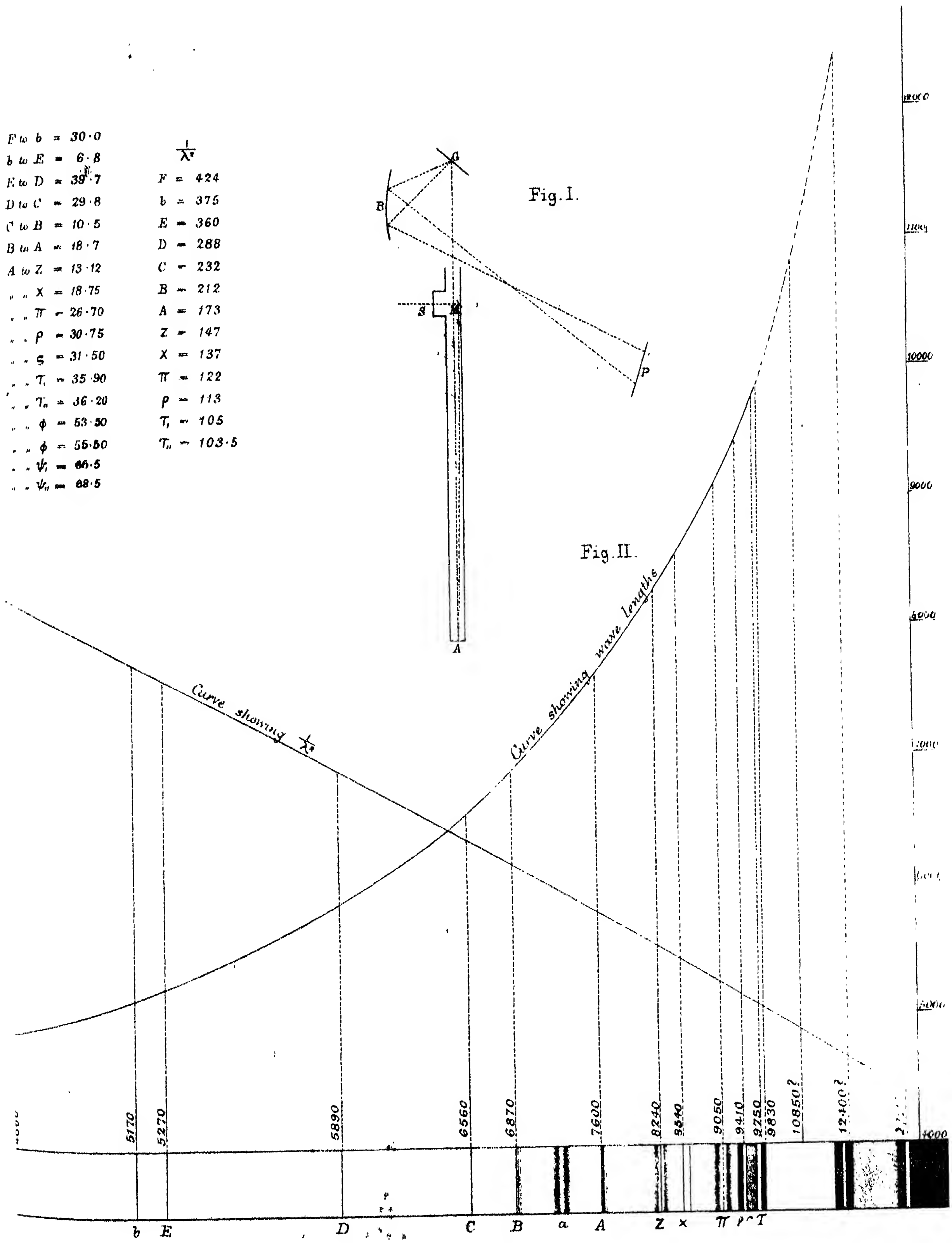
Fig. 30. Sect. XXIV.

Fig. 31. Sect. XXIV.

West, Newman & Co lith.

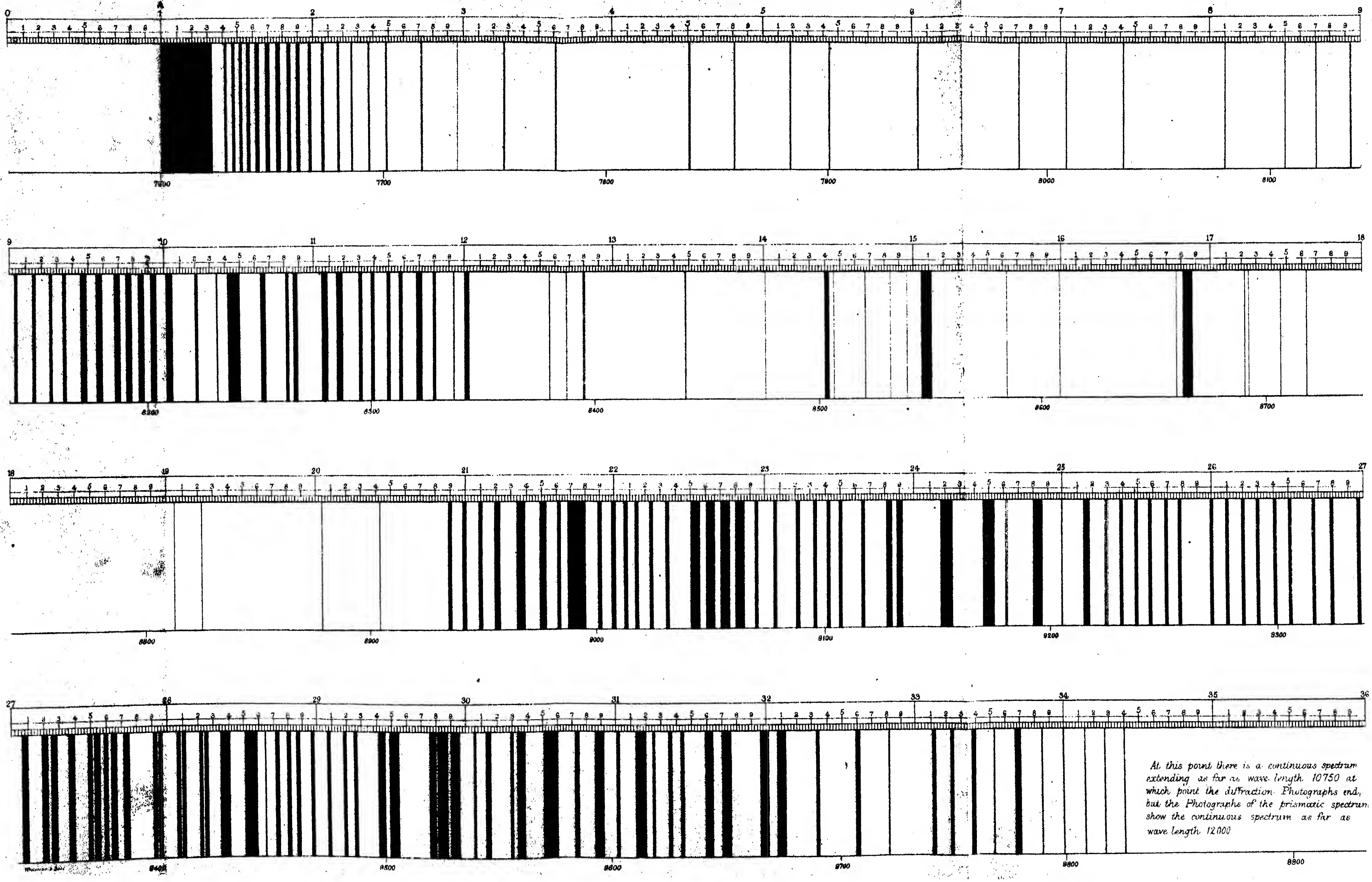


Prismatic Solar Spectrum,
measured from a Photograph

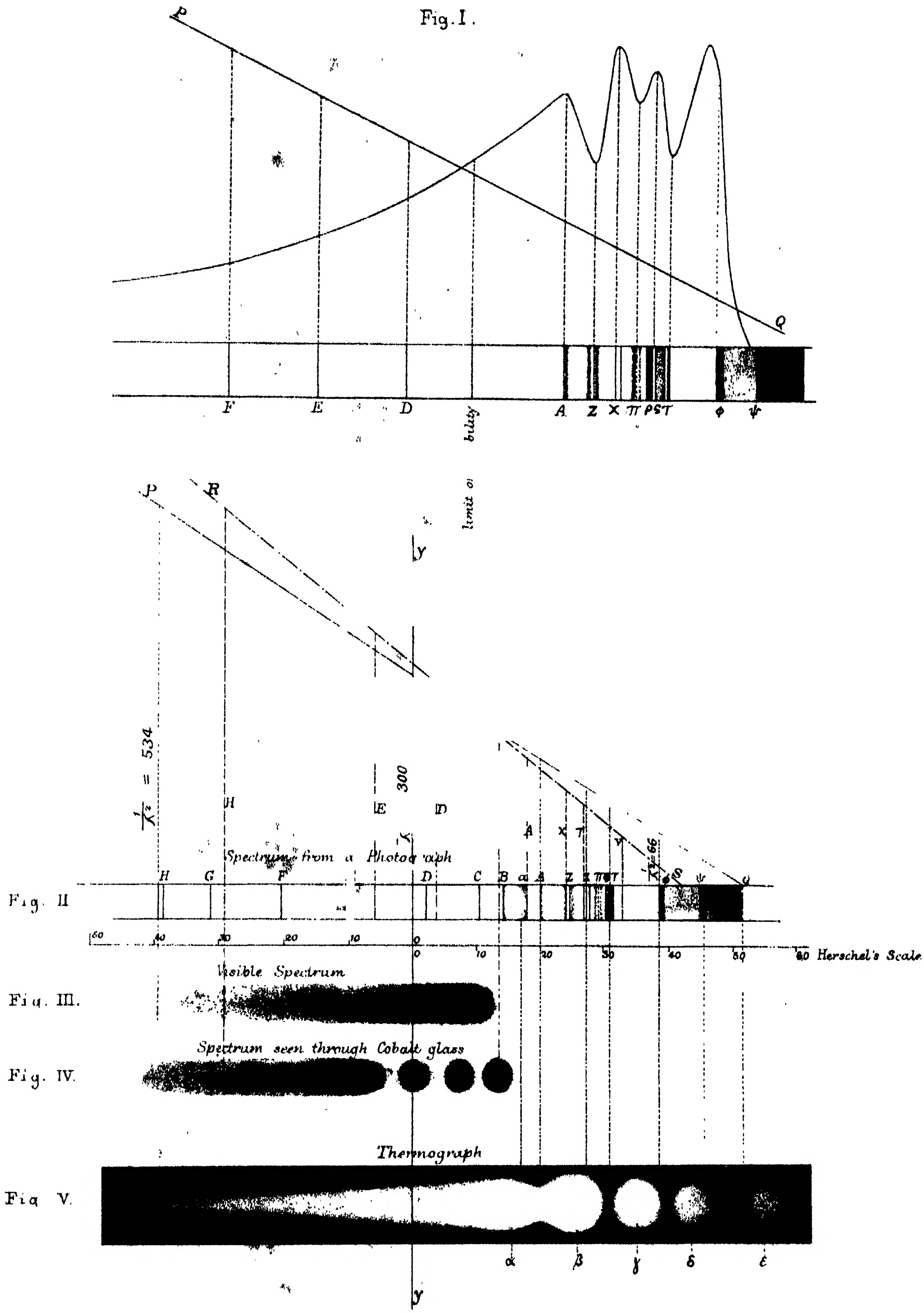
West, Newman & Co. lith.

MAP OF THE INFRA-RED END OF THE SOLAR SPECTRUM

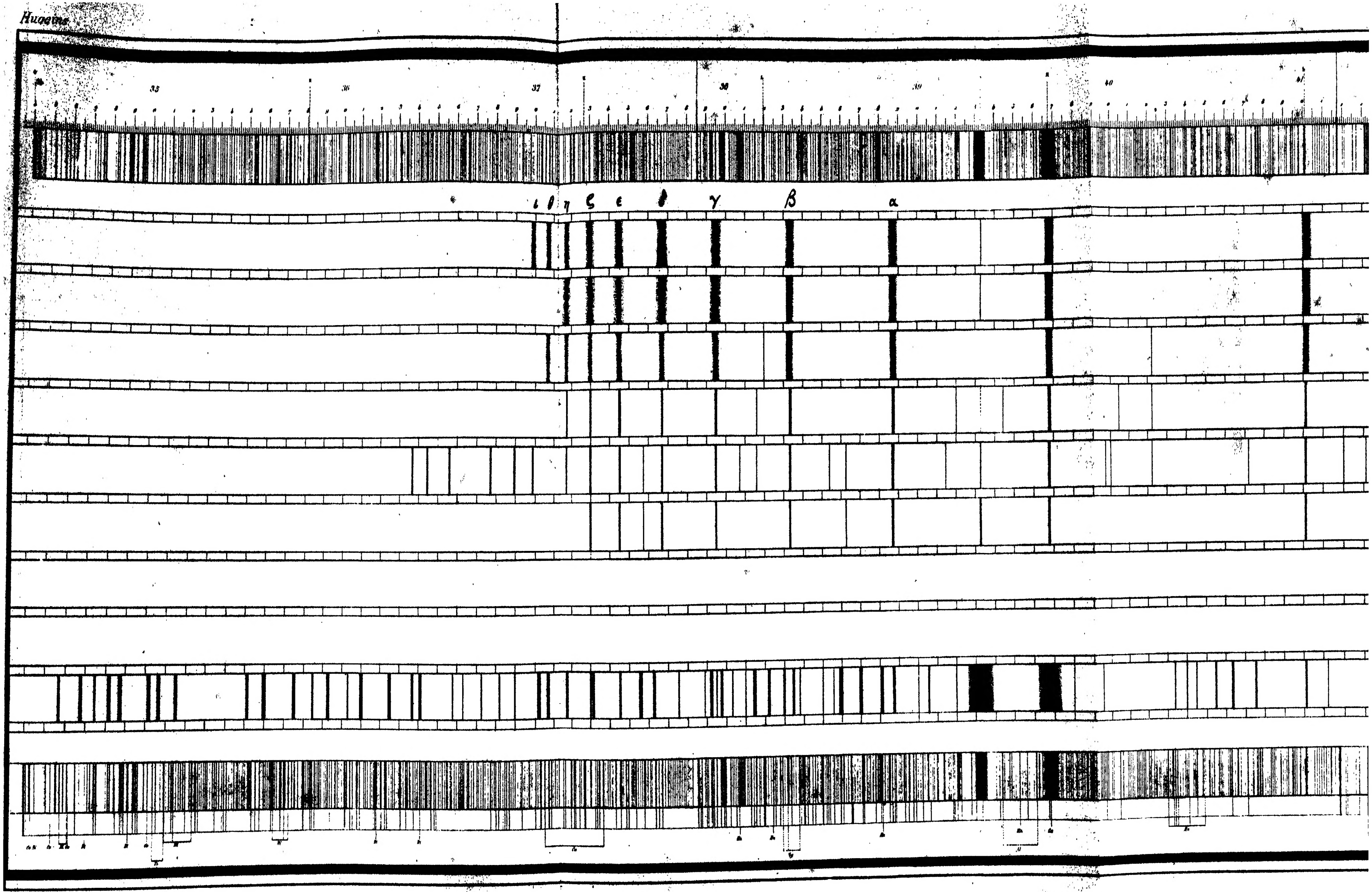
As measured from Photographs taken by the aid of Diffraction Gratings.



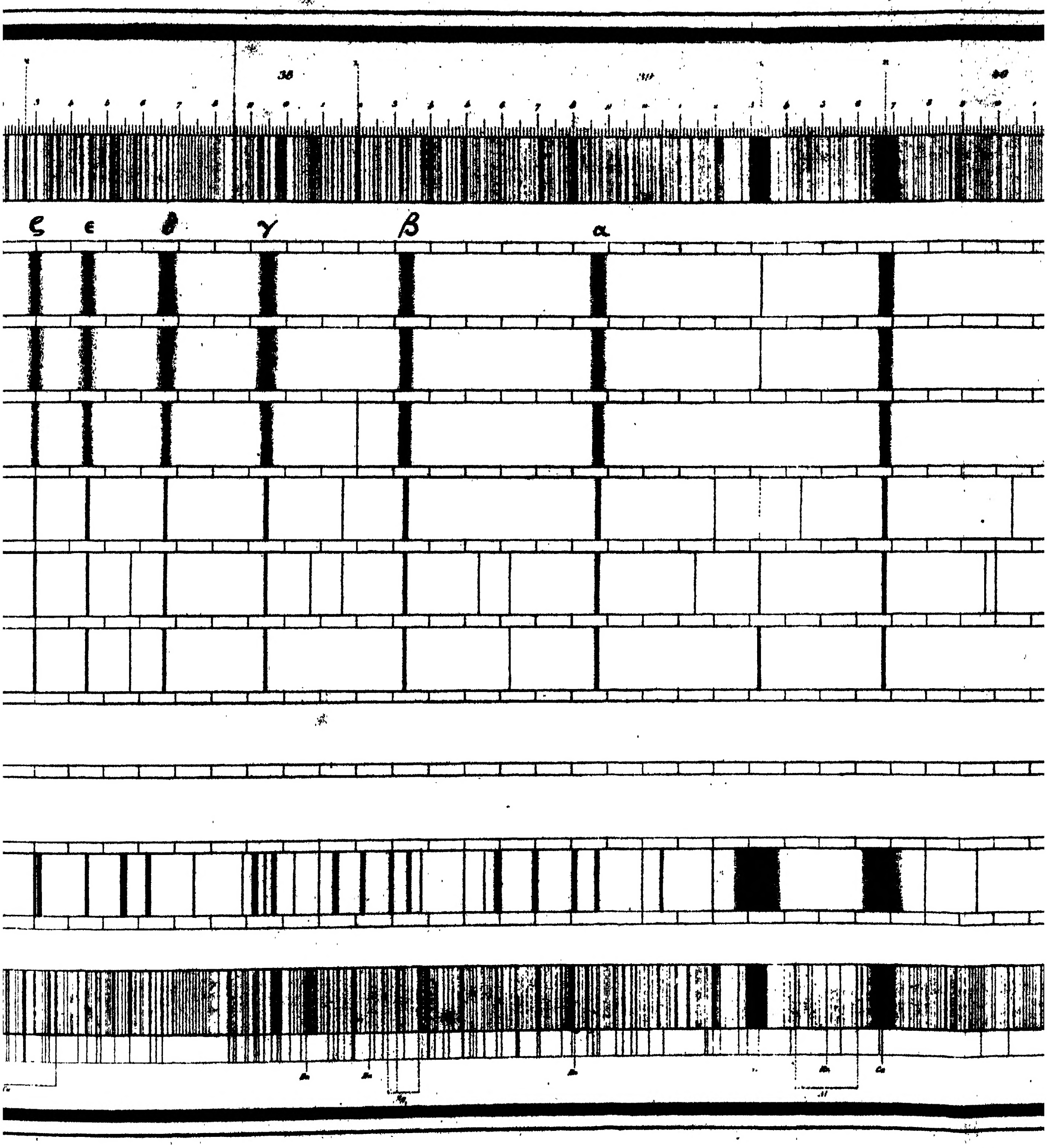
At this point there is a continuous spectrum extending as far as wave length 10750 at which point the diffraction Photographs end, but the Photographs of the prismatic spectrum show the continuous spectrum as far as wave length 12000

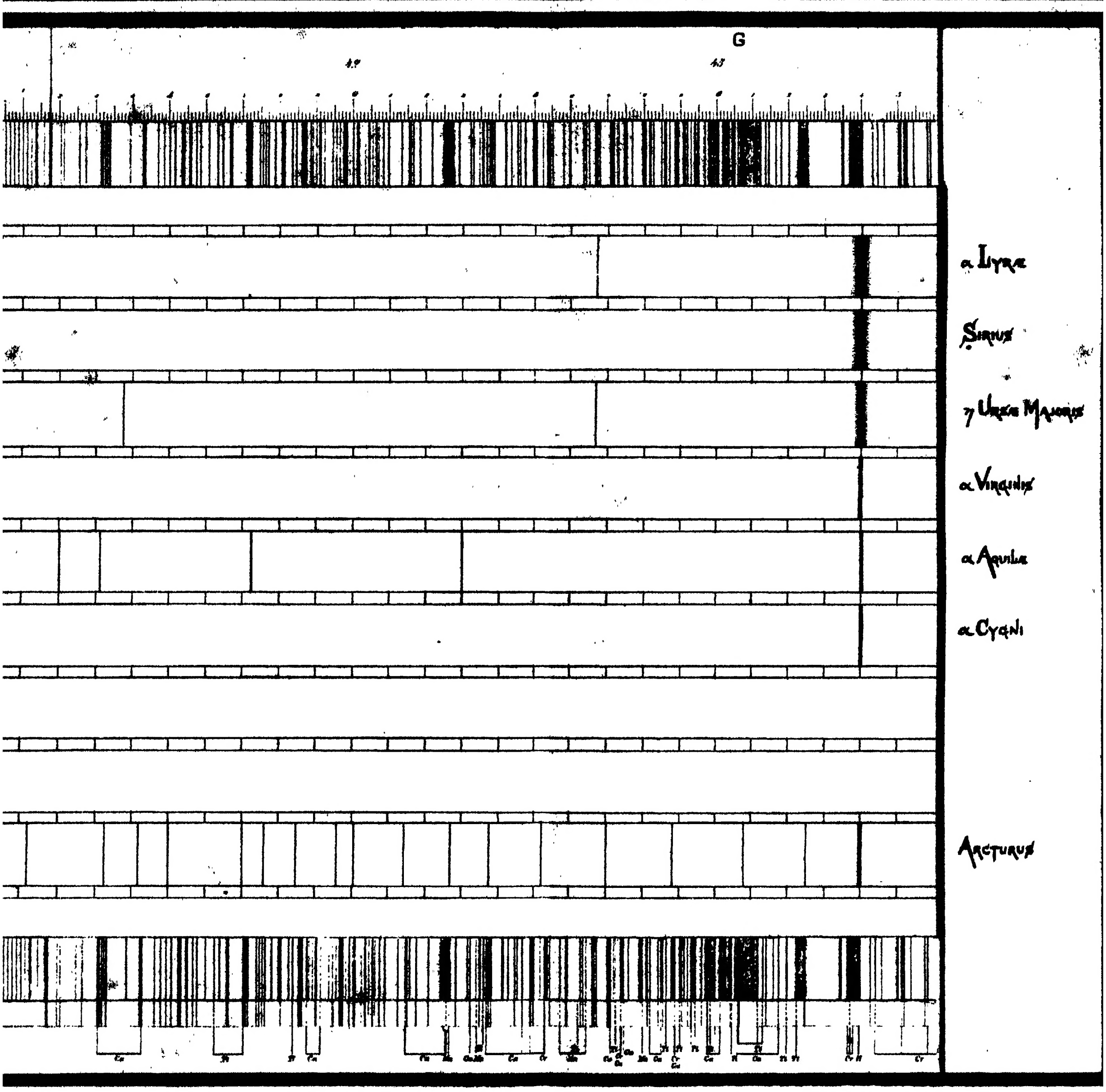


West Newman & Co. lith



Whiteman & Bass, Photo-Litho., 236 Holborn, London.


38
ε
γ
β
α


G
α Lyræ
Sirius
η Ursæ Majoris
α Virginis
α Aquilæ
α Cygni
Arcturus